Unlike Newton's Gravitational Acceleration, Einstein's is Velocity-Dependent; it Repels Sufficiently Near-c Objects, Obviating the Need for "Dark Energy"

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Abstract If a test body radially approaching a static point mass at near-c speed undergoes the same attractive gravitational acceleration as a nonrelativistic test body, its speed soon exceeds c. That doesn't occur because Einstein's gravitational acceleration by a static point mass is velocity-dependent; it counterintuitively repels a test body traveling radially at a speed sufficiently near c. Indeed, a basic feature of the gravitational refraction of light is that a radially-traveling light packet's speed increases monotonically toward c with its increasing radial distance from a static point mass, so a light packet traveling radially away from a static point mass is gravitationally accelerated toward radial speed c in the outward direction of its travel. Likewise, a test body traveling radially away from a static point mass at a speed sufficiently near c is counterintuitively gravitationally accelerated in the outward direction of its travel. The universe expands radially at a speed sufficiently near c to undergo such a counterintuitive gravitational acceleration in the outward direction of its radial expansion, "dark energy" isn't needed.

1. Newton's force of gravity exerted by a static point mass on a radially-moving test body

Newton's gravitational second-law equation of motion for a mass-m test body is,

$$m\ddot{\mathbf{x}} = -\nabla_{\mathbf{x}} V(\mathbf{x}), \tag{1.1a}$$

where $V(\mathbf{x})$ is Newton's gravitational potential energy of the mass-m test body, which satisfies the equation,

$$\nabla^2_{\mathbf{x}} V(\mathbf{x}) = 4\pi G m \rho(\mathbf{x}), \tag{1.1b}$$

where $\rho(\mathbf{x})$ is the gravitational source's static mass density. From Eqs. (1.1a) and (1.1b) we see that Newton's gravitational acceleration $\ddot{\mathbf{x}}$ of the test body *is independent of both its mass m and its velocity $\dot{\mathbf{x}}$*, so a test body traveling at a speed close enough to c *can be accelerated to a speed exceeding c.* To *prevent* that from occurring, *Einstein's gravitational force and acceleration must necessarily be velocity-dependent.*

For a gravitational source which is a static point mass M located at the origin $\mathbf{x} = \mathbf{0}$, $\rho(\mathbf{x}) = M\delta^{(3)}(\mathbf{x})$, and solving Eq. (1.1b) yields $V(\mathbf{x}) = (-GmM/|\mathbf{x}|) + \mathcal{V} - \mathbf{f}\cdot\mathbf{x}$, where $\mathcal{V}$ and $\mathbf{f}$ are arbitrary "gauge constants". *Imposition of the causal physical "gauge condition"* that $V(\mathbf{x})$ must vanish when the gravitational source's static mass density $\rho(\mathbf{x})$ vanishes *removes this "gauge-constant ambiguity" of $V(\mathbf{x})$, fixing the values of the "gauge constants" to $\mathcal{V} = 0$ and $\mathbf{f} = \mathbf{0}$.* Therefore for a gravitational source which is a static point mass M located at the origin, Newton's Eq. (1.1a) for an arbitrary-mass test body is specialized to,

$$\ddot{\mathbf{x}} = -\nabla_{\mathbf{x}}(-GM/|\mathbf{x}|) = -\big(GM\,\mathbf{x}/|\mathbf{x}|^3\big). \tag{1.1c}$$

We now combine the mathematical fact that $d|\dot{\mathbf{x}}|^2/dt = d(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}})/dt = (2\ddot{\mathbf{x}}\cdot\dot{\mathbf{x}})$ with Eq. (1.1c) to obtain,

$$d|\dot{\mathbf{x}}|^2/dt = (2\nabla_{\mathbf{x}}(GM/|\mathbf{x}|)\cdot\dot{\mathbf{x}}) = d(2GM/|\mathbf{x}|)/dt \implies |\dot{\mathbf{x}}|^2 - (2GM/|\mathbf{x}|) = \varepsilon, \tag{1.1d}$$

where ε is *a constant* (of dimension speed-squared), so Eq. (1.1d) is *a gravitational conservation relation.* When $\varepsilon < 0$, the test body *is gravitationally bound by the static point mass,* but when $\varepsilon \geq 0$, $\sqrt{\varepsilon}$ *is the asymptotic speed of the test body as* $|\mathbf{x}| \to \infty$. When the test body *moves only radially relative to the static point mass located at the origin,* the Eq. (1.1c) Newtonian gravitational acceleration equation *simplifies to,*

$$d^2r/dt^2 = -\big(GM/r^2\big), \text{ where } r \text{ is } |\mathbf{x}|, \tag{1.1e}$$

and the Eq. (1.1d) Newtonian gravitational conservation relation *simplifies to,*

$$(dr/dt)^2 = (2GM/r) + \varepsilon, \tag{1.1f}$$

whose test body speed $|dr/dt|$ *has no upper bound.* However, *Einstein's* gravity theory *enforces* $|dr/dt| \leq c$.

2. Einstein's force of gravity exerted by a static point mass on a radially-moving test body

Since mass, *unlike* charge, *isn't exactly conserved* (even *chemical reactions* don't *exactly* conserve mass), but energy and momentum *are exactly conserved* and *as well comprise a Lorentz-covariant four-vector,* the energy-momentum four vector, rather than mass, is regarded as gravity's source in Einstein's gravity theory, which can then be *tentatively* formulated in a Lorentz-covariant manner *that parallels electromagnetic theory.*

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Therefore we now *review equations of electromagnetism,* both equations that connect the Lorentz four-vector *electromagnetic potential* A^μ *to its Lorentz four-vector source, the density/flux of moving conserved charge* j^μ, which, because of local charge conservation, *must satisfy* $\partial_\mu j^\mu = 0$, and also the equation for the acceleration of a test body of mass m and charge e by an electromagnetic potential A^μ. One very well-known equation that connects A^μ to j^μ is mathematically inconsistent unless $\partial_\mu j^\mu = 0$, i.e.,

$$\partial_\nu\partial^\nu A^\mu - \partial^\mu(\partial_\sigma A^\sigma) = 4\pi j^\mu/c. \tag{2.1a}$$

Since $\partial_\mu\big(\partial_\nu\partial^\nu A^\mu - \partial^\mu(\partial_\sigma A^\sigma)\big) = 0$, Eq. (2.1a) isn't mathematically consistent unless $\partial_\mu j^\mu = 0$. However, since for *any* well-behaved scalar function χ, $\big(\partial_\nu\partial^\nu(\partial^\mu\chi) - \partial^\mu(\partial_\sigma(\partial^\sigma\chi))\big) = 0$, $(A^\mu + \partial^\mu\chi)$ *also solves* Eq. (2.1a); *this* χ*-function gauge covariance of* Eq. (2.1a) *obviously prevents its having a unique solution.*

Among *the infinite number of solutions of* Eq. (2.1a), there exists one which satisfies the "Coulomb gauge condition" $\nabla\cdot\mathbf{A}' = 0$; this particular $\mathbf{A}' = (\mathbf{A} + \nabla\chi)$ corresponds to a scalar function χ that satisfies the equation $-\nabla^2\chi = \nabla\cdot\mathbf{A}$. Since the Coulomb gauge condition $\nabla\cdot\mathbf{A} = 0$ *violates the Lorentz covariance of* A^μ, we are beginning to realize that Eq. (2.1a) *has a plethora of physically unacceptable solutions.*

Doubtless *the very simplest way* to narrow down the solutions of Eq. (2.1a) to ones *which definitely are Lorentz-covariant* is to impose on Eq. (2.1a) the *supplementary* "Lorentz condition" $\partial_\sigma(A')^\sigma = 0$, which for $(A')^\sigma = (A^\sigma + \partial^\sigma\chi)$ requires that χ satisfies the equation $\partial_\sigma\partial^\sigma\chi = -\partial_\sigma A^\sigma$. The *upshot of imposing this supplementary "Lorentz condition"* $\partial_\sigma A^\sigma = 0$ on Eq. (2.1a) *is its replacement by the following two equations,*

$$\partial_\sigma A^\sigma = 0, \tag{2.1b}$$

and,

$$\partial_\nu\partial^\nu A^\mu = 4\pi j^\mu/c. \tag{2.1c}$$

Eq, (2.1c) *still* doesn't have a *mathematically unique* solution because the differential operator $\partial_\nu\partial^\nu$ doesn't have a mathematically unique inverse. However, *only* its "retarded" inverse $(\partial_\nu\partial^\nu)_R^{-1}$ *makes causal physical sense.* Thus the unique physically sensible *causal* solution of Eq. (2.1c) is the "retarded" one,

$$A^\mu = (4\pi/c)(\partial_\nu\partial^\nu)_R^{-1} j^\mu, \tag{2.1d}$$

which in addition satisfies Eq. (2.1b) *because* $\partial_\mu j^\mu = 0$.

We see here that solution nonuniqueness due to a dynamic symmetry such as gauge covariance is resolved by imposing utterly basic physical requirements on the solution in the simplest way feasible.

We next exhibit the derivation from a simple Lorentz-invariant Lagrangian, which is written in terms of proper time τ instead of observer time t, of the equation of motion for a mass-m and charge-e test body acted on by an electromagnetic potential A^μ. For a *free* mass-m test body, the *simplest* such Lorentz-invariant Lagrangian written in terms of proper time τ instead of observer time t is,

$$L^{\rm invr}_{\rm free}(dx^\mu/d\tau) = -(m/2)\big(\eta_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau)\big), \tag{2.2a}$$

where $\eta_{\mu\nu}$ is the Minkowski metric of Lorentzian space-time,

$$\eta_{00} = +1,\ \eta_{11} = -1,\ \eta_{22} = -1,\ \eta_{33} = -1 \quad\text{and}\quad \eta_{\mu\nu} = 0 \ \text{ when } \ \mu \neq \nu. \tag{2.2b}$$

In the nonrelativistic limit $\tau \to t$, so the Eq. (2.2a) Lagrangian $L^{\rm invr}_{\rm free}(dx^\mu/d\tau) \to -(mc^2/2) + (m|\dot{\mathbf{x}}|^2/2)$, which, except for the dynamically inert constant term $-(mc^2/2)$, *is precisely the nonrelativistic Lagrangian for a free mass-m test body.* Of course *the Lorentz-invariant action* for a free mass-m test body is,

$$S_{\rm free} = \int L^{\rm invr}_{\rm free}(dx^\mu/d\tau)\, d\tau, \tag{2.2c}$$

and the equation of motion which follows from $L^{\rm invr}_{\rm free}(dx^\mu/d\tau) = -(m/2)\big(\eta_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau)\big)$ is,

$$\begin{aligned} &d\big[\partial L^{\rm invr}_{\rm free}(dx^\mu/d\tau)/\partial(dx^\lambda/d\tau)\big]/d\tau - \partial L^{\rm invr}_{\rm free}(dx^\mu/d\tau)/\partial x^\lambda = 0 \quad \Longrightarrow \\ &d\big[-(m/2)\big(2(dx_\lambda/d\tau)\big)\big]/d\tau = 0 \quad \Longrightarrow \quad -m\big(d^2x_\lambda/d\tau^2\big) = 0, \end{aligned} \tag{2.2d}$$

so the Lorentz-covariant acceleration of the free mass-m test body is zero, as of course *is expected.*

When the mass-m test body has charge e and is acted on by A^μ, its Lorentz-invariant *free* Lagrangian $L^{\rm invr}_{\rm free}(dx^\mu/d\tau)$ *acquires a Lorentz-invariant add-on term.* Since in the static limit the add-on term is $-eA_0(\mathbf{x})$, *the Lorentz-invariant add-on term must be* $-(e/c)A_\nu(x)(dx^\nu/d\tau)$, so the Lagrangian which includes A^μ is,

$$L^{\rm invr}(dx^\mu/d\tau,\ A_\nu(x)) = L^{\rm invr}_{\rm free}(dx^\mu/d\tau) - (e/c)A_\nu(x)(dx^\nu/d\tau). \tag{2.3a}$$

The add-on term *is velocity-dependent,* and *is also gauge covariant* since $-(e/c)(\partial\chi(x)/\partial x^\nu)(dx^\nu/d\tau) = d[-(e/c)\chi(x)]/d\tau$, *which is wholly a derivative with respect to* τ, *and therefore doesn't contribute to the dynamics.* It is *also directly seen that* $-(e/c)(\partial\chi(x)/\partial x^\nu)(dx^\nu/d\tau)$ *doesn't contribute to the dynamics,*

$$-(e/c)\big\{d\big[\partial\big((\partial\chi(x)/\partial x^\nu)(dx^\nu/d\tau)\big)/\partial(dx^\lambda/d\tau)\big]/d\tau - \partial\big((\partial\chi(x)/\partial x^\nu)(dx^\nu/d\tau)\big)/\partial x^\lambda\big\} =$$
$$-(e/c)\big\{d\big[(\partial\chi(x)/\partial x^\lambda)\big]/d\tau - (\partial^2\chi(x)/\partial x^\nu\partial x^\lambda)(dx^\nu/d\tau)\big\} =$$
$$-(e/c)\big\{(\partial^2\chi(x)/\partial x^\lambda\partial x^\nu)(dx^\nu/d\tau) - (\partial^2\chi(x)/\partial x^\nu\partial x^\lambda)(dx^\nu/d\tau)\big\} = 0. \tag{2.3b}$$

Gauge covariance of the part of the action which describes the coupling of A^μ to charged entities *ensures* that the functional derivative with respect to A^μ of that part of the action *has vanishing four-divergence,* and therefore *is a density/flux* j^μ *of moving conserved charge that is a physically legitimate source of electromagnetic potential* A^μ. Thus *gauge covariance* is a useful, necessary property of the part of the action which describes the coupling of A^μ to charged entities, but it is something of a nuisance for the part of the action which involves A^μ by itself, since gauge covariance of that part of the action produces an equation of motion for A^μ which, like Eq. (2.1a), has an infinite number of solutions for A^μ, including a plethora of solutions that violate basic physical requirements such as Lorentz covariance and causality. But, as Eqs. (2.1b) through (2.1d) show, this issue is resolved *by selecting the simplest member of the subset of the solutions of such a gauge-covariant equation for* A^μ *which adhere to all basic physical requirements.*

The *force on the charge e test body produced by the* Eq. (2.3a) *add-on term* $-(e/c)A_\nu(x)(dx^\nu/d\tau)$ is,

$$(e/c)\big\{d\big[\partial\big(A_\nu(x)(dx^\nu/d\tau)\big)/\partial(dx^\lambda/d\tau)\big]/d\tau - \partial\big(A_\nu(x)(dx^\nu/d\tau)\big)/\partial x^\lambda\big\} =$$
$$(e/c)\big\{d\big[A_\lambda(x)\big]/d\tau - (\partial A_\nu(x)/\partial x^\lambda)(dx^\nu/d\tau)\big\} =$$
$$(e/c)\big\{(\partial A_\lambda(x)/\partial x^\nu)(dx^\nu/d\tau) - (\partial A_\nu(x)/\partial x^\lambda)(dx^\nu/d\tau)\big\} =$$
$$(e/c)\big[\partial A_\lambda(x)/\partial x^\nu - \partial A_\nu(x)/\partial x^\lambda\big](dx^\nu/d\tau), \tag{2.3c}$$

so the equation of motion of a mass-m and charge-e test body subject to an electromagnetic potential A^μ is,

$$-m\big(d^2x_\lambda/d\tau^2\big) = (e/c)\big[\partial A_\lambda(x)/\partial x^\nu - \partial A_\nu(x)/\partial x^\lambda\big](dx^\nu/d\tau), \tag{2.3d}$$

whose standard form of presentation is,

$$m\big(d^2x^\mu/d\tau^2\big) = (e/c)\big[\partial A^\nu(x)/\partial x_\mu - \partial A^\mu(x)/\partial x_\nu\big](dx_\nu/d\tau). \tag{2.3e}$$

The electromagnetic force here *is easily shown to be gauge covariant, and it is obviously velocity-dependent.*

Just as local charge conservation implies the vanishing of the four-divergence of the four-vector charge density/flux j^μ, i.e., $\partial_\mu j^\mu = 0$, local energy-momentum conservation implies the vanishing of the four-divergence of the second-rank-tensor energy-momentum density/flux, *a symmetric second-rank tensor that is called the energy-momentum tensor and is written* $T^{\mu\nu}$, where $T^{\mu\nu} = T^{\nu\mu}$, and, of course, $\partial_\mu T^{\mu\nu} = 0$. Just as the divergence-free charge density/flux four-vector j^μ is the source of *the four-vector electromagnetic potential* A^μ, we assume that the divergence-free energy-momentum density/flux symmetric second-rank tensor $T^{\mu\nu}$ *entirely analogously* is the source of *a symmetric second-rank-tensor gravitational potential* $H^{\mu\nu}$. Furthermore, just as the dynamics of a mass-m and charge-e test body acted on by an electromagnetic potential A^μ is described by by the Eq. (2.3a) Lorentz-invariant Lagrangian, the dynamics of a mass-m test body acted on by a gravitational potential $H^{\mu\nu}$ is described by a Lorentz-invariant Lagrangian of the form,

$$L^{\mathrm{invr}}(dx^\mu/d\tau,\, H_{\mu\nu}(x)) = L^{\mathrm{invr}}_{\mathrm{free}}(dx^\mu/d\tau) - \mathcal{K}H_{\mu\nu}(x)(dx^\mu/d\tau)(dx^\nu/d\tau), \tag{2.4a}$$

whose Lorentz-invariant add-on term $-\mathcal{K}H_{\mu\nu}(x)(dx^\mu/d\tau)(dx^\nu/d\tau)$ *is clearly velocity-dependent.* Since the dimension of a Lagrangian is that of energy, the dimension of the constant $\mathcal{K}$ in this add-on term must be that of mass times the inverse of whatever the dimension of the gravitational potential $H_{\mu\nu}(x)$ is. Therefore since $L^{\mathrm{invr}}_{\mathrm{free}}(dx^\mu/d\tau) = -(m/2)\big(\eta_{\mu\nu}(dx^\mu/d\tau)(dx^\mu/d\tau)\big)$, we can reexpress Eq. (2.4a) as follows,

$$L^{\mathrm{invr}}(dx^\mu/d\tau,\, h_{\mu\nu}(x)) = -(m/2)\big(\eta_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau)\big) - (m/2)\big(h_{\mu\nu}(x)(dx^\mu/d\tau)(dx^\nu/d\tau)\big), \tag{2.4b}$$

where $h_{\mu\nu}(x) \stackrel{\mathrm{def}}{=} (2/m)\mathcal{K}H_{\mu\nu}(x)$ *is clearly dimensionless.* It is interesting to compare the nonrelativistic limit of the Eq. (2.4b) Lorentz-invariant gravitational Lagrangian $L^{\mathrm{invr}}(dx^\mu/d\tau,\, h_{\mu\nu}(x))$ for a mass-m test body with the nonrelativistic Lagrangian $L(\dot{\mathbf{x}},\, V(\mathbf{x})) = (m/2)|\dot{\mathbf{x}}|^2 - V(\mathbf{x})$ which produces the Eq. (1.1a) Newtonian gravitational equation of motion for that test body. If we assume that $h_{\mu\nu}(x)$ is time-independent, and we also assume the nonrelativistic asymptotic relations $|\dot{\mathbf{x}}| \ll c$ and $\tau \to t$, then,

$$L^{\mathrm{invr}}(dx^\mu/d\tau,\, h_{\mu\nu}(x)) \simeq -(mc^2/2) + (m/2)|\dot{\mathbf{x}}|^2 - (mc^2/2)h_{00}(\mathbf{x}). \tag{2.4c}$$

The completely constant term $-(mc^2/2)$ in Eq. (2.4c) doesn't contribute to the equation of motion, so,

$$h_{00}(\mathbf{x}) \simeq 2V(\mathbf{x})/(mc^2), \tag{2.4d}$$

in the nonrelativistic limit, where $V(\mathbf{x})$ is the mass-m test body's Newtonian gravitational potential energy.

Returning now to the Eq. (2.4b) Lorentz-invariant Lagrangian for a mass-m test body acted on by a gravitational potential, we immediately see that it can be reexpressed *in metric form,*

$$L^{\mathrm{invr}}(dx^\mu/d\tau,\, g_{\mu\nu}(x)) = -(m/2)\big(g_{\mu\nu}(x)(dx^\mu/d\tau)(dx^\nu/d\tau)\big), \tag{2.4e}$$

where $g_{\mu\nu}(x) \stackrel{\mathrm{def}}{=} \eta_{\mu\nu} + h_{\mu\nu}(x)$. The Eq. (2.4e) *metric form* of the gravitational Lagrangian for a mass-m test body changes the very concept of gravity *to that of effectively being a distortion of Lorentzian space-time* (which is caused by energy-momentum). *At the same time,* however, our *derivation* of Eq. (2.4e) *absolutely requires the gravitational space-timel metric $g_{\mu\nu}(x)$ to be Lorentz-covariant,* which brings on a clash with the differential-geometry idea that arbitrary transformations of the coordinates used don't "really" change the metric. We have, however, already pointed out a similar clash regarding the electromagnetic four-potential A^μ. It might be claimed that gauge transformations $A^\mu \to A^\mu + \partial^\mu\chi$ don't "really" change the electromagnetic four-potential A^μ, but gauge-covariant equations obviously *fail to have a unique solution for A^μ, and their solution sets typically include solutions which violate basic physical requirements such as Lorentz covariance or causality.* One therefore seeks *the simplest member of the solution subset whose members don't violate any basic physical requirement.* In practice that implies *imposition of the retarded Lorentz gauge condition,* which narrows the solution subset down to *a unique solution which definitely is Lorentz-covariant and causal.* Gauge-transformation covariance is an example of a "dynamic symmetry", which imposes absolutely crucial structural requirements on the equations of a particular theory, *but simultaneously prevents those equations from having a unique solution,* which makes *the physical-principle-guided winnowing of the consequent solution sets down to a unique solution unavoidable and absolutely essential.*

When differential geometers say that arbitrary transformations of the coordinates used don't "really" change the metric, the idea they are attempting to enunciate is that arbitrary transformations of the coordinates used *don't alter the six field degrees of freedom of the metric which comprise its "Einstein curvature".* The metric, however, *has four additional field degrees of freedom, and scientific observationalists and empiricists never in practice deal with a version of Nature which is other than strictly Lorentz covariant.* Three categories of experiments, taken together, confirm or invalidate the physical legitimacy of the Lorentz transformation: (1) the Michelson-Morley experiment category, (2) the Kennedy-Thorndike experiment category and (3) the Stilwell-Ives experiment category. All three of these experiment categories have been refined by many orders of magnitude since their original namesake experiments were performed well over a century ago. Therefore transformations to coordinates *which aren't Lorentz covariant,* such as the Galilean-covariant coordinates introduced to gravity theory by Alexandre Friedmann in 1922, aren't physically legitimate *unless,* of course, transformation to Lorentz-covariant coordinates is carried out on the results obtained—which is simply never done in practice. *In addition to Lorentz covariance, another condition applies to gravity theory,* one which arises from the way that the equation of motion implied by the Eq. (2.4e) Lagrangian is dealt with. We now work out that equation of motion,

$$-(m/2)\big\{d\big[\partial\big(g_{\mu\nu}(x)(dx^\mu/d\tau)(dx^\nu/d\tau)\big)/\partial(dx^\lambda/d\tau)\big]/d\tau - \partial\big(g_{\mu\nu}(x)(dx^\mu/d\tau)(dx^\nu/d\tau)\big)/\partial x^\lambda\big\} = 0 \quad\Longrightarrow$$

$$-(m/2)\big\{d\big[2g_{\lambda\nu}(x)(dx^\nu/d\tau)\big]/d\tau - (\partial g_{\mu\nu}(x)/\partial x^\lambda)(dx^\mu/d\tau)(dx^\nu/d\tau)\big\} = 0 \quad\Longrightarrow$$

$$-(m/2)\big\{2g_{\lambda\nu}(x)\big(d^2x^\nu/d\tau^2\big) + 2(\partial g_{\lambda\nu}(x)/\partial x^\mu)(dx^\mu/d\tau)(dx^\nu/d\tau) -$$

$$(\partial g_{\mu\nu}(x)/\partial x^\lambda)(dx^\mu/d\tau)(dx^\nu/d\tau)\big\} = 0 \quad\Longrightarrow$$

$$-mg_{\lambda\nu}(x)\left(d^2x^\nu/d\tau^2\right) = (m/2)\big[\partial g_{\lambda\nu}(x)/\partial x^\mu + \partial g_{\lambda\mu}(x)/\partial x^\nu - \partial g_{\mu\nu}(x)/\partial x^\lambda\big](dx^\mu/d\tau)(dx^\nu/d\tau). \tag{2.4f}$$

The Eq. (2.4f) equation of motion of a mass-m test body acted on by the gravitational metric potential $g_{\lambda\nu}(x)$ tells us that that tensor potential *endows the mass-m test body with the inertial mass $mg_{\lambda\nu}(x)$.* Lorentz-covariant field potentials *often modify the inertial masses of test bodies;* the Higgs scalar field potential is the most famous example of a Lorentz-covariant field potental which modifies the inertial masses of test bodies. *A notable exception* is the electromagnetic four-vector field potential A^μ; *the fact that it doesn't modify the inertial mass of a test body* is apparent from the Eq. (2.3e) equation of motion of a mass-m and charge-e test body acted on by an electromagnetic potential A^μ. In those cases where the gravitational metric potential $g_{\lambda\nu}(x)$ *possesses a matrix inverse* $g^{\kappa\lambda}(x)$ *everywhere in space-time,* its modification of the inertial mass m of the test body *is equivalent to a modification of the force which the gravitational metric potential* $g_{\lambda\nu}(x)$ *exerts on the mass-m test body,* since in those cases *multiplication of both sides of the* Eq. (2.4f) *result by the negative of the metric's matrix inverse, namely by* $-g^{\kappa\lambda}(x)$*, produces the modified equation of motion,*

$$m\big(d^2x^\kappa/d\tau^2\big) = -m\,\Gamma^\kappa_{\mu\nu}(x)\,(dx^\mu/d\tau)\,(dx^\nu/d\tau), \tag{2.4g}$$

where the affine connection $\Gamma^\kappa_{\mu\nu}(x)$ *is defined as,*

$$\Gamma^\kappa_{\mu\nu}(x) \stackrel{\text{def}}{=} (1/2)\,g^{\kappa\lambda}(x)\big[\partial g_{\lambda\nu}(x)/\partial x^\mu + \partial g_{\lambda\mu}(x)/\partial x^\nu - \partial g_{\mu\nu}(x)/\partial x^\lambda\big]. \tag{2.4h}$$

Because the affine connection *is nonlinear in the gravitational potential* $g_{\lambda\nu}(x)$*, so is the gravitational force on a mass-m test body* that appears on the right side of Eq. (2.4g). *We shall indeed require* $g_{\lambda\nu}(x)$ *to possess a matrix inverse* $g^{\kappa\lambda}(x)$ *everywhere in space-time in order to be able to regard the gravitational modification of a test body's inertial mass in* Eq. (2.4f) *as merely a modification of the gravitational force exerted on that test body, as shown in* Eqs. (2.4g) *and* (2.4h), albeit a modification of that gravitational force *which makes it nonlinear in the gravitational potential* $g_{\mu\nu}(x)$, as is apparent in the affine connection in Eq. (2.4h).

We therefore *require the determinant of* $g_{\lambda\nu}(x)$ *to be nonvanishing everywhere in space-time,* and of course we as well require $g_{\mu\nu}(x)$ to be Lorentz-covariant, as pointed out at length in the discussion following Eq. (2.4e) which runs up to Eq. (2.4f). *The simplest way by far to simultaneously fulfill both of these requirements is to merely require that* $\det(g_{\mu\nu}(x)) = k$ *everywhere in space-time, where k is a fixed nonzero constant,* because the square of the determinant of a Lorentz transformation is equal to unity. Because $\det(\eta_{\mu\nu}) = -1$, $k = -1$, *so the physically correct coordinate condition for metric gravity theory—which has the dynamic symmetry of general coodinate transformation covariance embedded into both the "geodesic equation" of* Eqs. (2.4g) *and* (2.4h), *and also into Einstein's field equation—is* $\det(g_{\mu\nu}(x)) = -1$*, which in addition implies that* $\Gamma^\kappa_{\kappa\lambda}(x) = 0$ *as a consequence of the fact that* $\partial\det(g_{\mu\nu}(x))/\partial x^\lambda = 0$.

Upon applying the coordinate condition $\det(g_{\mu\nu}(x)) = -1$ to a second-order approximate calculation of the metric of a static point mass (the sun) in his landmark November 18, 1915 paper, Einstein at last obtained a value for Mercury's perihelion shift which agreed with observation (after the vastly larger effect of the other planets on Mercury's perihelion shift is subtracted out). *Before he applied the* $\det(g_{\mu\nu}(x)) = -1$ *coordinate condition,* Einstein had struggled with perihelion-shift-calculation results which were smaller than what is observed. In that same November 18, 1915 paper, Einstein found that the coordinate condition $\det(g_{\mu\nu}(x)) = -1$ *doubled his previous result for the gravitational deflection of light by the sun;* a 1919 solar-eclipse expedition *confirmed that doubled result.* So Einstein's November 18, 1915 coordinate condition $\det(g_{\mu\nu}(x)) = -1$ *is supported by observation as well as by theory;* an English translation of Einstein's November 18, 1915 paper is given within the November 21, 2021 preprint "Einstein and the Perihelion Motion of Mercury" by Michel Janssen and Jürgen Renn (arXiv:2111.11238v1 [physics.hist-ph] 22 Nov 2021).

No reason exists to ever deviate from Einstein's November 18, 1915 coordinate condition $\det(g_{\mu\nu}(x)) = -1$ *for all x when making gravitational calculations for which Newtonian gravity is inadequate;* Alexandre Friedmann's 1922 coordinate condition $g_{00}(x) = 1$ for all x *in particular isn't capable of yielding results superior to those of Newtonian gravity.* For example, *the only way to in general reconcile* $g_{00}(x) = 1$ *for all x with the* Eq. (2.4d) *fact that* $g_{00}(\mathbf{x}) \simeq 1 + 2V(\mathbf{x})/(mc^2)$ *in the Newtonian gravitational limit,* where $V(\mathbf{x})$ is the Newtonian gravitational potential energy of a mass-m test body, *is to drive c to infinity, which ensures Newtonian gravitational physics.* Furthermore, *applying* $g_{00}(x) = 1$ *for all x to the exact general expression for gravitational time dilation, i.e.,*

$$\big[(\text{the tick rate of the clock at } x_2)/(\text{the tick rate of the clock at } x_1)\big] = \sqrt{g_{00}(x_2)/g_{00}(x_1)}\,, \tag{2.5}$$

completely eliminates gravitational time dilation, which likewise ensures Newtonian gravitational physics. Also, *the known exact gravitational solutions using* $g_{00}(x) = 1$ *for all x* display Newtonian-gravity character.

On January 13, 1916 Karl Schwarzschild published the metric whose determinant is -1 and exactly solves the Einstein field equation whose source is a static point mass M fixed to $|\mathbf{x}| = 0$. Using that exact metric, Schwarzschild could confirm and refine Einstein's second-order approximate metric results of November 18, 1915. Schwarzschild also exhibited the corresponding exact circular-orbit solution of the geodesic equation given by Eqs. (2.4g) and (2.4h). Regarding *the technical detail* of Schwarzschild's solution, it is *crucial* to note that just as the Newtonian gravitational potential energy $V(\mathbf{x}) = -GmM/|\mathbf{x}|$ of a mass-m test body acted on by a static point mass M fixed to $|\mathbf{x}| = 0$ *is nonsingular for all* $|\mathbf{x}| > 0$, *the nature of the Einstein equation guarantees that the physically-correct metric for a static point mass* M *fixed to* $|\mathbf{x}| = 0$ *is as well nonsingular for all* $|\mathbf{x}| > 0$. Technically this is because *the Ricci curvature tensor vanishes at all* $|\mathbf{x}| > 0$, *which guarantees that the components of the metric tensor are at least twice differentiable when* $|\mathbf{x}| > 0$, *but a coordinate transformation of a physically-correct solution of the Einstein equation, while also a solution of the Einstein equation, can definitely be physically incorrect because the fixed point* $|\mathbf{x}| = 0$ *is readily displaced by a coordinate transformation!* Schwarzschild, however, *carefully made certain that his metric is nonsingular for all* $|\mathbf{x}| > 0$, *and therefore is physically correct.* Schwarzschild's January 13, 1916 paper was translated into English by S. Antoci and A. Loinger in 1999, and posted by them on arXiv (arXiv:physics/9905030v1 [physics.hist-ph] 12 May 1999).

On May 27, 1916, J. Droste published the following exact metric solution for a static point mass M fixed to $|\mathbf{x}| = 0$, *which is algebraically much simpler than Schwarzshild's metric, but is physically flawed,*

$$(c\,d\tau)^2 = (1 - r_s/R)(c\,dt)^2 - (1/(1 - r_s/R))(dR)^2 - R^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big), \tag{2.6a}$$

where R is $|\mathbf{x}|$, and $r_s \overset{\text{def}}{=} 2GM/c^2$ is called the Schwarzschild radius. The famous mathematician David Hilbert *admired the algebraic simplicity of Droste's* Eq. (2.6a) *metric, and strongly promoted it in 1918.* The *upshot* of Hilbert's promotional effort *is that gravity textbooks universally feature Droste's physically-flawed* Eq. (2.6a) *metric, but very mistakenly attribute it to Schwarzschild.* In consequence, *Schwarzschild's physically-correct January 13, 1916 metric has been thrust into almost complete obscurity.*

The component $(1/(1 - r_s/R))$ of Droste's physically-flawed Eq. (2.6a) metric *has a singularity at* $R = r_s$ *which violates the metric smoothness consequence of the Ricci tensor's being zero at all* $R > 0$. Schwarzschild's physically-correct metric has no such singularity, as we have noted in the paragraph preceding the one that envelops Eq. (2.6a), but since Droste's metric and Schwarzschild's metric are solutions of the same Einstein equation, a radial coordinate transformation $R(r)$ exists which links them. *A key property of that transformation is that* $R = r_s$ *necessarily corresponds to* $r = 0$, *so* $R(r = 0) = r_s$. *Another key fact is that the determinant of Schwarzschild's metric is* -1, *Einstein's November 18, 1915 coordinate condition.* Inserting $R(r)$ into Droste's Eq. (2.6a) metric transforms it to Schwarzschild's metric,

$$(c\,d\tau)^2 = (1 - r_s/R(r))(c\,dt)^2 - (1/(1 - r_s/R(r)))(dR(r))^2 - (R(r))^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big), \tag{2.6b}$$

which is equivalent to,

$$(c\,d\tau)^2 = (1 - r_s/R(r))(c\,dt)^2 - (1/(1 - r_s/R(r)))(dR(r)/dr)^2(dr)^2 - (R(r)/r)^2 r^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big). \tag{2.6c}$$

For the determinant of Schwarzschild's metric to be -1, $(dR(r)/dr)^2(R(r)/r)^4$ must equal 1, which implies $R^2 dR = \pm\, r^2 dr$. On selecting $\pm = +$, we obtain $(R(r))^3 = r^3 + (r_0)^3$, where r_0 is an arbitrary constant, so $R(r) = (r^3 + (r_0)^3)^{\frac{1}{3}}$. Since $R(r = 0) = r_s$, $r_0 = r_s$, and therefore $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$, which implies that when $r > 0$, $R(r) > r_s$, so Schwarzschild's Eq. (2.6b) metric *is demonstrably singularity-free for all* $r > 0$, *in agreement with the metric smoothness consequence of the Ricci tensor's being zero when* $r > 0$.

The *contrast* between Droste's *physically-flawed* Eq. (2.6a) metric and Schwarzschild's *physically-correct* Eq. (2.6b) $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$ metric *is an object lesson in the fact that the coordinate-transformed members of the solution set of a given Einstein equation aren't physically equivalent,* just as the gauge-transformed members of the solution set of a given Eq. (2.1a) electromagnetic field equation,

$$\partial_\nu \partial^\nu A^\mu - \partial^\mu(\partial_\sigma A^\sigma) = 4\pi j^\mu/c,$$

aren't physically equivalent. Immensely useful and essential as *dynamic symmetries* such as gauge transformation covariance and general coordinate transformation covariance unquestionably are, *they nevertheless cannot alter by a single jot or tittle the fact that an adequately-posed classical-physics problem must, without exception, have a unique, well-defined solution.* Therefore *there always are physically-cogent criteria which winnow dynamic-symmetry solution sets down to a unique single member of that set.*

In particular, for a gravitational source which is a static point mass M fixed to $|\mathbf{x}| = 0$, Schwarzschild's January 13, 1916 Eq. (2.6b) $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$ *unique metric solution of determinant* -1 *that is demonstrably singularity-free for all* $r > 0$*, is the sole physical solution.*

We will now use Schwarzschild's Eq. (2.6b) $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$ metric in conjunction with the time component of the geodesic equation given by Eqs. (2.4g) and (2.4h) to work out the Einstein-gravity analog of the Eq. (1.1f) Newtonian-gravity equation of motion of a test body *which moves only radially relative to the static point mass* M *that is fixed to* $r = 0$. To carry this out, it will be convenient to temporarily abbreviate Eq. (2.6b) as follows,

$$(c\, d\tau)^2 = B(R(r))(c\, dt)^2 - A(R(r))(dR(r))^2 - (R(r))^2\big((d\theta)^2 + (\sin\theta\, d\phi)^2\big), \tag{2.6d}$$

where $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$, $B(R(r)) = 1 - r_s/R(r)$ and $A(R(r)) = 1/B(R(r))$. The Eq. (2.6d) metric *itself* immediately yields *the particular first-order equation of test-body gravitational motion,*

$$c^2 = c^2 B(R(r))(dt/d\tau)^2 - A(R(r))(dR(r)/d\tau)^2 - (R(r))^2\big((d\theta/d\tau)^2 + (\sin\theta(d\phi/d\tau))^2\big). \tag{2.6e}$$

Since the test body we consider *moves only radially relative to the static point mass* M *that is fixed to* $r = 0$, the Eq. (2.6e) *angular frequencies* $d\theta/d\tau$ *and* $d\phi/d\tau$ *are both equal to zero,* which *reduces* Eq. (2.6e) *to a first-order equation of test-body gravitational radial motion,*

$$c^2 = \big[c^2 B(R(r)) - A(R(r))(dR(r)/dt)^2\big](dt/d\tau)^2. \tag{2.6f}$$

To turn Eq. (2.6f) into a radial equation of motion, namely a differential equation for dr/dt, we need to evaluate the factor $(dt/d\tau)^2$. Evaluating $(dt/d\tau)$ requires integrating the time component of the second-order in τ four-vector geodesic equation which is given by Eqs. (2.4g) and (2.4h). For the Eq. (2.6d) metric this time component of the geodesic equation is,

$$\tfrac{d^2t}{d\tau^2} + \tfrac{dt}{d\tau}\tfrac{dB(R(r))/dR(r)}{B(R(r))}\tfrac{dR(r)}{d\tau} = 0, \tag{2.7a}$$

which can be written,

$$\tfrac{1}{dt/d\tau}\tfrac{d(dt/d\tau)}{d\tau} + \tfrac{dB(R(r))/dR(r)}{B(R(r))}\tfrac{dR(r)}{d\tau} = 0, \tag{2.7b}$$

which in turn can be written,

$$d\big(\ln\big(dt/d\tau\big) + \ln\big(B(R(r))\big)\big)/d\tau = 0, \tag{2.7c}$$

which implies that,

$$\ln\big((dt/d\tau)(B(R(r)))\big) = -C, \tag{2.7d}$$

where C is an arbitrary dimensionless constant. Eq. (2.7d) implies that,

$$dt/d\tau = 1/(KB(R(r))), \tag{2.7e}$$

where $K = \exp(C)$ is an arbitrary dimensionless positive constant. Inserting Eq. (2.7e) into Eq. (2.6f) yields,

$$\big(A(R(r))/(B(R(r)))^2\big)(dR(r)/dt)^2 - \big(c^2/B(R(r))\big) = -c^2K^2. \tag{2.8a}$$

The object $dR(r)/dt$ in Eq. (2.8a) is of course equal to $(dR(r)/dr)(dr/dt)$, and from $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$, $dR(r)/dr = r^2(r^3 + r_s^3)^{-\frac{2}{3}} = (r/R(r))^2$. Inserting this result along with $B(R(r)) = 1 - r_s/R(r)$ and $A(R(r)) = 1/B(R(r))$ into Eq. (2.8a) yields,

$$\big((r/R(r))^4/(1 - r_s/R(r))^3\big)(dr/dt)^2 - \big(c^2/(1 - r_s/R(r))\big) = -c^2K^2, \tag{2.8b}$$

where $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$ and $r_s = (2GM/c^2)$. We are now able to write down the Einstein-gravity analog of the Eq. (1.1f) Newtonian-gravity equation of motion $(dr/dt)^2 = (2GM/r) + \varepsilon$ of a test body *which moves only radially relative to the static point mass* M *that is fixed to* $r = 0$,

$$(dr/dt)^2 = c^2\big((R(r)/r)^2(1 - r_s/R(r))\big)^2\big[1 - K^2(1 - r_s/R(r))\big], \tag{2.8c}$$

where $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$. Defining the dimensionless variable q as $q \overset{\text{def}}{=} (r/r_s)$, we note that $(R(r)/r) = \big((q^3 + 1)^{\frac{1}{3}}/q\big)$ and $(1 - r_s/R(r)) = \big(1 - \big(1/(q^3 + 1)^{\frac{1}{3}}\big)\big)$, so Eq. (2.8c) becomes,

$$(dr/dt)^2 = c^2\big(\big((q^3+1)^{\frac{1}{3}}/q\big)^2\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\big)^2\left[1-K^2\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\right], \tag{2.8d}$$

where $q \stackrel{\text{def}}{=} (r/r_s)$. In the Newtonian-gravity case, we see from Eq. (1.1f), i.e., $(dr/dt)^2 = (2GM/r) + \varepsilon$, that $(dr/dt)^2$ *increases without bound as* $r \to 0$. Indeed, in the Newtonian-gravity case the radial speed $|dr/dt|$ *is asymptotic to* $\sqrt{2GM/r}$ *as* $r \to 0$.

To work out the asymptotic behavior of $(dr/dt)^2$ as $q \to 0$ in Eq. (2.8d), we note that as $q \to 0$, $(q^3+1)^{\frac{1}{3}}/q \simeq 1/q$ and $\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big) \simeq q^3/3$, so $\big((q^3+1)^{\frac{1}{3}}/q\big)^2\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big) \simeq q/3$, which together with Eq. (2.8d) yields that $((dr/dt)/c)^2 \simeq (q/3)^2$ as $q \to 0$. Thus $|dr/dt| \simeq c(q/3)$ as $q \to 0$, so,

$$\textit{the test body's radial speed } |dr/dt| \text{ is asymptotic to } (c/(3r_s))r \text{ as } r \to 0, \tag{2.8e}$$

which is precisely the opposite of the unbounded speed of the test body as $r \to 0$ *in the Newtonian-gravity case.* In the Einstein-gravity case, *the gravitational time-dilation effect of very strong gravity reduces speeds.*

We next verify that $(dr/dt)^2 < c^2$. We *first* show that $d\big(\big((q^3+1)^{\frac{1}{3}}/q\big)^2\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\big)/dq > 0$ when $q > 0$. Since $d\big(\big((q^3+1)^{\frac{1}{3}}/q\big)^2\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\big)/dq = \big[2+q^3-2(q^3+1)^{\frac{1}{3}}\big]/\big[q^3(q^3+1)^{\frac{2}{3}}\big]$, *we must show that* $2+q^3 > 2(q^3+1)^{\frac{1}{3}}$ when $q > 0$. We do so by exhibiting a chain of inequalities *which are logically equivalent to* $2+q^3 > 2(q^3+1)^{\frac{1}{3}}$, *where the final inequality in the chain is clearly valid when* $q > 0$,

$$2+q^3 > 2(q^3+1)^{\frac{1}{3}} \iff 1+(q^3/2) > (1+q^3)^{\frac{1}{3}} \iff 1+3(q^3/2)+3(q^3/2)^2+(q^3/2)^3 > 1+q^3$$
$$\iff (1/2)q^3+(3/4)q^6+(1/8)q^9 > 0 \text{ when } q > 0. \tag{2.8f}$$

Therefore $\big(\big((q^3+1)^{\frac{1}{3}}/q\big)^2\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\big)$ is a strictly increasing function of q when $q > 0$, so when $q > 0$, it is less than its $q \to \infty$ limit, which has the value unity. Consequently, from Eq. (2.8d), $(dr/dt)^2 < c^2\big[1-K^2\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\big] < c^2$ when $q > 0$, because $K^2 > 0$ and $\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big) > 0$ when $q > 0$. Thus, $(dr/dt)^2 < c^2$ when $q > 0$, and, when $q = 0$, Eq. (2.8e) implies that $(dr/dt)^2 = 0$, so $(dr/dt)^2 < c^2$ *under all circumstances; the test body never has a speed as great as c.*

We next investigate the radial speed $|dr/dt|$ of the test body as $r \to \infty$. From Eq. (2.8d) we see that as $q \to \infty$, $(dr/dt)^2 \to c^2\big(1-K^2\big)$, so,

$$|dr/dt| \to c\sqrt{1-K^2} \text{ as } r \to \infty. \tag{2.8g}$$

If $K^2 > 1$, Eq. (2.8g) tells us *that the test body is gravitationally bound, so it can't reach arbitrarily large values of* r. Conversely, *the closer* K^2 *is to zero, the closer the test body's* $r \to \infty$ *asymptotic speed is to c.*

Having obtained the radial speed $|dr/dt|$ of the test body as $r \to \infty$, we next work out *its radial acceleration* d^2r/dt^2 *as* $r \to \infty$. To do so, we rewrite Eq. (2.8d) as,

$$(dr/dt)^2 = c^2\big(\chi(q) - K^2\xi(q)\big), \tag{2.8h}$$

where,

$$\chi(q) = \big((q^3+1)^{\frac{1}{3}}/q\big)^4\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big)^2 \text{ and } \xi(q) = \big((q^3+1)^{\frac{1}{3}}/q\big)^4\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big)^3. \tag{2.8i}$$

Differentiating both sides of Eq. (2.8h) with respect to t yields,

$$2(dr/dt)(d^2r/dt^2) = c^2\big(d\chi(q)/dq - K^2 d\xi(q)/dq\big)\,(dq/dr)(dr/dt), \tag{2.8j}$$

where,

$$d\chi(q)/dq = \big(4/q^5\big)\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\big(1+(q^3/2)-(1+q^3)^{\frac{1}{3}}\big) \text{ and}$$
$$d\xi(q)/dq = \big(4/q^5\big)\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big)^2\big(1+(3q^3/4)-(1+q^3)^{\frac{1}{3}}\big). \tag{2.8k}$$

Since $q = (r/r_s)$ and $r_s = 2GM/c^2$, Eq. (2.8j) can be rewritten as follows,

$$d^2r/dt^2 = \tfrac{1}{2}(c^2/r_s)(r_s/r)^2q^2\big(d\chi(q)/dq - K^2 d\xi(q)/dq\big) = \big(GM/r^2\big)\big((q^2\,d\chi(q)/dq) - K^2(q^2\,d\xi(q)/dq)\big). \tag{2.8l}$$

From Eq. (2.8k) we see that as $q \to \infty$, $(q^2\,d\chi(q)/dq) \to 2$ and $(q^2\,d\xi(q)/dq) \to 3$, so from Eq, (2.8l),

$$\textit{the test body's radial acceleration } d^2r/dt^2 \textit{ is asymptotic to } \big(GM/r^2\big)\big(2-3K^2\big) \textit{ as } r \to \infty, \tag{2.8m}$$

which only agrees with the Eq. (1.1e) *Newtonian-gravity acceleration* $-(GM/r^2)$ *when* $0 \le (1-K^2) \ll 1$. When $0 < K^2 < (2/3)$, *the test body's acceleration becomes positive.* An *unexpected positive acceleration of the expansion of the universe has been reliably observed,* and its discoverers awarded a Nobel prize. *If the* *radially-moving test body is a packet of light, we see from the* $r \to \infty$ *asymptotic* Eq. (2.8g) *that* $K^2 = 0$. Putting K^2 to zero in Eqs. (2.8h) and (2.8l) produces,

$$(dr/dt)^2 = c^2\chi(q) \quad \text{and} \quad d^2r/dt^2 = \big(GM/r^2\big)\big(q^2\, d\chi(q)/dq\big), \tag{2.8n}$$

where $q = (r/r_s)$, and $\chi(q)$ *increases monotonically from zero at* $q = 0$ *toward unity as* $q \to \infty$. Therefore $d\chi(q)/dq > 0$, and *furthermore,* immediately below Eq. (2.8l) we learn that as $q \to \infty$, $(q^2\, d\chi(q)/dq) \to 2$. Therefore *the closer the radially-moving light packet is to the static point mass* M *at* $r = 0$*, the slower its speed is,* so the gravitational acceleration which the static point mass exerts on the radially-moving light packet *is always directed radially outward, which is the opposite direction of the gravitational acceleration that the static point mass exerts on a stationary or nonrelativistically-moving test body.* Moreover, when the radially-moving light packet is at asymptotically large distance from the static point mass M, the acceleration d^2r/dt^2 which the static point mass M exerts on it is $2(GM/r^2)$, *which not only is oppositely directed to the acceleration* $-(GM/r^2)$ *that that static point mass* M *exerts on a nonrelativistically-moving test body, but as well is twice as great in magnitude!* The effect of gravity on light is best regarded *as a purely refractive one; the stronger the gravitational field is, the slower light travels,* so the "gravitational lensing" terminology astronomers use is very appropriate indeed.

The tremendous *velocity dependence* of Einstein's gravity strongly calls into question the conceptual appropriateness of the supposed "Principle of Equivalence". The fundamental *curvature/tidal effects of gravitation have never coexisted comfortably with that supposed "Principle", so the realization that high enough speeds play havoc with the direction and magnitude of the acceleration of gravity ought to be the final nail in the coffin of the "Principle of Equivalence".* It would seem that the best way to *begin* thinking about gravity *is to place great emphasis on its parallels with electromagnetism,* but with the immensely important *distinction* that *its source* is the second-rank-symmetric-tensor density/flux of conserved energy-momentum *instead of* the four-vector density/flux of conserved charge. The *consequent second-rank-symmetric-tensor gravitational potential necessarily interacts with a test body's four-vector velocity in a way which startlingly mimics the structure of that test body's kinetic-energy term.* The natural consequent *conjoining of the two almost identically-structured terms* then reveals gravity to be *a local warping of Lorentzian space-time, an effect of space-time metrics rather than of potentials as in electromagnetism.* Given the great existing store of knowledge about metrics in differential geometry, the familiar elements of Einstein's gravity theory inevitably emerge, including its key dynamic symmetry of general coordinate transformation covariance which enforces gravity's physically-crucial nonlinear structure, and plays a role in gravity theory which closely parallels the role played in electromagnetic theory by that theory'a own dynamic symmetry of gauge-transformation covariance. Gravity theory *is a magnificent amalgam of the principles of Lorentzian electromagnetism with those of metric differential geometry;* its almost kaleidoscopic attributes defy characterization by something as simpleminded as the "Principle of Equivalence".

We next elucidate *the possible time dependences* $r(t)$ of a test body's radial distance r as $r \to 0$. We *first* treat *Newtonian gravity,* where $(dr/dt)^2 \simeq (2GM/r)$ as $r \to 0$. The two possible $r \to 0$ time dependences *follow from solutions of the two differential equations* $dr_B/dt = +\sqrt{2GM/r_B}$ *and* $dr_C/dt = -\sqrt{2GM/r_C}$ *that also satisfy* $r_B(t) \to 0$ *and* $r_C(t) \to 0$. Therefore $r_B(t)$, dr_B/dt, $r_C(t)$ and dr_C/dt are given by,

$$r_B(t) = \big((9/2)GM(t - t_B)^2\big)^{\frac{1}{3}} \quad \text{and} \quad dr_B/dt = \big((4/3)GM/(t - t_B)\big)^{\frac{1}{3}} \quad \text{as} \quad (t - t_B) \to 0+, \ \text{and,}$$

$$r_C(t) = \big((9/2)GM(t_C - t)^2\big)^{\frac{1}{3}} \quad \text{and} \quad dr_C/dt = -\big((4/3)GM/(t_C - t)\big)^{\frac{1}{3}} \quad \text{as} \quad (t_C - t) \to 0+, \tag{2.9a}$$

where $r_B(t)$ and dr_B/dt *give the* $r_B(t) \to 0$ *"Big Bang" asymptotic time behavior as* $(t - t_B) \to 0+$*, where* t_B *is the time earlier than* t *when the "Big Bang" occurred,* while $r_C(t)$ and dr_C/dt *give the* $r_C(t) \to 0$ *"gravitational collapse" asymptotic time behavior as* $(t_C - t) \to 0+$*, where* t_C *is the time later than* t *when the "gravitational collapse" will occur.* Both of these time dependences accord with $r \to 0$, *but they both as well accord with* $|dr/dt| \to \infty$*, which is the most extreme violation of* $|dr/dt| < c$ *conceivable.* Moreover, *all initial conditions for this model, when solved using purely Newtonian gravity, exhibit either a past "Big Bang" or a future "gravitational collapse" or both.*

We next treat the time dependences of a test body's radial distance r as $r \to 0$ *in Einstein gravity,* where $(dr/dt)^2 \simeq ((c/(3r_s))r)^2$ as $r \to 0$ (see Eq. (2.8e)). Therefore we find solutions of the two differential equations $dr_I/dt = +(c/(3r_s))r_I$ and $dr_D/dt = -(c/(3r_s))r_D$ that satisfy $r_I(t) \to 0$ and $r_D(t) \to 0$,

$$r_I(t)=r_I(0)\exp(ct/(3r_s)) \text{ and } dr_I/dt=c(r_I(0)/(3r_s))\exp(ct/(3r_s)) \text{ for } t\le 0 \ \&\ r_I(0)\to 0, \text{ and,}$$

$$r_D(t)=r_D(0)\exp(-ct/(3r_s)) \text{ and } dr_D/dt=-c(r_D(0)/(3r_s))\exp(-ct/(3r_s)) \text{ for } t\ge 0 \ \&\ r_D(0)\to 0, \quad (2.9b)$$

where the inflationary $r_I(t) = r_I(0)\exp(ct/(3r_s))$ *for* $t \le 0$ *increases exponentially with time* t*, and the deflationary* $r_D(t) = r_D(0)\exp(-ct/(3r_s))$ *for* $t \ge 0$ *decreases exponentially with time* t*.* Since $|dr_I/dt| \le c(r_I(0)/(3r_s))$ and $r_I(0) \to 0$, $|dr_I/dt| < c$, and since $|dr_D/dt| \le c(r_D(0)/(3r_s))$ and $r_D(0) \to 0$, $|dr_D/dt| < c$.

We also note that both the inflationary and the deflationary $r \to 0$ *asymptotic forms* $r_I(t)$ *and* $r_D(t)$ *manifest entirely positive acceleration.* This is *a prime example* of the fact that *gravitational time dilation, when it is strong enough, provokes acceleration opposite to that associated with Newtonian gravity.* It is apparent *that proper understanding of the universe's inflationary era can't be attained without appreciation of the fundamental underlying role of gravitational time dilation.*

Eqs. (2.8h) to (2.8l) describe the differential equations for a test body's radial motion in the gravitational field of a static point mass M. *For their transparent numerical treatment,* we now *restate* Eqs. (2.8h) *and* (2.8l) *in terms of the dimensionless variables* $q \stackrel{\text{def}}{=} (r/r_s)$ *and* $u \stackrel{\text{def}}{=} (ct/r_s)$*, where* $r_s = (2GM/c^2)$. These equations *as well involve the dimensionless constant* K^2. The first-order Eq. (2.8h) becomes,

$$(dq/du)^2 = \big(\chi(q) - K^2\xi(q)\big), \quad (2.10a)$$

where $\chi(q)$ and $\xi(q)$ are explicitly given by Eq. (2.8i). The second-order Eq. (2.8l) becomes,

$$d^2q/du^2 = (1/2)\left(d\chi(q)/dq - K^2 d\xi(q)/dq\right), \quad (2.10b)$$

where $d\chi(q)/dq$ and $d\xi(q)/dq$ are explicitly given by Eq. (2.8k).

According to the Birkhoff theorem, the purely radial gravitational motion of the surface of a spherical blob of uniform-density zero-pressure perfect fluid doesn't change when the blob's interior fluid is replaced by an equivalent static point mass M fixed to its center; i.e., Mc^2 is the total conserved energy of the blob's interior fluid. Thus all of the results that we have obtained so far apply *not only to the purely radial motion of a test body, but also to the purely radial motion of the spherical surface of such a uniform-density zero-pressure spherical blob of perfect fluid,* which is known as an Oppenheimer-Snyder cosmological model.

As noted below Eq. (2.8m), the fact that the universe's expansion is accelerating implies that $K^2 < (2/3)$. Since the universe's asymptotic expansion speed is $c\sqrt{1-K^2}$, its expansion speed therefore must be greater than $c/\sqrt{3} = .57735c$, which corresponds to a redshift of only about 1. We thus see that the order of magnitude of the universe's expansion speed can comfortably account for the acceleration of its expansion, *with no need for "dark energy"* (the presence of a cosmological constant term in the Einstein equation).

The *Einstein-gravity* Oppenheimer-Snyder model shows *the very great importance of gravitational time dilation,* both to *the observed acceleration of the expansion of the universe* and to *the inflationary character of the universe when it was sufficiently small.* Of course this *Einstein-gravity* model *permits only speeds less than c under all circumstances, so nothing remotely resembling a Big Bang, with its unbounded speeds, could ever have occurred.* Also, the *Einstein-gravity universe has existed forever,* albeit in a state of extreme gravitational time dilation "suspended animation" when it was far smaller than its Schwarzschild radius $r_s = (2GM/c^2)$, where Mc^2 is its total conserved energy. With *no Big Bang whatsoever, and its having existed forever, there is no reason why the universe shouldn't have a surplus of particles over antiparticles.*

3. Further ideas of a more speculative nature about the universe's evolution

A key property of the universe is that it is expanding, so it presumably was arbitrarily compact and dense in the sufficiently remote past; in particular it was far inside its Schwarzschild radius $r_s = (2GM/c^2)$, where Mc^2 is the universe's conserved total energy. In that era its behavior would have been dominated by gravitational time dilation, so all physical processes would have been greatly slowed and radiation frequencies greatly reduced; it would have been dark and cold with almost paralyzed physical processes, even its expansion rate would have been greatly reduced. *Going further back in time only further accentuates its "suspended animation" character.* Going *forward* in time eventually brings it to a radius of the order of r_s. The accompanying *decrease in gravitational time dilation* would have *increased its expansion rate,* which would have *further reduced gravitational time dilation,* causing *a futher increase in its expansion rate,* etc.

Thus when the universe reached a radius of the order of r_s it was on the cusp *of a rapid increase in its expansion rate.* Physical process rates in that era *would have also rapidly increased* as the dead hand of extreme gravitational time dilation fell away. *Notwithstanding its rapid expansion, the universe would still have been vastly, vastly more compact and dense than today's universe,* which has undergone billions of years *of additional expansion.* So *dense* a universe, *which was liberated from extreme gravitational time dilation,* would have been able to give birth to every conceivable kind of young star at an utterly enormous rate, with particular emphasis on immensely massive, extremely hot and short-lived giants. However *considering how much even denser than that the universe was when it neared the liberating radius* r_s, only a quite small fraction of its matter *would have been able to participate directly in those fireworks; by far the greatest part of its matter would have been compelled to take the form of primordial black holes* (but do bear in mind that black holes *absolutely do not have event horizons*). However those primordial black holes *profoundly modulated* the spectacular star-formation fireworks underway by, for example, becoming the active nuclei of galaxies, with the primordial black holes of lesser mass *being utterly crucial to galaxy formation by supplying the necessary cold, dark gravitational "glue".* When the compact, dense universe's star-formation fireworks was at its zenith, the universe was obviously extremely hot, so the black-body cosmic microwave background is the frequency-downshifted remnant of the universe's immense black-body radiation of that intense star-formation era. With its continued expansion, the universe's density of course diminished, diminishing its rate of star and galaxy formation. The James Webb Space Telescope may possibly be registering evidence of rapid galaxy formation in the early universe.