No blowup for the Navier–Stokes equations

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A proposed solution to the millennium problem on the existence and smoothness of the Navier–Stokes equations.

1. Introduction

The Navier–Stokes equations are thought to govern the motion of a fluid in $\mathbb{R}^3$, see [1,3]. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^3$ be the fluid velocity and let $p = p(\mathbf{x}, t) \in \mathbb{R}$ be the fluid pressure, each dependent on position $\mathbf{x} \in \mathbb{R}^3$ and time $t \geqslant 0$. I take the externally applied force acting on the fluid to be identically zero. The fluid is assumed to be incompressible with constant viscosity $\nu > 0$ and to fill all of $\mathbb{R}^3$. The Navier–Stokes equations can then be written as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \nabla^2 \mathbf{u} - \nabla p, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

with initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^\circ \tag{3}$$

where $\mathbf{u}^\circ = \mathbf{u}^\circ(\mathbf{x}) \in \mathbb{R}^3$. In these equations

$$\nabla = (\frac{\partial}{\partial \mathbf{x}_1}, \frac{\partial}{\partial \mathbf{x}_2}, \frac{\partial}{\partial \mathbf{x}_3}) \tag{4}$$

is the gradient operator and

$$\nabla^2 = \sum_{i=1}^{3} \frac{\partial^2}{\partial {\mathbf{x}_i}^2} \tag{5}$$

is the Laplacian operator. When $\nu = 0$ equations (1), (2), (3) are called the Euler equations. Solutions of (1), (2), (3) are to be found with

$$\mathbf{u}^\circ(\mathbf{x} + e_i) = \mathbf{u}^\circ(\mathbf{x}) \tag{6}$$

for $1 \leqslant i \leqslant 3$ where e_i is the i^{th} unit vector in $\mathbb{R}^3$. The initial condition $\mathbf{u}^\circ$ is a given C^∞ divergence-free vector field on $\mathbb{R}^3$. A solution of (1), (2), (3) is then accepted to be physically reasonable [3] if

$$\mathbf{u}(\mathbf{x} + e_i, t) = \mathbf{u}(\mathbf{x}, t), \quad p(\mathbf{x} + e_i, t) = p(\mathbf{x}, t) \tag{7}$$

on $\mathbb{R}^3 \times [0, \infty)$ for $1 \leqslant i \leqslant 3$ and

$$\mathbf{u}, p \in C^\infty(\mathbb{R}^3 \times [0, \infty)). \tag{8}$$

2. Solution to the Navier–Stokes problem

I provide a proof of the following theorem, see [2,3,5,6].

Theorem. Take $\nu > 0$. Let $\mathbf{u}^\circ$ be any smooth, divergence-free vector field satisfying (6). Then there exist smooth functions $\mathbf{u}$, p on $\mathbb{R}^3 \times [0, \infty)$ that satisfy (1), (2), (3), (7), (8).

Proof. Let $\mathbf{u}$, p be given by

$$\mathbf{u} = \sum_{\mathbf{L}=-\infty}^{\infty} \mathbf{u_L} \mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}}, \tag{9}$$

$$p = \sum_{\mathbf{L}=-\infty}^{\infty} p_\mathbf{L} \mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}} \tag{10}$$

respectively. Here $\mathbf{u_L} = \mathbf{u_L}(t) \in \mathbb{C}^3$, $p_\mathbf{L} = p_\mathbf{L}(t) \in \mathbb{C}$, $\mathrm{i} = \sqrt{-1}$, $k = 2\pi$, and $\sum_{\mathbf{L}=-\infty}^{\infty}$ denotes the sum over all $\mathbf{L} \in \mathbb{Z}^3$. The initial condition $\mathbf{u}^\circ$ is a Fourier series [2] of which is convergent for all $\mathbf{x} \in \mathbb{R}^3$. Equation (1) implies

$$\begin{aligned}\sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial \mathbf{u_L}}{\partial t} \mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}} + \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u_L} \cdot \mathrm{i}k\mathbf{M})\mathbf{u_M} \mathrm{e}^{\mathrm{i}k(\mathbf{L}+\mathbf{M})\cdot\mathbf{x}} \\ = -\sum_{\mathbf{L}=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 \mathbf{u_L} \mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}} - \sum_{\mathbf{L}=-\infty}^{\infty} \mathrm{i}k\mathbf{L} p_\mathbf{L} \mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}}.\end{aligned} \tag{11}$$

Equating like powers of the exponentials in (11) yields

$$\frac{\partial \mathbf{u_L}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u_{L-M}} \cdot \mathrm{i}k\mathbf{M})\mathbf{u_M} = -\nu k^2 |\mathbf{L}|^2 \mathbf{u_L} - \mathrm{i}k\mathbf{L} p_\mathbf{L} \tag{12}$$

on using the Cauchy product type formula [4]

$$\sum_{l=-\infty}^{\infty} a_l x^l \sum_{m=-\infty}^{\infty} b_m x^m = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{l-m} b_m x^l. \tag{13}$$

Equation (2) implies

$$\sum_{\mathbf{L}=-\infty}^{\infty} \mathrm{i}k\mathbf{L} \cdot \mathbf{u_L} \mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}} = 0. \tag{14}$$

Equating like powers of the exponentials in (14) yields

$$\mathbf{L} \cdot \mathbf{u_L} = 0. \tag{15}$$

Applying $\mathbf{L}\cdot$ to (12) and noting (15) leads to

$$p_{\mathbf{L}} = -\sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \hat{\mathbf{L}})(\mathbf{u}_{\mathbf{M}} \cdot \hat{\mathbf{L}}) \tag{16}$$

where $p_{\mathbf{0}}$ is arbitrary and $\hat{\mathbf{L}} = \mathbf{L}/|\mathbf{L}|$ is the unit vector in the direction of $\mathbf{L}$. Then substituting (16) into (12) gives

$$\frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} = -\sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \mathrm{i}k\mathbf{M})\mathbf{u}_{\mathbf{M}} - \nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} + \sum_{\mathbf{M}=-\infty}^{\infty} \mathrm{i}k\mathbf{L}(\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \hat{\mathbf{L}})(\mathbf{u}_{\mathbf{M}} \cdot \hat{\mathbf{L}}) \tag{17}$$

where $\mathbf{u}_{\mathbf{0}} = \mathbf{u}_{\mathbf{0}}(0)$. Without loss of generality [2], I take $\mathbf{u}_{\mathbf{0}} = \mathbf{0}$. This is due to the Galilean invariance property of solutions to the Navier–Stokes equations. The equations for $\mathbf{u}_{\mathbf{L}}$ are to be solved for all $\mathbf{L} \in \mathbb{Z}^3$. Here we can find a representation of the solution $\mathbf{u}, p$ and show that $\mathbf{u}$ can not have a finite time singularity when $\mathbf{u}^\circ(\mathbf{x})$ is smooth.
First note that the solution $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ to

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{0}, \tag{18}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^\circ(\mathbf{x}) \tag{19}$$

can be represented by

$$\mathbf{u} = \mathbf{u}^\circ(\mathbf{X}), \tag{20}$$

$$\mathbf{X} = \mathbf{x} + t\mathbf{u}^\circ(\mathbf{X}). \tag{21}$$

This can be checked as follows via the chain rule.

$$\begin{aligned} \frac{\partial \mathbf{u}^\circ(\mathbf{X})}{\partial t} &= \frac{\partial \mathbf{u}^\circ(\mathbf{X})}{\partial \mathbf{X}} \frac{\partial \mathbf{X}}{\partial t} \\ &= \frac{\partial \mathbf{u}^\circ(\mathbf{X})}{\partial \mathbf{X}} \mathbf{u}^\circ(\mathbf{X}) \\ &= -\frac{\partial \mathbf{u}^\circ(\mathbf{X})}{\partial \mathbf{x}} \mathbf{u}^\circ(\mathbf{X}). \end{aligned} \tag{22}$$

Therefore (18), (19) are satisfied. We see here that this $\mathbf{u}$ satisfying (18), (19) can not have a finite time singularity when $\mathbf{u}^\circ(\mathbf{x})$ is smooth.
The solution $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, $p = p(\mathbf{x}, t)$ to (1), (2), (3), (6), (7) can be represented by

$$\mathbf{u} = \mathbf{u}^\circ(\mathbf{X}), \tag{23}$$

$$p = -\nabla_{\mathbf{X}}^{-2}\{\nabla_{\mathbf{X}} \cdot [(\mathbf{u}^\circ(\mathbf{X}) \cdot \nabla_{\mathbf{X}})\mathbf{u}^\circ(\mathbf{X})]\} = P(\mathbf{X}), \tag{24}$$

$$\mathbf{X} = \mathbf{x} + t[\mathbf{u}^\circ(\mathbf{X}) - \mathbf{u}^s(\mathbf{X})] \tag{25}$$

where $\mathbf{u}^s(\mathbf{X})$ is a representation of the implicit solution $\mathbf{u}^\circ(\mathbf{X})$ to

$$-\frac{\partial \mathbf{u}^\circ(\mathbf{X})}{\partial \mathbf{X}}\mathbf{u}^\circ(\mathbf{X}) = \nu\nabla^2_{\mathbf{X}}\mathbf{u}^\circ(\mathbf{X}) + \nabla_{\mathbf{X}}P(\mathbf{X}). \tag{26}$$

In these equations $\nabla_{\mathbf{X}}$, $\nabla^2_{\mathbf{X}}$, and $\nabla^{-2}_{\mathbf{X}}$ denote the gradient, Laplacian, and inverse Laplacian with respect to the variable $\mathbf{X}$ respectively. Note it is true that

$$\mathbf{u}^s(\mathbf{X}) = \frac{1}{\nu}\nabla^{-2}_{\mathbf{X}}[-\frac{\partial \mathbf{u}^\circ(\mathbf{X})}{\partial \mathbf{X}}\mathbf{u}^\circ(\mathbf{X}) - \nabla_{\mathbf{X}}P(\mathbf{X})]. \tag{27}$$

It is also true that $\mathbf{u}^s(\mathbf{X})$ can be represented by

$$\mathbf{u}^s(\mathbf{X}) = -(\frac{\partial \mathbf{u}^\circ(\mathbf{X})}{\partial \mathbf{X}})^{-1}[\nu\nabla^2_{\mathbf{X}}\mathbf{u}^\circ(\mathbf{X}) + \nabla_{\mathbf{X}}P(\mathbf{X})] \tag{28}$$

in cases where the nonlinearity is not identically equal to zero. The solution can be checked as follows via the chain rule.

$$\begin{aligned}
\frac{\partial \mathbf{u}^\circ(\mathbf{X})}{\partial t} &= \frac{\partial \mathbf{u}^\circ(\mathbf{X})}{\partial \mathbf{X}}\frac{\partial \mathbf{X}}{\partial t} \\
&= \frac{\partial \mathbf{u}^\circ(\mathbf{X})}{\partial \mathbf{X}}[\mathbf{u}^\circ(\mathbf{X}) - \mathbf{u}^s(\mathbf{X})] \\
&= \frac{\partial \mathbf{u}^\circ(\mathbf{X})}{\partial \mathbf{X}}\{\mathbf{u}^\circ(\mathbf{X}) - \frac{1}{\nu}\nabla^{-2}_{\mathbf{X}}[-\frac{\partial \mathbf{u}^\circ(\mathbf{X})}{\partial \mathbf{X}}\mathbf{u}^\circ(\mathbf{X}) - \nabla_{\mathbf{X}}P(\mathbf{X})]\} \\
\text{or} \quad & \frac{\partial \mathbf{u}^\circ(\mathbf{X})}{\partial \mathbf{X}}\{\mathbf{u}^\circ(\mathbf{X}) + (\frac{\partial \mathbf{u}^\circ(\mathbf{X})}{\partial \mathbf{X}})^{-1}[\nu\nabla^2_{\mathbf{X}}\mathbf{u}^\circ(\mathbf{X}) + \nabla_{\mathbf{X}}P(\mathbf{X})]\} \\
&= -\frac{\partial \mathbf{u}^\circ(\mathbf{x})}{\partial \mathbf{x}}\{\mathbf{u}^\circ(\mathbf{X}) - \frac{1}{\nu}\nabla^{-2}[\frac{\partial \mathbf{u}^\circ(\mathbf{X})}{\partial \mathbf{x}}\mathbf{u}^\circ(\mathbf{X}) + \nabla P(\mathbf{X})]\} \\
\text{or} \quad & -\frac{\partial \mathbf{u}^\circ(\mathbf{X})}{\partial \mathbf{x}}\{\mathbf{u}^\circ(\mathbf{X}) - (\frac{\partial \mathbf{u}^\circ(\mathbf{X})}{\partial \mathbf{x}})^{-1}[\nu\nabla^2\mathbf{u}^\circ(\mathbf{X}) - \nabla P(\mathbf{X})]\}.
\end{aligned} \tag{29}$$

Therefore (1), (3) are satisfied. We also have

$$p = -\nabla^{-2}\{\nabla \cdot [(\mathbf{u}\cdot\nabla)\mathbf{u}]\} \tag{30}$$

which is obtained by applying $\nabla\cdot$ to (1) of which ensures (2). Due to the form of the solution we see that $\mathbf{u}$ can not have a finite time singularity when $\mathbf{u}^\circ(\mathbf{x})$ is smooth. Therefore the theorem is true. □

References

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