ON THE EXACT GENERAL SOLUTION TO THE 3D NAVIER–STOKES EQUATIONS

Daniel Thomas Hayes

February 25, 2024

A new strategy is explored for finding the exact general solution to the 3D Navier–Stokes equations. This strategy is shown for the case of periodic boundary conditions. The general solution obtained shows that solutions to the 3D Navier–Stokes equations do not exhibit blowup in finite time.

Theorem 1

Let $\mathbf{U} = \mathbf{U}(\mathbf{x}, t) \in \mathbb{R}^3$, $F = F(\mathbf{x}, t) \in \mathbb{R}$, and $\mathbf{Q} = \mathbf{Q}(\mathbf{x}, t) \in \mathbb{R}^3$, each dependent on $\mathbf{x} \in \mathbb{R}^3$ and $t \geqslant 0$. We can find the general solution $\mathbf{U}, F$ to

$$\frac{\partial \mathbf{U}}{\partial t} + \bigcirc(\mathbf{U}^2) = \epsilon \bigcirc^2 (\mathbf{U}) + \bigcirc(\nabla F + \mathbf{Q}), \tag{1}$$

$$\nabla \cdot \mathbf{U} = 0 \tag{2}$$

with periodic boundary conditions

$$\mathbf{U}(\mathbf{x} + e_i, t) = \mathbf{U}(\mathbf{x}, t), \tag{3}$$

$$F(\mathbf{x} + e_i, t) = F(\mathbf{x}, t), \tag{4}$$

$$\mathbf{Q}(\mathbf{x} + e_i, t) = \mathbf{Q}(\mathbf{x}, t) \tag{5}$$

for $1 \leqslant i \leqslant 3$ where e_i is the i^{th} unit vector in $\mathbb{R}^3$. Here $\bigcirc$ is an operator such that

$$\bigcirc(\log(\mathbf{a})) = \frac{\bigcirc(\mathbf{a})}{\mathbf{a}} \tag{6}$$

for any vector $\mathbf{a}$.

Proof

Let

$$\mathbf{U} = c \bigcirc (\log(\mathbf{A})) = c\frac{\bigcirc(\mathbf{A})}{\mathbf{A}} \tag{7}$$

where c is a constant. Equation (1) implies

$$c\frac{\partial}{\partial t} \bigcirc (\log(\mathbf{A})) + c^2 \bigcirc [(\frac{\bigcirc(\mathbf{A})}{\mathbf{A}})^2] = c\epsilon \bigcirc [\frac{\bigcirc^2(\mathbf{A})}{\mathbf{A}} - (\frac{\bigcirc(\mathbf{A})}{\mathbf{A}})^2] + \bigcirc(\nabla F + \mathbf{Q}). \tag{8}$$

Let $c = -\epsilon$. Equation (8) implies

$$-\epsilon\frac{\partial}{\partial t} \bigcirc (\log(\mathbf{A})) = -\epsilon^2 \bigcirc [\frac{\bigcirc^2(\mathbf{A})}{\mathbf{A}}] + \bigcirc(\nabla F + \mathbf{Q}) \tag{9}$$

which implies

$$-\nabla^{-2}\nabla \times \nabla \times \epsilon\frac{\partial}{\partial t} \bigcirc (\log(\mathbf{A})) = -\nabla^{-2}\nabla \times \nabla \times \epsilon^2 \bigcirc [\frac{\bigcirc^2(\mathbf{A})}{\mathbf{A}}] + \nabla^{-2}\nabla \times \nabla \times \bigcirc\mathbf{Q} \tag{10}$$

which implies

$$-\nabla^{-2}\nabla \times \nabla \times \epsilon\frac{\partial}{\partial t}(\log(\mathbf{A})) = -\nabla^{-2}\nabla \times \nabla \times \epsilon^2[\frac{\bigcirc^2(\mathbf{A})}{\mathbf{A}}] + \nabla^{-2}\nabla \times \nabla \times \mathbf{Q} + \mathbf{g} \tag{11}$$

where

$$\bigcirc(\mathbf{g}) = \mathbf{0}. \tag{12}$$

Equation (11) implies

$$\epsilon\frac{\partial}{\partial t}(\log(\mathbf{A})) = \epsilon^2\frac{\bigcirc^2(\mathbf{A})}{\mathbf{A}} - \mathbf{Q} + \mathbf{G} \tag{13}$$

where

$$\nabla^{-2}\nabla \times \nabla \times \mathbf{G} = -\mathbf{g}. \tag{14}$$

Equation (13) implies

$$\epsilon\frac{\partial \mathbf{A}}{\partial t} = \epsilon^2 \bigcirc^2 (\mathbf{A}) + \mathbf{A}\mathbf{H} \tag{15}$$

where

$$\mathbf{H} = -\mathbf{Q} + \mathbf{G}. \tag{16}$$

Equation (15) is a linear PDE for $\mathbf{A}$ and is solvable. $\therefore$ $\mathbf{U}$ can be found.
Equation (2) implies

$$\nabla \cdot \bigcirc(\mathbf{U}^2) = \bigcirc(\nabla^2 F + \nabla \cdot \mathbf{Q}) \tag{17}$$

which implies

$$\nabla \cdot (\mathbf{U}^2) = \nabla^2 F + \nabla \cdot \mathbf{Q} + h \tag{18}$$

where

$$\bigcirc(h) = 0 \tag{19}$$

which implies

$$\nabla^2 F = \nabla \cdot (\mathbf{U}^2) - \nabla \cdot \mathbf{Q} - h \tag{20}$$

which implies

$$F = \nabla^{-2}[\nabla \cdot (\mathbf{U}^2) - -\nabla \cdot \mathbf{Q} - h]. \tag{21}$$

$\therefore$ F can be found. $\square$
Theorem 1 implies we can solve

$$\frac{\partial \mathbf{U}}{\partial t} = \diamond(\mathbf{U}^2) + \epsilon\diamond^2(\mathbf{U}) - \diamond(\mathbf{Q}) \tag{22}$$

and (3),(5) for $\mathbf{U}$ where

$$\diamond = \nabla^{-2}\nabla \times \nabla \times \bigcirc. \tag{23}$$

Theorem 2

Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^3$, $p = p(\mathbf{x}, t) \in \mathbb{R}$, and $\mathbf{f} = \mathbf{f}(\mathbf{x}, t) \in \mathbb{R}^3$, each dependent on $\mathbf{x} \in \mathbb{R}^3$ and $t \geqslant 0$. We can find the general solution $\mathbf{u}, p$ to the 3D Navier–Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu\nabla^2\mathbf{u} - \nabla p + \mathbf{f}, \tag{24}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{25}$$

with periodic boundary conditions

$$\mathbf{u}(\mathbf{x} + e_i, t) = \mathbf{u}(\mathbf{x}, t), \tag{26}$$

$$p(\mathbf{x} + e_i, t) = p(\mathbf{x}, t), \tag{27}$$

$$\mathbf{f}(\mathbf{x} + e_i, t) = \mathbf{f}(\mathbf{x}, t) \tag{28}$$

for $1 \leqslant i \leqslant 3$ where e_i is the i^{th} unit vector in $\mathbb{R}^3$. We find that the 3D Navier–Stokes equations are regular, see [1–4]. Here the equation for $\mathbf{u}$ is

$$\frac{\partial \mathbf{u}}{\partial t} = \nabla^{-2}\nabla \times \nabla \times [(\mathbf{u} \cdot \nabla)\mathbf{u}] + \nu\nabla^2\mathbf{u} - \nabla^{-2}\nabla \times \nabla \times \mathbf{f}. \tag{29}$$

Proof

Theorem 1 implies we can solve

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} &= \nabla^{-2}\nabla\times\nabla\times[(\mathbf{u}\cdot\nabla)\mathbf{u}] + \nu\nabla^2\mathbf{u} - \nabla^{-2}\nabla\times\nabla\times\mathbf{f} \\ &+ \diamondsuit(\mathbf{u}^2) + \epsilon\diamondsuit^2(\mathbf{u}) - \diamondsuit(\mathbf{Q}) \\ &- \nabla^{-2}\nabla\times\nabla\times[(\mathbf{u}\cdot\nabla)\mathbf{u}] - \nu\nabla^2\mathbf{u} + \nabla^{-2}\nabla\times\nabla\times\mathbf{f}\end{aligned} \tag{30}$$

and (26),(28),(5) for $\mathbf{u}$. We let

$$\diamondsuit(\mathbf{u}^2) + \epsilon\diamondsuit^2(\mathbf{u}) - \diamondsuit(\mathbf{Q}) - \nabla^{-2}\nabla\times\nabla\times[(\mathbf{u}\cdot\nabla)\mathbf{u}] - \nu\nabla^2\mathbf{u} + \nabla^{-2}\nabla\times\nabla\times\mathbf{f} = \mathbf{0} \tag{31}$$

which implies

$$\bigcirc(\mathbf{u}^2) - \epsilon\bigcirc^2(\mathbf{u}) - \bigcirc\mathbf{Q} - (\mathbf{u}\cdot\nabla)\mathbf{u} + \nu\nabla^2\mathbf{u} + \mathbf{f} = \mathbf{r} \tag{32}$$

where

$$\nabla^{-2}\nabla\times\nabla\times\mathbf{r} = \mathbf{0}. \tag{33}$$

We choose

$$\mathbf{f} = \mathbf{r} - \bigcirc(\mathbf{u}^2) + \epsilon\bigcirc^2(\mathbf{u}) + \bigcirc\mathbf{Q} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\nabla^2\mathbf{u} \tag{34}$$

which is still arbitrary since $\mathbf{Q}$ appears. $\therefore$ $\mathbf{u}$ can be found. Equation (25) implies

$$\nabla^2 p = -\nabla\cdot[(\mathbf{u}\cdot\nabla)\mathbf{u}] + \nabla\cdot\mathbf{f} \tag{35}$$

which implies

$$p = \nabla^{-2}\{-\nabla\cdot[(\mathbf{u}\cdot\nabla)\mathbf{u}] + \nabla\cdot\mathbf{f}\}. \tag{36}$$

$\therefore$ p can be found. Finally, in light of the general solution obtained we find that the 3D Navier–Stokes equations are regular because equations (1),(2),(3),(4),(5),(6) are regular. $\square$

References

[1] Doering C. 2009. The 3D Navier–Stokes problem. *Annu. Rev. Fluid Mech.* **41**: 109–128.
[2] Fefferman C. 2000. Existence and smoothness of the Navier–Stokes equation. *Clay Mathematics Institute.* Official problem description.
[3] Ladyzhenskaya O. 1969. *The mathematical theory of viscous incompressible flows.* Gordon and Breach, New York.
[4] Tao T. Why global regularity for Navier–Stokes is hard. Wordpress.