

# Dimensions of a Graph

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## Abstract

We introduce eight infinite sets of constants. Some we calculate. Roughly speaking, we seek graphs as small as possible. The graphs serve as examples for different kinds of ‘dimensions’.

In the second part ‘Points’, we place points on the plane, and from this, we ask infinite many questions.

In a third part, we introduce for each graph a sequence called the ‘Thuerey Numbers’. At the end, we summarize the open questions.

*Keywords and phrases:* graph, dimension

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## 1 Introduction

In the book [1] we have read on page 93 Erdős’s Open Problem 13.12: ‘What is the smallest number of edges in a graph  $G$ , such that  $\dim G = 4$ ?’.

That gives the cause to introduce some constants, which are all elements of the natural numbers  $\mathbb{N} := \{1, 2, 3, \dots\}$ .

We assume that the reader is familiar with the concepts of graphs, edges, and vertices. A *display* of an abstract graph  $G$  means an isomorphic graph  $H$  in  $\mathbb{R}^m$  such that the edges are line segments, and  $H$  has a finite number of intersection points.

We repeat the definitions of four ‘dimensions’.

Let  $G$  be a connected finite graph.

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**Definition 1.** The dimension  $\dim(G)$  is the minimum natural number  $k$  such that there is a display of an isomorphic graph  $H$  in  $\mathbb{R}^k$ , and all pairs of adjacent vertices in  $H$  have the same distance. Please see [1] and [2].

The symbol  $Edim(G)$  means the same, except that all pairs of vertices of  $G$  which are adjacent have the same distance, while pairs that are not adjacent have other distances. See [1].

The *edge distance* of two vertices  $\vec{a}$  and  $\vec{b}$  in a graph  $G$  is defined as the minimum number of edges in a way from  $\vec{a}$  to  $\vec{b}$ .

The dimension  $mdim(G)$  is defined as the smallest cardinality of a resolving set  $R$ , where a resolving set is a subset of the vertices of  $G$ . It has the property that for each pair  $(\vec{a}, \vec{b})$  of vertices of  $G$  where  $\vec{a} \neq \vec{b}$  there is  $\vec{r} \in R$  such that the edge distance between  $\vec{a}$  and  $\vec{r}$  differs from the edge distance between  $\vec{b}$  and  $\vec{r}$ . In the case that there is no resolving set we write  $mdim(G) = \infty$ . See the papers [3] and [4].

The distance from a vertex  $\vec{v}$  to an edge  $e$  between vertices  $\vec{a}$  and  $\vec{b}$  is the minimum of the edge distances between  $\vec{a}$  and  $\vec{v}$  and  $\vec{b}$  and  $\vec{v}$ , respectively. We call this number  $d(\vec{v}, e)$ .

The number  $edim(G)$  is defined as the smallest cardinality of an *edge resolving set*  $E$ , where an edge resolving set is a subset of the vertices of  $G$ . It has the property that for each pair  $(f, g)$  of edges of  $G$  where  $f \neq g$  there is  $\vec{r} \in E$  such that  $d(\vec{r}, f) \neq d(\vec{r}, g)$ . In the case that there is no edge resolving set we write  $edim(G) = \infty$ . See [5] or [6].

We define eight sequences of natural numbers. To avoid problems we define the denomination  $\mathbb{R}^0$  for the set of a single real number.

The number  $n$  is an element of  $\mathbb{N}$ .

**Definition 2.**

*vertex dim(n) :=* The smallest natural number of the cardinality of the set of vertices of a graph  $G$  such that  $\dim(G) = n$ .

*vertex Edim(n) :=* The smallest natural number of the cardinality of the set of vertices of a graph  $G$  such that  $Edim(G) = n$ .

*vertex mdim(n) :=* The smallest natural number of the cardinality of the set of vertices of a graph  $G$  such that  $mdim(G) = n$ .

*vertex edim(n) :=* The smallest natural number of the cardinality of the set of vertices of a graph  $G$  such that  $edim(G) = n$ .

*edge dim(n) :=* The smallest natural number of the cardinality of the set of edges of a graph  $G$  such that  $\dim(G) = n$ .

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*edge mdim(n) :=* The smallest natural number of the cardinality of the set of edges of a graph  $G$  such that  $mdim(G) = n$ .

*edge edim(n) :=* The smallest natural number of the cardinality of the set of edges of a graph  $G$  such that  $edim(G) = n$ .

Let  $S$  be a graph that has no edges. We define

$$\dim(S) := Edim(S) := mdim(S) := edim(S) := 1$$

## 2 Some Dimensions

### Proposition 1.

It holds  $\text{vertex dim}(1) = \text{vertex Edim}(1) = 2$  and  $\text{vertex dim}(2) = \text{vertex Edim}(2) = 3$  and  $\text{vertex dim}(3) = \text{vertex Edim}(3) = 4$

*Proof.* The complete graphs  $K_2$ ,  $K_3$ , and  $K_4$  establish the proposition.  $\square$

**Proposition 2.** It holds  $\text{edge dim}(1) = \text{edge Edim}(1) = 1$ ,  $\text{edge dim}(2) = \text{edge Edim}(2) = 3$ , and  $\text{edge dim}(3) = \text{edge Edim}(3) = 6$

*Proof.* The first two claims are trivial.

Graph  $K_4$  has a dimension of 3, see [1], p. 88, while graphs with fewer edges and four vertices have a smaller dimension since we can embed them in  $\mathbb{R}^2$ . This shows  $\text{edge dim}(3) = \text{edge Edim}(3) = 6$ , the number of edges of  $K_4$ .  $\square$

We can calculate  $\text{vertex dim}(n)$ .

### Proposition 3. It holds

$\text{vertex dim}(n) = n + 1$  for all natural numbers  $n$ .

*Proof.* From [1], p. 88, we get  $\dim(K_{n+1}) = n$ . The graph  $K_{n+1}$  has  $n + 1$  vertices. Every graph with fewer vertices is isomorphic to a subgraph of  $K_n$ .  $\square$

**Conjecture 1.** For all  $n \in \mathbb{N}$  it holds  $\text{vertex Edim}(n) = n + 1$  and  $\text{edge dim}(n) = \text{edge Edim}(n) = \frac{1}{2} \cdot n \cdot (n + 1)$ . We believe that we need  $K_{n+1}$ .

## 3 Points

There are a well-known riddle ‘Place 9 points in the plane, such that 8 of them form a square, and the 9. point is the center, such that there are 8 sets of 3 collinear points. Now draw 4 line segments through all points. The line segments are connected.’ We generalize the riddle to an infinite set of questions.

Let  $n$  be a natural number. We place  $n$  points anywhere in the plane on fixed positions. We call this  $P(n)$ . Note that for each  $n$ ,  $P(n)$  describes infinite many placements.

We define that a *polygon line* consists of a finite number of line segments and it is homeomorphic to a line segment.

We define that a *zig-zag line* consists of a finite number of line segments. Every line segment is connected at both ends with another line segment (with the exception of the first and the last one). It may self-intersect.

We define a *right-angle line* as a polygon line and each piece either is horizontal or vertical.

We define a *rectangle line* as a zig-zag line such that each piece either is horizontal or vertical. It may self-intersect.

We define for each placement  $P(n)$  of  $n$  points eight constants.

Let denote  $C(P(n))$  be the least natural number  $k$  such that all points of  $P(n)$  are a subset of a polygon line, which consists of  $k$  line segments.

Let denote  $D(P(n))$  be the least natural number  $k$  such that all points of  $P(n)$  are a subset of a zig-zag line, which consists of  $k$  line segments.

Let denote  $E(P(n))$  be the least natural number  $k$  such that all points of  $P(n)$  are a subset of a right-angle line, which consists of  $k$  line segments.

Let denote  $F(P(n))$  be the least natural number  $k$  such that all points of  $P(n)$  are a subset of a rectangle line, which consists of  $k$  line segments.

Let denote  $G(P(n))$  be the least natural number  $k$  such that all points of  $P(n)$  are a subset of a polygon line, which consists of  $k$  line segments, and each point is a subset of a single line segment from the polygon line.

Let denote  $H(P(n))$  be the least natural number  $k$  such that all points of  $P(n)$  are a subset of a zig-zag line, which consists of  $k$  line segments, and each point is a subset of a single line segment.

Let denote  $I(P(n))$  be the least natural number  $k$  such that all points of  $P(n)$  are a subset of a right-angle line, which consists of  $k$  line segments, and each point is a subset of a single line segment.

Let denote  $J(P(n))$  be the least natural number  $k$  such that all points of  $P(n)$  are a subset of a rectangle line, which consists of  $k$  line segments, and each point is a subset of a single line segment.

**Proposition 4.** For a natural number  $n$  all just defined eight constants are less or equal  $2 \cdot n$ .

*Proof.* We will construct a right-angle line that consists at most of  $2 \cdot n$  line segments, where the points are a subset of the line, and every point is a subset of a single line segment. This example fulfills the conditions for the situations to estimate all constants which we just have defined above.

We assume  $n$  points, which are placed anywhere on the plane. Every point is defined by two coordinates  $x$  and  $y$ . We assume that the points are ordered by their  $x$  coordinate. At first we assume that all  $x$  coordinates are different. We start with the first point. We go vertically and then horizontally to the second point. In this way, we connect all points. We use only horizontal and vertical line segments. In the case that some points have equal  $x$  coordinates, we draw one vertical line segment to connect them.

This means that we draw at most  $2 \cdot n$  line segments, which do not intersect. We have drawn a right-angle line. The proof is finished.  $\square$

Figure 1

On the right-hand side  
is on the left a zig-zag line  
and on the right a polygon line.

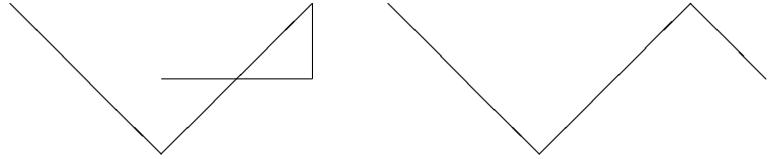


Figure 2

On the right-hand side  
on the left, we see a rectangle line  
and on the right, a right-angle line.



## 4 The Thuerey Numbers

Let  $G \subset \mathbb{R}^k$  be a graph with edges. If  $G$  has no subgraph  $K_n$  for all  $n > 2$  we define  $T(G)_n := 1$  for  $n > 2$ . Otherwise, let  $N$  be the natural number  $> 2$  such that there is a subgraph  $K_N$  of  $G$ , and for all  $n > N$  it holds that there is no subgraph  $K_n$  of  $G$ . (It may be that there is no  $N$ ). We define  $T(G)_n := 1$  for all  $n > N$ . For  $N \geq n > 2$  we define  $T(G)_n$  as the minimum natural number of colors such that there is a possibility to color the edges of  $G$  and there is no monochromatic subgraph  $K_n$  of  $G$ . In the case that there is no such number we write  $T(G)_n = \infty$ .

We define  $T(X)_1 := T(X)_2 := 0$  for all graphs  $X$ . If we define  $T(L)_n := 0$  for all  $n$  for a graph  $L$  without edges, we have defined a sequence  $(T(X)_n)_{n \in \mathbb{N}}$  for each graph  $X$ . We call ' $T(X)_n (n \in \mathbb{N})$ ' the 'Thuerey Numbers'.

We summarize the open problems.

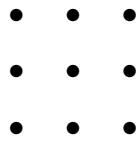
## 5 Open Problems

We ask for the sequences of Definition 2. We ask for the constants  $xxx(P(n))$  for any placement  $P(n)$ , where  $xxx$  is from  $\{C, D, E, F, G, H, I, J\}$ . Further, for a graph  $X$  we ask for the sequence  $(T(X)_n)_{n \in \mathbb{N}}$ .

Figure 3

The riddle:

Show  $D(P(9)) = 4$



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