

# Necessary Dimensionality Conditions for the Detection of Stimuli

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## Abstract

Transfer functions are used for modelling the behavior of sensors. Dimensionally, each transfer function defines a relation between the dimensions of the sensor parameters and the dimension of the response signal. The sensor's ability to detect stimuli depends on the stimuli's ability to vary the parameters of the sensor. We state the necessary dimensionality conditions that need to be imposed on the stimuli, parameters, transfer function and response signal in order for stimuli to be detected by a sensor.

A sensor is a device that receives a stimulus and responds with an electric signal [1]. When a stimulus is received by a sensor, the sensor's ability to sense the stimulus depends upon the effect the stimulus can have on the parameters of the sensor. For example, for a sensor whose output format is electric potential, a necessary condition for a stimulus to be detected is the ability of the stimulus to vary the sensor's parameters such that the variation results in a recognizable change in the electric potential of the sensor. From a dimensional standpoint, we observe that a stimulus can have an effect on the sensor's parameters only if the dimension of the stimulus contains the dimension of the time derivative of one of the parameters of the sensor. For example, one of the parameters of a capacitive touch sensor is the distance between the plates. The dimension of this parameter is  $L$  and therefore, this parameter can only be varied by a stimulus having dimensions containing the time derivative of length  $LT^{-1}$ , such as force, whose dimensions are  $M(LT^{-1})^2$ . Thus the necessary dimensionality condition for a stimulus to be detected is

that its dimension contains the dimension of the time derivative of atleast one of the sensor's parameters. On the sensor side, the dimensions of the parameters have to be mapped to the dimension of the response signal, which implies that the dimensions of the parameters have to be combined such that the resulting dimension is that of the response signal. We will denote this combination as the transfer function's dimension. To consolidate, there are four dimensions associated with a sensor: the dimension of the stimulus, the dimension of the parameters, the dimension of the transfer function, and the dimension of the response signal. We would like to find the dimensional relations that need to be satisfied between: stimuli parameters, parameters transfer function, and transfer function response signal.

The space of fundamental and derived physical units forms a vector space over the field of real numbers [2]. The SI base units are the basis vectors, the multiplication of physical units is the vector addition operation, and raising the units to powers is the scalar multiplication operation. Formally, we have  $V = \mathbf{R}^7$  as the vector space of physical units with basis  $B = \{T, L, M, I, \Theta, N, J\} \equiv \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\}$ , with the operations,

$$\begin{aligned} u + v &= (T^{u_1} L^{u_2} M^{u_3} I^{u_4} \Theta^{u_5} N^{u_6} J^{u_7}) (T^{v_1} L^{v_2} M^{v_3} I^{v_4} \Theta^{v_5} N^{v_6} J^{v_7}) \\ &= T^{u_1+v_1} L^{u_2+v_2} M^{u_3+v_3} I^{u_4+v_4} \Theta^{u_5+v_5} N^{u_6+v_6} J^{u_7+v_7}, \quad \forall u, v \in V \\ ru &= (T^{u_1} L^{u_2} M^{u_3} I^{u_4} \Theta^{u_5} N^{u_6} J^{u_7})^r \\ &= T^{ru_1} L^{ru_2} M^{ru_3} I^{ru_4} \Theta^{ru_5} N^{ru_6} J^{ru_7}, \quad \forall r \in \mathbf{R}, u \in V \end{aligned}$$

Let  $x_1, \dots, x_n$  be the dimensions of the parameters of a sensor. Since a stimulus can be received only if it changes one or more of the parameters of the sensor with respect to time, it is necessary for the stimulus to be of the form  $x_i - \mathbf{e}_1$  for some  $i \in \{1, \dots, n\}$ . However, it is not required for the stimulus to be exactly equal to this expression; the necessary criteria is that it should contain this expression. To illustrate, we will use the capacitive touch sensor example. The dimension of the distance between the plates is  $\mathbf{e}_2$ , and thus, the dimension of its time derivative is  $\mathbf{e}_2 - \mathbf{e}_1$ . The dimension of force is  $-2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3 = 2(\mathbf{e}_2 - \mathbf{e}_1) + \mathbf{e}_3$ . This expression contains  $\mathbf{e}_2 - \mathbf{e}_1$ , and hence from a dimensional standpoint, the force can be detected by the sensor. The process of checking whether the dimension of the stimulus contains the time derivative of one or more of the parameters can be expressed using projection operators [3]. Let  $S = \bigoplus_{i=1}^n \langle x_i - \mathbf{e}_1 \rangle$  be the subspace spanned by the time

derivatives of the parameters of the sensor, and let  $\bar{S}$  be its complement. Define a linear operator  $\rho : V \rightarrow V$  such that  $\rho(s + \bar{s}) = s$ . Thus, a stimulus having dimension  $v \in V$  can be detected iff  $\rho(v) \neq 0$ , that is,  $v \notin \ker(\rho) = \bar{S}$ . Therefore, the relation that needs to be satisfied between the dimension of the stimulus  $v$  and the dimension of the parameters is  $v \in V - \bar{S}$ .

Let the dimension of the response signal of the sensor be  $r$ . Let the dimensions of the  $n$  parameters of the sensor be represented by a matrix  $A = [x_1 \dots x_n] \in M_{7 \times n}(\mathbf{R})$ . The sensor must combine the dimensions of the parameters, such that it results in the dimension of the response signal. That is, we are looking for the solution of the equation  $Ax = r$ . The unique solution of minimum norm is given by  $x = A^+r$ , where  $A^+$  is the Moore-Penrose inverse [3] of  $A$ . We denote the solution  $x$  as the dimension of the transfer function. Therefore, the relation that needs to be satisfied between the dimension of the parameters and the dimension of the transfer function is  $x = A^+r$ . Now, in order to calculate the dimension of a function, we will need to define a functional that obeys the rules of dimensional calculation. The rule of dimensional homogeneity states that only physical quantities having the same dimension may be compared, equated, added, or subtracted. In contrast, physical quantities having arbitrary dimensions can be multiplied and divided; this rule also applies for the derivative and integral. In addition, the dot product and cross product of the physical quantity with the divergence operator only retains a length dimension decreased by 1, whereas the dimension of the gradient of a physical quantity is one less the dimension of the quantity in its length component. Formalizing these rules, let  $C = \{f, g, \dots\}$  be a function space of functions having  $n$  parameters, and let the base field of the space be  $F = \{c, d, \dots\}$ . A dimension function  $D$  is a function that satisfies,

$$\begin{aligned}
D : C &\rightarrow V \\
c &\mapsto 0 \\
f \pm c &\mapsto D(c) \iff c \neq 0 \\
f \pm g &\mapsto D(f) \iff D(f) = D(g) \\
fg &\mapsto D(f) + D(g) \\
\frac{f}{g} &\mapsto D(f) - D(g)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial g} &\mapsto D(f) - D(g) \\
\int f \, dg &\mapsto D(f) + D(g) \\
\nabla \cdot f &\mapsto ([D(f)]_2 - 1)\mathbf{e}_2 \\
\nabla \times f &\mapsto ([D(f)]_2 - 1)\mathbf{e}_2 \\
\nabla f &\mapsto D(f) - \mathbf{e}_2
\end{aligned}$$

Thus, a function  $f \in \mathcal{C}$  can be a transfer function only if  $D(f) = A^+r$ . To find the functions in  $\mathcal{C}$  that satisfy this condition, we decompose each function  $f \in \mathcal{C}$  into a sum of functions  $f = \sum_{i=1}^k f_i$ . By the rule of dimensional homogeneity,  $f$  cannot be a transfer function if  $D(f_i) \neq D(f_j)$  for some  $i \neq j$ . However, if the function satisfies this rule, it follows that we can determine whether  $D(f) = A^+r$  by determining whether  $D(f_i) = A^+r$ , for any  $i$ . We can calculate  $D(f_i)$  by decomposing  $f_i$  into a product  $\prod_{j=1}^m f_{i,j}$ , which gives  $D(f_i) = \sum_{j=1}^m D(f_{i,j})$ , and hence,  $A^+r = D(f_{i,1}) + \dots + D(f_{i,m})$ . It follows that the set of decompositions of  $A^+r$  into a sum of  $m$  vectors  $v_j \in V$  correspond to the set of valid transfer functions  $f_{i,j}$ , and thus the set  $A_m = \{(v_1, \dots, v_m) \in V^m \mid \sum_{j=1}^m v_j = A^+r\}$  is the set of required decompositions. Let  $A = \bigcup_{i=1}^\infty A_i$ , and  $F: A \rightarrow \mathcal{C}$  be an injective function. We observe that  $F(A)$  is a basis for the valid transfer functions in  $\mathcal{C}$ , since each valid  $f \in \mathcal{C}$  can be written as  $\sum_{i=1}^n f_i$ , where  $f_i \in F(A)$ .

In conclusion, given a sensor having parameter dimensions  $A = [x_1, \dots, x_n]$ , transfer function  $f$  and response signal dimension  $r$ , when a stimulus of dimension  $v$  is recieved by the sensor, the necessary dimensionality conditions for its detection are:

- $v \in V - \overline{\bigoplus_{i=1}^n \langle x_i - \mathbf{e}_1 \rangle}$
- $D(f) = A^+r$

## References

1. Jacob Fraden. *Handbook of Modern Sensors: Physics, Designs, and Applications*. Springer International Publishing Switzerland, 5th edition, 2016.

2. SymPy Development Team. *SymPy Documentation: Release 1.12rc1*. 2023.
3. Steven Roman. *Advanced Linear Algebra*. Springer, New York, 2nd edition, 2005.