# THE RADIATIVE CORRECTIONS TO THE COULOMB LAW AND BOHR ENERGY

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March 16, 2023

**Abstract**

The one-loop radiative correction to the photon propagator can be graphically represented by the Feynman diagram of the second order. The physical meaning of this diagram is the process $\gamma \rightarrow (e^- + e^+) \rightarrow \gamma$, where $\gamma$ is denotation for photon, and $e^-, e^+$ is the electron-positron pair. It means that photon can exist in the intermediate state with $e^+, e^-$ virtual particles. The photon propagation function based on such process with electron-positron pair $e^-, e^+$, is determined from the effective emission and absorption sources. The Schwinger source methods of quantum field theory is applied. Then, the Coulomb potential and Bohr energy with radiative corrections is determined.

## 1 Introduction

The theory of the electrostatic field is based on Coulomb's law that summarizes experimental data. This law states that two charged bodies with infinitely small dimensions (two point charges) repel each other if they have like charges and attract each other if they have unlike charges. The force of their interaction force is proportional to $\sim \frac{q_1 q_2}{R_{12}}$, where $q_1$ and $q_2$ are charges of the first and second bodies, respectively and $R_{12}$ is distance between them.

Next, we consider of an electrostatic field in a vacuum. A perfect vacuum cannot naturally be achieved in experiments, and a certain amount of air always remains in the vessels being evacuated. This does not at all mean, however, that the laws of an electric field in a vacuum cannot be studied experimentally Tamm (1979).

The force of interaction of charges being inversely proportional to the square of the distance between them can be directly verified experimentally. It can be verified by sequentially measuring the forces of interaction between pairs of charges.

As regards the sign of charges, it is pure convention that the charges which appear on glass when it is rubbed with silk or flannel are positive. Hence, the charges that are repelled by these charges on the glass are also positive.

It is very important that Coulomb's law holds only for the interaction of point charges, i.e. charged particles of infinitely small dimensions.

The expression "infinitely small" should naturally not be understood here in its strictly mathematical sense. In physics, the expression "infinitely small" (or "infinitely great") quantity is always understood in the sense of "sufficiently small" (or "sufficiently great") quantity-sufficiently small with respect to another quite definite physical quantity. (Tamm, 1979).

In the formulation of Coulomb's law, the infinitely small (point) value of the dimensions of charged bodies is understood in the sense that they are sufficiently small relative to the distance between these bodies, sufficiently small in the sense that with the given distance between the bodies the force of their interaction no longer changes within the limits of the preset accuracy of measurements upon a further reduction of their dimensions and an arbitrary change in their shape (Tamm, 1979).

When determining the resultant of electric forces, we must naturally take account of the circumstance that these forces are vectors. $\mathbf{R}_{12}$ stands for a radius-vector drawn from point 1 to point 2, and $R_{12} = |\mathbf{R}_{12}|$ for the numerical value of the distance between points 1 and 2. It is obvious that $\mathbf{R}_{12} = -\mathbf{R}_{21}$.

Coulomb's law, as in general of any law on which the relevant branch of theoretical physics is based, belongs not only to the direct experimental verification of this law. It also belongs, and this is much more significant, to the agreement with experimental data of the entire complex of theoretical conclusions having this law as one of their cornerstones Tamm (1979).

The radiative corrections to the Coulomb potential follows from the quantum electrodynamics and cannot be determined by the classical mathematical procedures of the classical electromagnetism. So, we explain in the next section the method of the determination of the Green function of photon from which the radiative corrections to the Coulomb potential follow.

## 2 The modified propagation function of photon

It is known from the traditional theory of the Feynman propagator of photon that the one-loop radiative correction to the photon propagator can be graphically represented by the Feynman diagram of the second order. The physical meaning of this diagram is the process $\gamma \;\rightarrow\; (e^- + e^+) \;\rightarrow\; \gamma$, where $\gamma$ is denotation for photon, and $e^-, e^+$ is the electron-positron pair. It means that photon can exist in the intermediate state with $e^+, e^-$ virtual particles.

The modified photon propagation function involving only the two-particle exchange process between sources is then diagrammatic expressed by the analogical way in the Schwinger source theory of QED.

Now, the goal of this contribution is to determine, in the framework of the source methods, the photon propagator corresponding to the intermediate electron-positron pair. The emission photon source emits by manner of the inter medial virtual photon the electron-positron pair which is absorbed by the detection source. Such process is possible when a source emits too much energy to produce only a photon. For the virtual photon the relation $k^2 \neq 0$ and the excitation cannot propagate very far because the balance of energy and momentum is broken.

We can split the process into two parts. The lower part and the upper part. The lower part is the emission part and the upper one is the absorption part. The emission part corresponds to the emission effective source. The effective two-particle source is here the electromagnetic vector potential.

There is no renormalization procedure necessary, neither for the mass nor for the charge. Here is the way from free traveling photon ($k^2 = 0$) to the modified effective photon propagator which experiences from the source an excess of energy ($k^2 = -M^2$), so that after an extremely short time, it can produce an electron-positron pair. Everything happens between the "vacua" $< 0_-|$ and $|0_+ >$. These are not the vacua with particle-antiparticle pairs, etc. They are absolutely empty until an external source delivers or takes the necessary attributes of energy, momentum, spin, etc. to or from the particles to be produced or annihilated (Dittrich, eBook).

The vacuum amplitude corresponding to the primitive interaction that occurs is involved in the vacuum to vacuum amplitude with $i/\hbar = i$, for $\hbar = 1$ (Dittrich, 1978)

$$\langle 0_+|0_-\rangle = e^{iW_{int}}, \tag{1}$$

where

$$W_{int} = \int (dx) j^\mu(x) A_\mu(x) \tag{2}$$

with

$$j^\mu(x) = \frac{1}{2}\psi(x)\gamma^0 eq\gamma^\mu\psi(x) \tag{3}$$

and the vacuum amplitude corresponding to the considered process is

$$\langle 0_+|0_-\rangle = i\int (dx)\psi(x)\gamma^0 eq\gamma^\mu\psi(x)A_\mu(x), \tag{4}$$

where $q$ is the charge matrix: $q = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

The vacuum amplitude for the non-interacting spin 1/2 particle is involved in the general formula (Dittrich, 1978)

$$\langle 0_+|0_-\rangle = \langle 0_+|0_-\rangle^\eta \langle 0_+|0_-\rangle^\eta = e^{iW(\eta)}e^{iW(\eta)} \tag{5}$$

with

$$W(\eta) = \int \psi\gamma^0\eta, \tag{6}$$

from which we extract the vacuum amplitude for the two non-interacting spin 1/2 particles in the form (Schwinger, 1970; 2018)

$$\langle 0_+|0_-\rangle = \frac{1}{2}\left[i\int (dx)\psi(x)\gamma^0\eta(x)\right]^2 \quad = -\frac{1}{2}\int (dx)(dx')\psi(x)\gamma^0\eta(x)\eta(x')\gamma^0\psi(x') \quad \to$$

$$-\frac{1}{2}\int \psi_1(x)\gamma^0\eta_2(x)\eta_2(x')\gamma^0\psi_1(x'). \tag{7}$$

The comparison of eq. (7) with eq. (4) supplies the matrix

$$i\eta_2(x)\eta_2(x')|_{eff\ emiss} = eq\gamma^\mu\gamma^0 A_{2\mu}\delta(x-x'), \tag{8}$$

or, in the momentum representation

$$i\eta_2(p)\eta_2(p')|_{eff\ emiss} = eq\gamma^\mu\gamma^0 A_{2\mu}(k) \tag{9}$$

with $k = p + p'$.

By the same procedure just performed we get for the absorption effective source the following formula:

$$i\eta_1(p)\eta_1(p')|_{eff\ abs} = eq\gamma^\mu\gamma^0 A_{1\mu}(-k) \tag{10}$$

with $k = p + p'$.

Let us remark that the antisymmetry of the left side of eq. (8) for all indices expressing the Fermi-Dirac statistics is involved in the charge matrix $q$.

The amplitude which describes emission and absorption of the two noninteracting particles which propagate freely between the effective source can be separated from the vacuum amplitude

$$\langle 0_+|0_-\rangle = \exp\left\{\int (dx)(dx')\eta_1(x)\gamma^0 G_+(x-x')\eta_2(x')\right\} \tag{11}$$

as its quadratic term of its expansion. Here

$$G_+(x-x') = i\int d\omega_p e^{ip(x-x')}(m-\gamma p), \tag{12}$$

where $d\omega_p = \frac{d(\mathbf{p})}{2\pi^3}\frac{1}{2p^0}$ and $p^0 = +\sqrt{\mathbf{p}^2 + m^2}$ (Dittrich, 1978).

Then,

$$\langle 0_+|0_-\rangle = \frac{1}{2}\int d\omega_p \int d\omega_{p'}\left[\eta_1(-p)\gamma^0(m-\gamma p)\eta_2(p)\eta_2(p')\gamma^0(m+\gamma p')\eta_1(p')\right] \tag{13}$$

Using relation

$$\eta_{1a}(-p)M_{ab}\eta_{1b}(-p') = -M_{ab}\eta_{1b}(-p')\eta_{1a}(-p) \quad = -\mathrm{tr}[M\eta_1(-p')\eta_1(-p)], \tag{14}$$

where $a$ and $b$ are indexes for the eight-dimensional mathematical object $\eta$ and symbol tr denotes the eight-dimensional trace.

Using eq. (14) we can write

$$\langle 0_+|0_-\rangle = \frac{1}{2}\int d\omega_p \int d\omega_{p'} \mathrm{tr}\left[(m-\gamma p)\eta_2(p)\eta_2(p')\gamma^0(-m-\gamma p')\eta_1(-p)\eta_1(-p)\gamma^0\right]. \tag{15}$$

After inserting of the effective emission and absorption sources with $k = p + p'$

$$i\eta_2(p)\eta_2(p')|_{eff\ emiss} = eq\gamma^\mu\gamma^0 A_{2\mu}(k) \tag{16}$$

$$i\eta_1(-p')\eta_1(-p)|_{eff\ abs} = eq\gamma^\mu\gamma^0 A_{1\mu}(-k) \tag{17}$$

into eq. (13) we get with $(\gamma^0)^2 = 1$,

$$\langle 0_+|0_-\rangle \quad = -\frac{1}{2}\int d\omega_p \int d\omega_{p'} \mathrm{tr}\left[(m-\gamma p)eq\gamma A_2(k)(-m-\gamma p')eq\gamma A_1(-k)\right]. \tag{18}$$

Substituting the unit factor

$$1 = (2\pi)^3 \int dM^2 d\omega_k \delta(k-p-p'), \tag{19}$$

we find

$$\langle 0_+|0_-\rangle = -e^2 \int dM^2 d\omega_k A_1^\mu(-k) I_{\mu\nu}(k) A_2^\nu(k), \tag{20}$$

where

$$I_{\mu\nu}(k) = I_{\nu\mu}(k) \quad = (2\pi)^3 \int d\omega_p d\omega_{p'} \delta(k-p-p').\mathrm{tr}[\gamma_\mu(m-\gamma p)\gamma_\nu(-m-\gamma p')]. \tag{21}$$

Using relations $p^2 + m^2 = 0$, $p'^2 + m^2 = 0$ we find

$$\mathrm{tr}[\gamma_\mu(m-\gamma p)\gamma_\nu(-m-\gamma p')] = \mathrm{tr}[(\gamma p + m)\gamma_\mu(\gamma p' - m)(m-\gamma p)\gamma_\nu(-m-\gamma p')]. \tag{22}$$

It may be easily seen that

$$k^\mu I_{\mu\nu}(k) = 0, \tag{23}$$

which implies the gauge invariance of $\langle 0_+|0_-\rangle$ in the form

$$A_\mu(k) \to A_\mu(k) + ik_\mu\lambda(k). \tag{24}$$

The symmetrical tensor constructed from the vector $k_\mu$ is

$$I_{\mu\nu} = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) I(M^2), \tag{25}$$

where $I(M^2)$ can be calculated from relation

$$3I(M^2) = (2\pi)^3 \int d\omega_p d\omega_{p'} \delta(p + p' - k) \mathrm{tr}[\gamma^\mu (m - \gamma p)\gamma_\nu(-m - \gamma p')]. \tag{26}$$

Then, using (Dittrich, 1978)

$$\gamma^\mu \gamma_\mu = -4; \quad \gamma^\mu \gamma p \gamma_\mu = 2\gamma p \tag{27}$$

$$\mathrm{tr}[\gamma_\mu \gamma_\nu] = -4g_{\mu\nu}; \quad \mathrm{tr}\gamma_\mu = 0 \tag{28}$$

$$-M^2 = k^2 = (p + p')^2 = -2m^2 + 2pp' \tag{29}$$

and

$$(2\pi)^3 \int d\omega_p d\omega_{p'} \delta(p + p' - k) = \frac{1}{(4\pi)^2}\left(1 - \frac{4m^2}{M^2}\right)^{1/2}, \tag{30}$$

we obtain

$$I(M^2) = \frac{4}{3}\left(M^2 + 2m^2\right)\frac{1}{(4\pi)^2}\left(1 - \frac{4m^2}{M^2}\right)^{1/2}. \tag{31}$$

Now, we can write the vacuum amplitude in the form

$$\langle 0_+|0_-\rangle = -e^2 \int dM^2 d\omega_k A_1^\mu(-k) \quad \times$$

$$\left(g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\right)\frac{4}{3}\left(M^2 + 2m^2\right)\frac{1}{(4\pi)^2}\left(1 - \frac{4m^2}{M^2}\right)^{1/2} A_2^\nu(k). \tag{32}$$

Since $k^2 A^\mu(k) = J^m$ and $k^2 = -M^2$, we have for the effective sources $J_{1,2}(\mp k)$:

$$A_2^\mu(k) = -\frac{1}{M^2}J_2^\mu(k); \quad A_1^\mu(-k) = -\frac{1}{M^2}J_1^\mu(-k) \tag{33}$$

and after substitution of eq. (33) into vacuum amplitude (32) we get

$$\langle 0_+|0_-\rangle \quad = i\frac{\alpha}{3\pi}\int \frac{dM^2}{M^2}\left(1 + \frac{2m^2}{M^2}\right)\left(1 - \frac{4m^2}{M^2}\right)^{1/2} i d\omega_k J_1^\mu(-k) J_{2\mu}(k). \tag{34}$$

Now, we substitute for the momentum representation of $J^\mu(k)$

$$J_1^\mu(-k) = \int (dx) J_1^\mu(x) e^{ikx} \tag{35}$$

$$J_2^\mu(-k) = \int (dx') J_2^\mu(x') e^{ikx'} \tag{36}$$

and put

$$\Delta_+(x-x';M^2) = i\int d\omega_k e^{ik(x-x')}. \tag{37}$$

Then,

$$\langle 0_+|0_-\rangle = i\frac{\alpha}{3\pi}\int \frac{dM^2}{M^2}\left(1+\frac{2m^2}{M^2}\right)\left(1-\frac{4m^2}{M^2}\right)^{1/2} \quad \times$$

$$\int (dx)(dx')J_1^\mu(x)\Delta_+(x-x';M^2)J_{2\mu}(x'). \tag{38}$$

The amplitude (38) involves the electron-positron pair production and the complete radiation process is described by the amplitude

$$\langle 0_+|0_-\rangle = \int (dx)(dx')J_1^\mu(x)\tilde{D}_+(x-x';M^2)J_{2\mu}(x'), \tag{39}$$

where the momentum representation of $\tilde{D}_+(x-x')$ can be now written with the regard to eq. (38) in the form:

$$\tilde{D}_+(k) = \frac{1}{k^2-i\varepsilon} \quad +$$

$$\frac{\alpha}{3\pi}\int_{4m^2}^\infty \frac{dM^2}{M^2}\left(1+\frac{2m^2}{M^2}\right)\left(1-\frac{4m^2}{M^2}\right)^{1/2}\frac{1}{k^2+M^2-i\varepsilon}, \tag{40}$$

or,

$$\tilde{D}_+(k) = \frac{1}{k^2-i\varepsilon} + \int dM^2\frac{a(M^2)}{k^2+M^2-i\varepsilon} \tag{41}$$

where

$$a(M^2) = \frac{\alpha}{3\pi}\frac{1}{M^2}\left(1+\frac{2m^2}{M^2}\right)\left(1-\frac{4m^2}{M^2}\right)^{1/2}. \tag{42}$$

is the weight function of the $e^+e^-$ - particle production. Let us remark that for $M \gg 2m$ the radiative correction to the Green function of the free photon $\tilde{D}_+$ behave like

$$\int_{4m^2}^\infty \frac{dM^2}{M^2}\frac{1}{k^2+M^2} \tag{43}$$

and therefore there is no convergence problem of integral in eq. (41).

# 3 The modified Coulomb potential

The introduction of the modified propagation function implies the change of the interaction between static charges which originally interact by manner of the Coulomb law. Let us first recall the definition of the potential by means of the Green function. This definition of the potential is not involved in the Roche majestic article on the historical development of the potential (Roche, 2003).

So, from the Green function of the massive scalar particle (Schwinger, 1970; 2018)

$$\Delta_+(x-x') = i\int d\omega_p e^{ip(x-x')-ip^0|x^0-x'^0|} \tag{44}$$

we get as the consequence of it the massless Green function (Schwinger, 1970; 2018; 2-3.91)

$$D_+(x-x') = \Delta_+(x-x', m=0) = D_+(\mathbf{x}-\mathbf{x}', \tau) \quad = \tag{45}$$

$$\frac{i}{4\pi^2}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\int_0^\infty dp^0 \sin\left(p^0|\mathbf{x}-\mathbf{x}'|\right) e^{-ip^0|\tau|} \tag{46}$$

with $\tau = x^0 - x'^0$. The potential corresponding to the Green function (3) is then defined by the following way (Schwinger, 1970; 2018; 2-3.92):

$$V(\mathbf{x}-\mathbf{x}') = \int_{-\infty}^{\infty} d\tau D_+(\mathbf{x}-\mathbf{x}', \tau) \quad =$$

$$\frac{1}{2\pi^2}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\int_0^\infty dp^0 \frac{\sin(p^0|\mathbf{x}-\mathbf{x}'|)}{p^0} = \frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'|}. \tag{47}$$

Replacing $D_+(\mathbf{x}-\mathbf{x}', \tau)$ by its modified spin 1/2 version $\tilde{D}_+(\mathbf{x}-\mathbf{x}', \tau)$, we get the modified Coulomb potential (Schwinger, 1970; 2018):

$$\tilde{V}(x) = \frac{1}{4\pi|\mathbf{x}|} + \frac{\alpha}{3\pi}\int_{(2m)^2}^\infty \left(1+\frac{2m^2}{M^2}\right)\left(1-\frac{4m^2}{M^2}\right)^{1/2}\frac{e^{-M|\mathbf{x}|}}{4\pi|\mathbf{x}|} \quad =$$

$$\frac{1}{4\pi|\mathbf{x}|}\left[1+\frac{\alpha}{\pi}\int_0^1 du\frac{u^2(1-u^2/3)}{1-u^2}\exp\left\{-\frac{2m|\mathbf{x}|}{(1-u^2)^{1/2}}\right\}\right], \tag{48}$$

where

$$u = \left(1-\frac{4m^2}{M^2}\right)^{1/2}. \tag{49}$$

So, we have seen that the four variable Green function is reduced by time integration to the the tree-variable Green function and the exponential part of it is the Green function corresponding to operator $-\Delta + M^2$. It can be obtained by the contour integration with the result

$$\int \frac{(d\mathbf{p})}{(2\pi)^3}\frac{e^{i\mathbf{p}\mathbf{x}}}{p^2+M^2-i\varepsilon} = \frac{e^{-M|\mathbf{x}|}}{4\pi|\mathbf{x}|}. \tag{50}$$

Now, let us consider two cases in eq. (49):

$$\tilde{V}(\mathbf{x}) \approx V(\mathbf{x}); \quad 2m|\mathbf{x}| \gg 1 \tag{51}$$

$$\tilde{V}(|\mathbf{x}|) \approx \frac{1}{4\pi|\mathbf{x}|}\left[1+\frac{2\alpha}{3\pi}\left(\lg\frac{1}{m|\mathbf{x}|} - C - \frac{5}{6}\right\}\right]; \quad 2m|\mathbf{x}| \ll 1, \tag{52}$$

where

$$C = 0,57721.... \tag{53}$$

is the Euler constant. The additional logarithmic behavior under these circumstances comes from the interval of $M$-integration such that $|\mathbf{x}|^{-1} \gg M \gg 2m$. The evaluation of $\tilde{V}(|\mathbf{x}|)$ for $2m|\mathbf{x}| \ll 1$ is obtained by partitioning the integral at some value of $M$ that satisfies the considered inequality.

Let us remark that the used methods of this article can be applied in the area of the general theory of the potential (Gunter, 1953).

## 4 Discussion

The effect we are discussing here, is usually named as vacuum polarization. It increases the strength of the Coulomb interaction with diminishing distance. The increase is quite small, however, at any realizable distance. Thus, with $2m|\mathbf{x}| \sim \mathbf{10^{-3}}$, which represents a distance of roughly $10^{-14}$ cm when the electron mass is used, it is approximately one percent. In view of the logarithmic dependence on distance, this order of magnitude cannot be changed significantly by any conceivable improvement in experimental process; a ten-percent increase in interaction strength requires dropping to a distance $\sim 10^{-37}$ cm. And long before such distances could be approached, the situation would change qualitatively through the growing importance of particles that are heavier than the electron.

Nevertheless, vacuum polarization effects are measurable at the present level of experimental technique. The most elementary situation is that of hydrogen atoms where the strengthened attraction between electron and nucleus depresses the energy values of zero orbital angular momentum states, these being the ones in which the electron spends appreciable time near the nucleus (Schwinger, 1983; 2018).

Simple perturbation theory can be applied to the change in interaction energy,

$$\delta V(\mathbf{x}) = -Ze^2 \delta\mathcal{D}(\mathbf{x}), \tag{54}$$

where $\delta\mathcal{D}(\mathbf{x})$ represents the difference between $\tilde{\mathcal{D}}(\mathbf{x})$ and

$$D(\mathbf{x}) = \frac{1}{4\pi}|\mathbf{x}|. \tag{55}$$

In a state with non-relativistic wave function $\psi(\mathbf{x})$, appropriate to the restriction $Z\alpha \ll 1$, we have

$$\delta E = \int d(\mathbf{x})\delta V(\mathbf{x})|\psi(\mathbf{x})|^2 \cong -4\pi Z\alpha|\psi(0)|^2 \int d(\mathbf{x})\delta\mathcal{D}(\mathbf{x}), \tag{56}$$

which uses the fact that the perturbation is significant only over distances that are small compared with atomic dimensions. The integration that appears here is equivalent to evaluating the zero momentum limit of $\delta D_+(k)$, and

$$\int d(\mathbf{x})\delta\mathcal{D}(\mathbf{x}) = \frac{\alpha}{3\pi}\int_{4m^2}^{\infty}\frac{dM^2}{M^4}\left(1+\frac{2m^2}{M^2}\right)\left(1-\frac{4m^2}{M^2}\right)^{1/2} =$$

$$\frac{\alpha}{\pi}\frac{1}{(2m)^2}\int_0^1 dvv^2\left(1-\frac{1}{3}v^2\right)=\frac{\alpha}{15\pi}\frac{1}{m^2}. \tag{57}$$

Only s-states need to be considered. For principal quantum number $n$

$$|\psi_{ns}(0)|^2=\frac{1}{\pi}\left(\frac{Z\alpha}{n}m\right)^3 \tag{58}$$

and

$$\delta E_{ns}=-\frac{4}{15\pi}\frac{Z^4\alpha^5}{n^3}m, \tag{59}$$

or,

$$\frac{\delta E_{ns}}{\left(\frac{1}{2}\frac{Z^2\alpha^2}{n^2}m\right)}=-\frac{8}{15\pi}\frac{Z^2\alpha^3}{n}, \tag{60}$$

the latter giving a comparison with the Bohr energy values. More details will not be supplied now since, this effect is rather minor compared to another that displaces the s-states in the opposite sense. Let us remark that quantum numbers can take the following values: $n=1,2,3,...$ - (principal quantum number), $l=0,1,2,...,n-1$ - (azimuthal quantum number) and $m=-l,...\ l,$ - (magnetic quantum number) (Merzbacher, 1988). In our case, Only s-states are considered being nonzero.

The existence of the vacuum polarization effect must be inferred from the quantitative comparison with experiment; in its absence a small but significant discrepancy with experiment would remain (Schwinger, 1983; 2018).

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