On the connection between the powers of natural numbers and the factorial

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Abstract: The article describes the relationship between the power of natural numbers and the factorial, established as a result of applying the binomial transformation method to the sequence of the of natural numbers.

1 Introduction

As a result of applying the binomial transformation method [1] to a sequence of powers of natural numbers, while searching for solutions to known problems of mathematics, we have established a correlation between the power of natural numbers and the factorial. This pattern is closely related to the Euler transform [2], which is the result of applying the binomial transformation to a sequence, in which the direct differences of the sequence are calculated. It also has a connection with the theorem on the mean value of divided differences for a polynomial function [3]. At the same time, in textbooks and reference materials on mathematics, as well as in popular books such as [4], which describes the patterns of numbers and the factorial function, there is no information about the relationship between the power of natural numbers and the factorial, so we hope that this work will be interesting and useful to readers.

It should be noted that between the sequence of natural numbers and other functions there are many interesting previously undiscovered relationships.

2 Difference of two natural numbers that have the same powers

First, we study the difference of two natural numbers that have the same degree. To do this, consider a sequence of squares of integers, for example, 0, 1, 4, 9, 16, 25, 36. Then we take two adjacent squares and subtract the smaller square from the larger square:1-0=1, 4-1=3, 9-4=5, 16-9=7, 25-16=9, 36-25=11 resulting in the following sequence 1, 3, 5, 7, 9 , 11. After that, we again take two adjacent numbers of the sequence and subtract the smaller one from the larger one, then we get a sequence of twos 3-1=2, 5-3=2, 7-5=2, 9-7=2, 11-9=2 . In this case, after two iterations, we got a constant number of 2.

Key words: power, factorial, binomial, relationship.
MSC Numbers:11A25

Next, we take a sequence of cubes of integers, for example 0, 1, 8, 27, 64, 125, 216. Then we take two adjacent cubes and subtract the smaller cube from the larger cube: 1-0=1, 8 -1=7, 27-8 =19, 64-27=37, 125-64=61, 216-125=91, as a result we get the following sequence 1, 7, 19, 37, 61, 91. After that, again we take two neighboring numbers of the sequence and from the larger subtract the smaller, then we get the sequence 7-1=6, 19-7=12, 37-19=18, 61-37=24, 91- 61=30, then repeat the procedure for this sequence and get the sequence of one natural number 12- 6=6, 18-12=6, 24-18=6, 30-24=6. Thus, for the cube, we got a constant number of 6 after three iterations.

If the above procedure (binomial transformation method) is carried out for the fourth and fifth powers of natural numbers until a sequence of one number is obtained, then as a result we obtain, respectively, sequences of constant numbers 24 and 120, which are obtained, respectively, after four and five iterations.

It is easy to see that the numbers that form a sequence of one number are equal to factorials whose arguments correspond to the degree of the natural number under consideration: 2=2!, 6=3!, 24=4! and 120=5!. Note that if we consider the difference of natural numbers of the first degree, i.e. the difference between adjacent natural numbers, then we get a sequence of constant number 1, which corresponds to the factorial of number 1.

Thus, we can write the following relationships between powers of natural numbers and the factorial, $(a+1)^1 - a^1 \leftrightarrow 1!$; $(a+1)^2 - a^2 \leftrightarrow 2!$; $(a+1)^3 - a^3 \leftrightarrow 3!$; $(a+1)^4 - a^4 \leftrightarrow 4!$; $(a+1)^5 - a^5 \leftrightarrow 5!$ or for the general form we have the following formula

(1) $$(a+1)^n - a^n \leftrightarrow n!$$

Note. The sign $\leftrightarrow$ in the above formulas means that there is a relationship between the difference in the powers of natural numbers and the factorial.

Table 1, as an example, shows the results of applying the binomial transformation to the sequence of the fifth power of natural numbers, and obtaining numbers equal to 5!.

Note that the number of iterations i, which must be carried out to obtain a number equal to the factorial, is equal to the degree of the considered numbers, i.e. $i = n$ for equal degrees. And if we write the difference of powers of natural numbers in the following form $(a+q)^n - a^n$, where $a, q, n \in \mathbb{N}$, then we obtain a correspondence between the difference of equal powers of two natural numbers of the following form

(2) $$(a+q)^n - a^n \leftrightarrow q \cdot n!$$

Table 1. The difference of the fifth power of natural numbers

a	a^5	$(a+1)^5 - a^5, i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
1	**1**					
2	**32**	31				
3	**243**	211	180			
4	**1024**	781	570	390		
5	**3125**	2101	1320	750	360	
6	**7776**	4651	2550	1230	480	120
7	**16807**	9031	4380	1830	600	120
8	**32768**	15961	6930	2550	720	120
9	**59049**	26281	10320	3390	840	120
10	**100000**	40951	14670	4350	960	120

3 Difference of two natural numbers with different powers

Above, we considered the difference of equal powers of two natural numbers, so the question arises: what happens if the two numbers in question have different powers?

In order to get an answer to the question posed, we further study the difference of two natural numbers having different degrees, i.e., consider the difference of the following form $a^m - (a+q)^n$, where $m > n;\ q = 0, 1, 2, \dots$.

If the powers of the natural numbers under consideration are different $m > n$, then the constant number equal to the factorial does not depend on q,

(3) $a^m - (a+1)^n \leftrightarrow m!, i = m + 1;$

(4) $a^m - (a+q)^n \leftrightarrow m!, i = m + 1.$

At the same time, the number of iterations $i = m + 1$ also does not depend on the difference of the exponents ($m - n = j$), for example, for $a^6 - (a+q)^3$ and $a^6 - (a+q)^2$, where the power difference differs (6-3=3, 6-2=4) constant number 720=6! formed after 7 iterations.

The pattern described above is also true for the cases $(a+1)^m - a^n$ and $(a+q)^m - a^n$, i.e. and in this case the constant number equal to the factorial does not depend on q,

(5) $(a+1)^m - a^n \leftrightarrow m!, i = m + 1;$

(6) $(a+q)^m - a^n \leftrightarrow m!, i = m + 1$, where $m > n, q = 0, 1, 2, \dots$

And in this case, the number of iterations $i = m + 1$ also does not depend on the difference of the exponents, for example, for $a^5 - (a+q)^3$ and $a^5 - (a+q)^2$, where the difference in degrees different, constant number 120=5! formed after 6 iterations.

Table 2 shows the results of applying the binomial transformation to the sequence of the fifth power and the square of natural numbers, and obtaining numbers equal to 5!.

Table 2. The difference of the fifth power and the square of natural numbers

a	a^5	a^2	$(a+2)^5 - a^2, i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$
0	**0**	**0**						
1	**1**	**1**						
2	**32**	**4**	32					
3	**243**	**9**	242	210				
4	**1024**	**16**	1020	778	568			
5	**3125**	**25**	3116	2096	1318	750		
6	**7776**	**36**	7760	4644	2548	1230	480	
7	**16807**	**49**	16782	9022	4378	1830	600	120
8	**32768**	**64**	32732	15950	6928	2550	720	120
9	**59049**	**81**	59000	26268	10318	3390	840	120
10	**100000**	**100**	99936	40936	14668	4350	960	120

In Table 3, an example is given for the case $a^n - a^m \leftrightarrow m!, i = m + 1$, where $m > n, q = 0, 1, 2, \dots$, when the difference is a negative number. As follows from Table 3, and for these cases, the constant number obtained as a result of applying the binomial transformation corresponds to the factorial, although it has a negative sign.

Table 3. The difference between the cube and the fifth power of natural numbers

a	a^5	a^3	$a^3 - a^5, i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$
0	**0**	**0**	0					
1	**1**	**1**	0					
2	**32**	**8**	-24	-24				
3	**243**	**27**	-216	-192	-168			
4	**1024**	**64**	-960	-744	-552	-384		
5	**3125**	**125**	-3000	-2040	-1296	-744	-360	
6	**7776**	**216**	-7560	-4560	-2520	-1224	-480	-120
7	**16807**	**343**	-16464	-8904	-4344	-1824	-600	-120
8	**32768**	**512**	-32256	-15792	-6888	-2544	-720	-120
9	**59049**	**729**	-58320	-26064	-10272	-3384	-840	-120
10	**100000**	**1000**	-99000	-40680	-14616	-4344	-960	-120

Conclusion

The difference of any two natural numbers that have equal or different natural degrees is correlated with a factorial function, while the argument of the factorial will be equal to the exponent. If two numbers have different degrees, then the factorial argument will be equal to the larger exponent. The described pattern is also valid for cases where the difference of the powers of two natural numbers is negative.

References

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