A new proof that the reals are uncountable

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Abstract. We show that the reals are uncountable using Russell's Paradox, in a proof reminiscent of Gödel's proof of the Incompleteness Theorem. This simple proof could be offered as an alternative to people who find it difficult to accept Cantor's Diagonalization argument.

Let us assume, for the sake of contradiction, that the real numbers between 0 and 1 (inclusive) are countable. Since *"…the continuum of numbers, or real numbers system … is the totality of infinite decimals,"* [1] then a listing of the real numbers between 0 and 1 in base 3 (a ternary numeral system) would have all possible sequences of the digits 0, 1, and 2 after the radix point.[2]

Without loss of generality:

- Every ternary expansion is represented on a row of the countable list by a string of base 3 digits.
- Every digit 2 is replaced by a space to separate strings of digits on a row. Multiple spaces are replaced with a single space.
- The ternary expansions encompass *all possible strings* of base 2 digits (*i.e.,* all possible combinations of the digits 0 and 1)
- Each finite string in base 2 of a row can refer to itself or another row on the countable list.
- There is a row *among all possible rows* that lists all the rows that do not list themselves.

The last bullet is similar to Russell's Paradox.[3] Does this row that catalogues "all the rows that do not include themselves" include itself? If it does, then it must not list itself. If it doesn't, then it must include itself in the list.

So, there is one row that leads to a contradiction when the strings are given meaning.[4] Therefore, we must reject the assumption that a countable listing of all the real numbers between 0 and 1 can be complete. Accordingly, the real numbers cannot be countable.

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[1] Courant, Richard, and Herbert Robbins. 1996. *What Is Mathematics? : An Elementary Approach to Ideas and Methods*. Oxford: Oxford University Press. p.68.

[2] Note that each real number is not expressed uniquely on the list (*i.e.*, since 1 = 0.222… in base 3). However, the list can be converted to a countably unique list by going through the list and eliminating duplicate real numbers that end in 222…

[3] Weisstein, Eric W. "Russell's Antimony." From *MathWorld*—A Wolfram Web Resource. https://mathworld.wolfram.com/RussellsAntinomy.html (accessed Dec. 18, 2021).

[4] This is reminiscent of Kurt Gödel's 1931 mathematically rigorous proof that:

> … we could not set out a complete set of rules for arithmetic. Gödel showed that we could use any set of possible rules to create sentences similar to the sentence, "this sentence is false." If it's true, then it's false. But if it's false, then it's true. Any attempt to create rules would either allow sentences like, "this sentence is unprovable" to be proven. And so, we would have sentences that can be proved but are false. We would have just proven the sentence that says it can't be proven. Alternatively, we could strengthen our rules to exclude these sentences. But then, because we could no longer prove the sentence, the sentence, "this sentence is unprovable" would be true. And so, we would have true sentences that we can't prove in our system, making our system incomplete.

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