Gravitational Wave Colliding with a Small Mass Having Path Not Approximately a Geodesic

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Abstract

We consider a system of a gravitational plane wave pulse colliding with a point mass of small mass. The path of the mass is shown not to be approximately a geodesic.

1 Gravitational plane wave pulse metric

Define $u = t - x$ and let the metric $g_{\mu\nu}$ of the gravitational plane wave pulse be determined by [1]

$$ds^2 = -dt^2 + dx^2 + [L(u)]^2 \Big[e^{2\beta(u)} dy^2 + e^{-2\beta(u)} dz^2 \Big] \tag{1}$$

and $g_{\mu\nu}(u) = \eta_{\mu\nu}$ for $u < 0$. From the equation $R_{\mu\nu} = 0$ the only relation between L and β is

$$\frac{d^2L}{du^2}(u) + \left[\frac{d\beta}{du}(u)\right]^2 L(u) = 0 \tag{2}$$

Let $L(0) = 1$ and $\beta \neq 0$. We then have by (2) that $L(u)$ will decrease and become zero at some point $u_0 > 0$. Consequently $g_{22}(u) > 0$ for $u < u_0$.

2 Proper Lorentz transformation

Consider a coordinate transformation from t, x, y, z to t', x', y', z' coordinates that is a composition of a rotation by θ about the z axis followed by a boost by $2\cos\theta/(1+\cos^2\theta)$ in the x direction followed by a rotation by $\theta + \pi$ about the z axis. For θ/π not an integer this is a proper Lorentz transformation such that

$$\begin{aligned}
t &= t'(1 + 2\cot^2\theta) - 2x'\cot^2\theta + 2y'\cot\theta & (3)\\
x &= 2t'\cot^2\theta + x'(1 - 2\cot^2\theta) + 2y'\cot\theta & (4)\\
y &= 2t'\cot\theta - 2x'\cot\theta + y' & (5)\\
z &= z' & (6)
\end{aligned}$$

By (3) and (4) we have $u = t - x = t' - x' = u'$. For the metric (1) and transformation(3)-(6) define the metric $g'_{\mu\nu}(u')$ by

$$g'_{\mu\nu}(u') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(u) \tag{7}$$

hence we get

$$\begin{aligned}
ds^2 = {} & \Big\{ -1 - 4[1 - g_{22}(u')]\cot^2\theta \Big\} dt'^2 + 8[1 - g_{22}(u')]\cot^2\theta\, dt' dx' \\
+ {} & \Big\{ 1 - 4[1 - g_{22}(u')]\cot^2\theta \Big\} dx'^2 - 4[1 - g_{22}(u')]\cot\theta\, dt' dy' \\
+ {} & 4[1 - g_{22}(u')]\cot\theta\, dx' dy' + g_{22}(u') dy'^2 + g_{33}(u') dz'^2 \qquad (8)
\end{aligned}$$

Since $g_{\mu\nu} = \eta_{\mu\nu}$ for $u < 0$ we have $g'_{\mu\nu}(u') = \eta_{\mu\nu}$ for $u' < 0$. The metric $g'_{\mu\nu}(u')$ satisfying $R'_{\mu\nu} = 0$ and $g'_{\mu\nu}(u') = \eta_{\mu\nu}$ for $u' < 0$ is then also the metric of a gravitational plane wave pulse.

3 A geodesic of the metric $g'_{\mu\nu}$

The curve

$$t'(\lambda) = (1+2\cot^2\theta)\lambda - 2\cot^2\theta \int_0^\lambda \frac{dw}{g_{22}(w)} \tag{9}$$

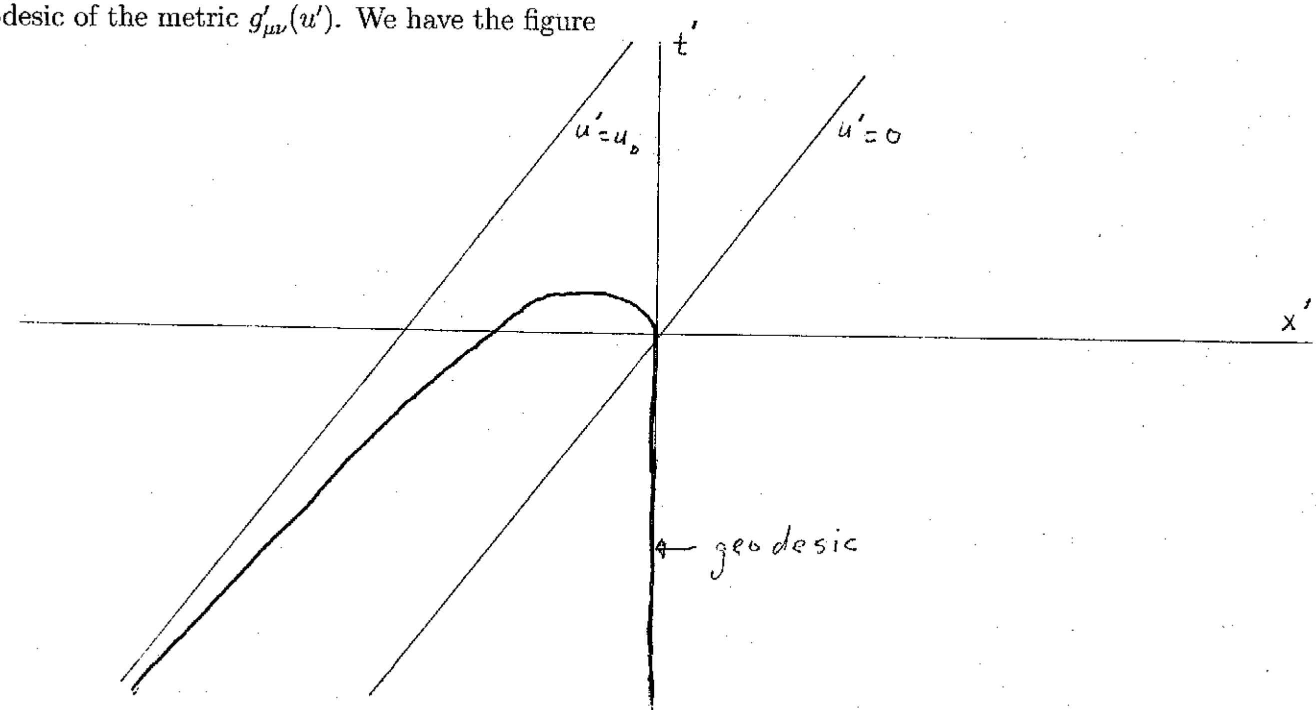
$$x'(\lambda) = 2\cot^2\theta\lambda - 2\cot^2\theta \int_0^\lambda \frac{dw}{g_{22}(w)} \tag{10}$$

$$y'(\lambda) = -2\cot\theta\lambda + 2\cot\theta \int_0^\lambda \frac{dw}{g_{22}(w)} \tag{11}$$

$$z'(\lambda) = 0 \tag{12}$$

is a geodesic of the metric $g'_{\mu\nu}(u')$. We have the figure

Fig.



4 Path of particle is not approximately a geodesic

Consider a system of a gravitational plane wave pulse that collides with a point mass A initially at rest at the origin. Let $\tilde{g}'_{\mu\nu}(t', x', y', z')$ be the metric of the combined system of wave and A. The wave comes from infinity so for points having large negative t' and $x' < t'$ the wave is far from A and so is little affected by A. Consequently $\tilde{g}'_{\mu\nu}(t', x', y', z')$ is approximately $g'_{\mu\nu}(t' - x')$ at points having large negative t' and $x' < t'$. Now $g'_{\mu\nu}(t' - x')$ is finite at all points hence $\tilde{g}'_{\mu\nu}(t', x', y', z')$ is finite at points having large negative t' and $x' < t'$.

Assume the path of A is approximately the curve (9)-(12) for an A of small mass. We then have using the figure that A will reach a point p having large negative t' and $x' < t'$. By previous paragraph $\tilde{g}'_{\mu\nu}(t', x', y', z')$ is then finite at p. Since A is a point mass $\tilde{g}'_{\mu\nu}(t', x', y', z')$ at p is not finite. We then have $\tilde{g}'_{\mu\nu}(t', x', y', z')$ is both finite and not finite at p. This is a contradiction. The path of A is then not approximately a geodesic.

References

[1] C. Misner, K. Thorne, J. Wheeler, *Gravitation*, p. 957

[2] K. De Paepe, viXra:1711.0414

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