BIVARIATE GENERATING FUNCTIONS FOR NON-ATTACKING WAZIRS ON RECTANGULAR BOARDS

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Abstract. A wazir is a fairy chess piece that attacks the 4 neighbors to the North, East, South and West of the chess board. This work constructs the bivariate generating functions for the number of placing w mutually non-attacking wazirs on rectangular boards of shape $r \times c$ at fixed c. The equinumerous setup counts binary $\{0, 1\}$ arrays of dimension $r \times c$ which have w 1's with mutual L1 (Manhattan) distances > 1.

1. Non-attacking Wazirs

A wazir is a chess figure that attacks the 4 squares directly attached to its square in the four principle compass directions. It has the narrow range of the king [6]—which attacks its surrounding 8 squares—but the directions of attack inherited from the rook. [Placed on the edge or in the corner of the board, the wazir attacks only 3 or 2 squares.] One may also say that a wazir attacks the squares in the von-Neumann neighborhood or that it attacks the four squares at L1 (Manhattan) distance equal to 1.

Definition 1. *$W(r, c, w)$ is the number of arrangements of w non-attacking wazirs on a $r \times c$ rectangular chess board.*

$W(r, c, w)$ is also the number of placements of w monominoes on $r \times c$ boards such that no two of them could be bonded edge-to-edge into dominoes or higher polyominoes.

Since we do not consider rotations or flips of the entire configuration along board middle axes, diagonals or through the center, the role of rows and columns may be interchanged:

$$W(r, c, w) = W(c, r, w). \tag{1}$$

If no wazir is present, the empty board is the only solution:

$$W(r, c, 0) = 1. \tag{2}$$

A single wazir can be placed on any square, because the constraint on neighbors is irrelevant then:

$$W(r, c, 1) = rc. \tag{3}$$

A single wazir can be placed in one of 4 corners which leaves $rc - 3$ non-attacked squares for the second. A single wazir can be placed on one of the $2(r-2)+2(c-2)$ edges which leaves $rc - 4$ squares for the second. A single wazir can be placed on

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one of $(r-2)(c-2)$ non-border squares which leaves $rc-5$ squares for the second. $4(rc-3)+[2r+2c-8](rc-4)+(r-2)(c-2)(rc-4)$ is a factor 2 too large because each pair is counted twice:

$$W(r,c,2) = r + c + \frac{rc}{2}(rc-5). \tag{4}$$

The encoding of a configuration is a stack of r 2-letter words of the alphabet $\{0,1\}$ of length c, where 1's indicate that a wazir occupies a square, 0's indicate the square is empty. For a single row, the non-attacking request is closely related to the Zeckendorf representation and to words avoiding the 11 subword.

Example 1. *The words of length 1 are 0 and 1. The words of length 2 are 00, 01, and 10. The words of length 3 are 000, 001, 010, 100 and 101. The words of length 4 are 0000, 0001, 0010, 0100, 0101, 1000, 1001, and 1010* [2, A014417].

The number of such words is F_{c+2} where F are the Fibonacci numbers.

The densest packing of non-attacking wazirs is achieved by placing them on the subgrid reachable by moves of a bishop. So

$$W(r,c,w) = 0 \quad \text{if} \quad w > \lceil rc/2 \rceil. \tag{5}$$

These trailing zeros are not printed in the tables of $W(r,c,w)$ in this manuscript.

Definition 2. *(Bivariate GF) A bivariate generating function keeping the number of columns fixed is*

$$\hat{W}_c(x,y) \equiv \sum_{r=0}^{\infty}\sum_{w=0}^{\infty} W(r,c,w)x^r y^w. \tag{6}$$

2. Transfer Matrices

W may be counted by recursively attaching a new row to an already existing binary array of c columns, registering only those rows (binary words) in the new row which are compatible with the non-attacking requirement; compatibility means the bitwise `and` of the new binary word and the binary row of the previous row must be zero.

This is a typical Markov chain requirement: compatibility is defined related to knowledge of a single previous row. A state diagram (automaton) is defined where each of the

$$s = F_{c+2} \tag{7}$$

binary words is a node—a trivial labeling is the integer value of the binary word—, and a digraph is constructed where two nodes are connected by an arc if they are compatible.

Remark 1. *Compatibility in this problem is commutative. So the digraph is a symmetric digraph.*

The s nodes can be considered ordered, for example by just using their integer labels as a measure—which are unique because c is supposed to be fixed.

The accumulation of new rows in an array of c columns, starting with a (virtual) top row of the 000…0 word which is compatible with any other binary word, is a walk along arcs of the digraph.

$r\backslash w$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2					
3	1	3	1				
4	1	4	3				
5	1	5	6	1			
6	1	6	10	4			
7	1	7	15	10	1		
8	1	8	21	20	5		
9	1	9	28	35	15	1	
10	1	10	36	56	35	6	
11	1	11	45	84	70	21	1
12	1	12	55	120	126	56	7

TABLE 1. The number $W(r,1,w)$ of placing w non-attacking wazirs on $r \times 1$ boards [2, A011973]. Essentially a slanted version of the Pascal Triangle of binomial coefficients.

The Transfer Matrix method is the standard tool to track the number of new wazirs [5]. It is a $s \times s$ matrix $T(n,m)$. $T(n,m) = 0$ if the nth node is not compatible with the mth node, otherwise it is xy^b where

- y^b indicates the number of wazirs has increased by b, the number of bits (or wazirs) in node n,
- and the x^1 indicates the number of rows has increased by 1.

Walks of finite length are associated with powers of the T-matrix, and the bivariate generating function $\hat{W}$ is top left element of the inverse $(1-T)^{-1}$ [1, 3].

The nature of the transfer matrix method proves that the $\hat{W}$ are rational polynomials in x and y because all elements of T are polynomials in x and y.

3. RESULTS

The generating function associated with boards 1 column wide, Table 1, is

$$\hat{W}_1(x,y) \equiv p_1(x,y)/q_1(x,y) = (1+xy)/(1-x-x^2y). \tag{8}$$

The generating function associated with boards 2 columns wide, Table 2, is

$$\hat{W}_2(x,y) \equiv p_2(x,y)/q_2(x,y) = (1+xy)/(1-x-xy-x^2y). \tag{9}$$

The generating function associated with boards 3 columns wide, Table 3, is

$$\begin{aligned} \hat{W}_3(x,y) &\equiv p_3(x,y)/q_3(x,y); \\ p_3(x,y) &= (1+xy)\left(-x^2y^3+xy+xy^2+1\right); \\ q_3(x,y) &= x^4y^4+x^3y^4-x^3y^2-3x^2y^2-2x^2y-x^2y^3-xy-x+1. \end{aligned} \tag{10}$$

$r\backslash w$	0	1	2	3	4	5	5	6	8	9	10	11	12
0	1												
1	1	2											
2	1	4	2										
3	1	6	8	2									
4	1	8	18	12	2								
5	1	10	32	38	16	2							
6	1	12	50	88	66	20	2						
7	1	14	72	170	192	102	24	2					
8	1	16	98	292	450	360	146	28	2				
9	1	18	128	462	912	1002	608	198	32	2			
10	1	20	162	688	1666	2364	1970	952	258	36	2		
11	1	22	200	978	2816	4942	5336	3530	1408	326	40	2	
12	1	24	242	1340	4482	9424	12642	10836	5890	1992	402	44	2

Table 2. The number $W(r,2,w)$ of placing w non-attacking wazirs on $r\times 2$ boards [2, A035607][4]. Row sums in [2, A001333].

$r\backslash w$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1	1	3	1										
2	1	6	8	2									
3	1	9	24	22	6	1							
4	1	12	49	84	61	18	2						
5	1	15	83	215	276	174	53	9	1				
6	1	18	126	442	840	880	504	158	28	2			
7	1	21	178	792	2023	3063	2763	1478	472	93	12	1	
8	1	24	239	1292	4176	8406	10692	8604	4374	1416	297	38	2

Table 3. The number $W(r,3,w)$ of placing w non-attacking wazirs on $r\times 3$ boards [2, A371967][4].

Its Taylor expansion with respect to y starts

$$\hat{W}_3(x,y) = \frac{1}{1-x} + \frac{3x}{(1-x)^2}y + \frac{x(1+5x+3x^2)}{(1-x)^3}y^2 + \frac{x^2(2+14x+8x^2+3x^3)}{(1-x)^4}y^3 + \frac{x^3(6+31x+31x^2+10x^3+3x^4)}{(1-x)^5}y^4 + \cdots . \tag{11}$$

The coefficients $[y^0]\hat{W}_c(x,y)$ and $[y^1]\hat{W}_c(x,y)$ are not interesting because they merely echo (2) and (3). The coefficient's $[y^j]$ denominators are powers of $1-x$, so the $W(r,c,w)$ are polynomials in r "down the columns" if c and w are kept constant. Each term $\propto x^l/(1-x)^k$ in the univariate generating function contributes $\propto \binom{r+k-l-1}{k-1}$ to the polynomial [7].

Remark 2. *The matrices $1-T$ have (i) a top-left entry $1-x$ where the $000\ldots0$ word is compatible with itself, (ii) at least one factor y in the left column where the $000\ldots0$ word is compatible with any other word with at least one 1, (iii) 1 on the diagonal where no word with at least one 1 is compatible with itself. Here is the*

$r\backslash w$	0	1	2	3	4	5	6	7	8
0	1								
1	1	4	3						
2	1	8	18	12	2				
3	1	12	49	84	61	18	2		
4	1	16	96	276	405	304	114	20	2
5	1	20	159	652	1502	1998	1537	678	170
6	1	24	238	1276	4072	8052	10010	7836	3846
7	1	28	333	2212	9091	24238	42864	50726	40235
8	1	32	444	3524	17791	60168	140050	227456	259289
9	1	36	571	5276	31660	130318	379247	793690	1205457
10	1	40	714	7532	52442	255052	895062	2310740	4439121
11	1	44	873	10356	82137	461646	1902326	5869438	13739384
12	1	48	1048	13812	123001	785312	3723486	13406168	37187238

TABLE 4. The number $W(r,4,w)$ of placing w non-attacking wazirs on $r\times 4$ boards. Columns for $r\geq 5$ not printed in full.

example for $c=4$ with $s=8$ nodes labeled 0000, 0001, 0010, 0101,... 1010 in the digraph:

$$1-T=\begin{pmatrix} 1-x & -x & -x & -x & -x & -x & -x & -x \\ -xy & 1 & -xy & -xy & 0 & -xy & 0 & -xy \\ -xy & -xy & 1 & -xy & -xy & -xy & -xy & 0 \\ -xy & -xy & -xy & 1 & 0 & -xy & -xy & -xy \\ -xy^2 & 0 & -xy^2 & 0 & 1 & -xy^2 & 0 & -xy^2 \\ -xy & -xy & -xy & -xy & -xy & 1 & 0 & 0 \\ -xy^2 & 0 & -xy^2 & -xy^2 & 0 & 0 & 1 & 0 \\ -xy^2 & -xy^2 & 0 & -xy^2 & -xy^2 & 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

The matrix inverse $(1-T)^{-1}$ puts the determinant in the denominator. By the quotient rule, the coefficients $[y^j]\hat{W}_c(x,y)$ are obtained by building the j-th partial derivatives with respect to y—essentially generating the jth power in the denominators—then setting $y=0$. Laplace Expansion of $1-T$ along the first column and each submatrix along their first column proves that in the limit $y\to 0$ the determinant has the $(1-x)^{1+j}$ format needed to generate the polynomials "down the columns."

$$W(r,3,2) = \frac{9}{2}r^2-\frac{13}{2}r+3, \quad r\geq 1. \tag{13}$$

$$W(r,3,3) = \frac{9}{2}r^3-\frac{39}{2}r^2+32r-20, \quad r\geq 2. \tag{14}$$

$$W(r,3,4) = \frac{27}{8}r^4-\frac{117}{4}r^3+\frac{829}{8}r^2-\frac{715}{4}r+126, \quad r\geq 3. \tag{15}$$

The generating function for the row sums is

$$\hat{W}_3(x,1)=\frac{(1+x)(1+2x-x^2)}{1-2x-6x^2+x^4}. \tag{16}$$

$i\backslash j$	0	1	2	3	4	5	6
0	1						
1	0	2	2				
2	0	0	1	0	-1		
3	0	0	0	0	-2	-2	
4	0	0	0	0	0	0	1

$i\backslash j$	0	1	2	3	4	5	6
0	1						
1	-1	-2	-1				
2	0	-2	-5	-2			
3	0	0	-1	0	4	2	
4	0	0	0	0	2	2	
5	0	0	0	0	0	0	-1

TABLE 5. The coefficients $\alpha_{4,i,j}$ (left) and $\beta_{4,i,j}$ (right) for $\hat{W}_4(x,y)$.

The writeup of the rational polynomials of x and y in the generating functions is lengthy for larger c. Concise notation tabulates the coefficients α and β in numerator and denominator:

Definition 3. *(polynomial coefficients of rational g.f.)*

$$\hat{W}_c(x,y) \equiv \frac{\sum_{i,j} \alpha_{c,i,j} x^i y^j}{\sum_{i,j} \beta_{c,i,j} x^i y^j}. \tag{17}$$

The generating function associated with boards 4 columns wide, Table 4 is

$$\hat{W}_4(x,y) \equiv p_4(x,y)/q_4(x,y); \tag{18}$$
$$p_4(x,y) = -x^2y^4 + 2xy^2 + x^4y^6 - 2x^3y^5 - 2x^3y^4 + x^2y^2 + 2xy + 1;$$
$$q_4(x,y) = -x^5y^6+2x^4y^5+2x^4y^4+2x^3y^5+4x^3y^4-x^3y^2-2x^2y^3-5x^2y^2-2x^2y-x-2xy-xy^2+1,$$

p_4 and q_4 rephrased in Table 5.

Its Taylor expansion with respect to y starts

$$\hat{W}_4(x,y) = \frac{1}{1-x} + \frac{4x}{(1-x)^2}y + \frac{x(3+9x+4x^2)}{(1-x)^3}y^2 + \frac{4x^2(3+9x+3x^2+x^3)}{(1-x)^4}y^3 + \frac{x^2(2+51x+120x^2+67x^3+12x^4+4x^5)}{(1-x)^5}y^4 + \cdots. \tag{19}$$

$$W(r,4,2) = 8r^2 - 9r + 4, \quad r \ge 1. \tag{20}$$
$$W(r,4,3) = \frac{32}{3}r^3 - 36r^2 + \frac{148}{3}r - 28, \quad r \ge 2. \tag{21}$$
$$W(r,4,4) = \frac{32}{3}r^4 - 72r^3 + \frac{1235}{6}r^2 - \frac{599}{2}r + 187, \quad r \ge 3. \tag{22}$$

The generating function for the row sums is

$$\hat{W}_4(x,1) = \frac{1+4x-4x^3+x^4}{1-4x-9x^2+5x^3+4x^4-x^5}. \tag{23}$$

The generating function associated with boards 5 columns wide, Table 6, is $\hat{W}_5(x,y)$ gathered in Table 7. Its Taylor expansion with respect to y starts

$$\hat{W}_5(x,y) = \frac{1}{1-x} + \frac{5x}{(1-x)^2}y + \frac{x(6+14x+5x^2)}{(1-x)^3}y^2 + \frac{x(1+34x+69x^2+16x^3+5x^4)}{(1-x)^4}y^3 + \frac{x^2(16+196x+282x^2+114x^3+12x^4+5x^5)}{(1-x)^5}y^4 + \cdots. \tag{24}$$

$r\backslash w$	0	1	2	3	4	5	6	7	8
0	1								
1	1	5	6	1					
2	1	10	32	38	16	2			
3	1	15	83	215	276	174	53	9	1
4	1	20	159	652	1502	1998	1537	678	170
5	1	25	260	1474	5024	10741	14650	12798	7157
6	1	30	386	2806	12792	38438	78052	108354	103274
7	1	35	537	4773	27381	107004	293409	573797	807161
8	1	40	713	7500	51991	251354	875407	2239218	4255370
9	1	45	914	11112	90447	522528	2217382	7060833	17101603
10	1	50	1140	15734	147199	990816	4972570	19034728	56415728
11	1	55	1391	21491	227322	1748883	10150982	45519984	160254659
12	1	60	1667	28508	336516	2914894	19231904	99049302	404967606

TABLE 6. The number $W(r,5,w)$ of placing w non-attacking wazirs on $r \times 5$ boards. Columns for $r \geq 4$ not printed in full.

$i\backslash j$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1														
1	0	3	5	1											
2	0	0	3	6	2										
3	0	0	0	1	-3	-16	-15	-4							
4	0	0	0	0	0	-4	-11	-7	-1	-1					
5	0	0	0	0	0	0	0	6	20	20	7				
6	0	0	0	0	0	0	0	0	0	-4	-9	-3	1		
7	0	0	0	0	0	0	0	0	0	0	0	1	-1	-3	
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

$i\backslash j$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1														
1	-1	-2	-1												
2	0	-3	-12	-14	-6	-1									
3	0	0	-3	-10	-4	9	7	1							
4	0	0	0	-1	3	26	43	26	7	1					
5	0	0	0	0	0	4	11	1	-17	-12	-2				
6	0	0	0	0	0	0	0	-6	-20	-22	-11	-4	-1		
7	0	0	0	0	0	0	0	0	0	4	9	5	1	1	
8	0	0	0	0	0	0	0	0	0	0	0	-1	1	3	1
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1

TABLE 7. The coefficients $\alpha_{5,i,j}$ (top) and $\beta_{5,i,j}$ (bottom) for $\hat{W}_5(x,y)$.

$$W(r,5,2) = \frac{25}{2}r^2 - \frac{23}{2}r + 5, \quad r \geq 1. \tag{25}$$

$$W(r,5,3) = \frac{125}{6}r^3 - \frac{115}{2}r^2 + \frac{206}{3}r - 36, \quad r \geq 2. \tag{26}$$

$$W(r,5,4) = \frac{625}{24}r^4 - \frac{574}{4}r^3 + \frac{8327}{24}r^2 - \frac{1765}{4}r + 249, \quad r \geq 3. \tag{27}$$

$r\backslash w$	0	1	2	3	4	5	6	7	8
0	1								
1	1	6	10	4					
2	1	12	50	88	66	20	2		
3	1	18	126	442	840	880	504	158	28
4	1	24	238	1276	4072	8052	10010	7836	3846
5	1	30	386	2806	12792	38438	78052	108354	103274
6	1	36	570	5248	31320	127960	368868	763144	1143638
7	1	42	790	8818	65272	339330	1280832	3581924	7514182
8	1	48	1046	13732	121560	769820	3612344	12842256	35093344
9	1	54	1338	20206	208392	1559038	8774380	38035756	129022058
10	1	60	1666	28456	335272	2896704	19049692	97720664	397650884
11	1	66	2030	38698	513000	5030426	37898664	224960724	1070686062
12	1	72	2430	51148	753672	8273476	70311824	474630304	2591238920

TABLE 8. The number $W(r,6,w)$ of placing w non-attacking wazirs on $r\times 6$ boards. Columns for $r\geq 3$ not printed in full.

The generating function for the row sums is

$$\hat{W}_5(x,1)=\frac{1+9x+11x^2-37x^3-24x^4+53x^5-15x^6-3x^7+x^8}{1-4x-36x^2+105x^4-15x^5-64x^6+20x^7+4x^8-x^9}. \tag{28}$$

The generating function associated with boards 6 columns wide, Table 8, is $\hat{W}_6(x,y)$ gathered in Table 9. Its Taylor expansion with respect to y starts

$$\hat{W}_6(x,y)=\frac{1}{1-x}+\frac{6x}{(1-x)^2}y+\frac{2x(5+10x+3x^2)}{(1-x)^3}y^2 + \frac{2x(2+36x+57x^2+3x^4+10x^3)}{(1-x)^4}y^3+\frac{2x^2(33+255x+266x^2+86x^3+5x^4+3x^5)}{(1-x)^5}y^4+\cdots. \tag{29}$$

The generating function for $r\times 7$ boards is $\hat{W}_7(x,y)$ gathered in Tables 10–13. Its Taylor expansion with respect to y starts

$$\hat{W}_7(x,y)=\frac{1}{1-x}+\frac{7x}{(1-x)^2}y+\frac{x(15+27x+7x^2)}{(1-x)^3}y^2 + \frac{x(10+130x+172x^2+24x^3+7x^4)}{(1-x)^4}y^3+\frac{x(1+187x+1073x^2+886x^3+241x^4+6x^5+7x^6)}{(1-x)^5}y^4+\cdots. \tag{30}$$

$i\backslash j$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	1																			
1	0	3	7	3																
2	0	0	3	9	4	-5	-2													
3	0	0	0	1	-3	-32	-58	-38	-7											
4	0	0	0	0	0	-5	-23	-23	11	22	8									
5	0	0	0	0	0	0	0	10	60	121	110	53	14							
6	0	0	0	0	0	0	0	0	0	-10	-55	-98	-71	-19	1					
7	0	0	0	0	0	0	0	0	0	0	0	5	17	6	-22	-19	-2			
8	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	3	17	17	3		
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	-2	1	
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1

$i\backslash j$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	1																			
1	-1	-3	-3	-1																
2	0	-3	-16	-26	-14	-3														
3	0	0	-3	-14	-9	31	48	22	3											
4	0	0	0	-1	3	47	125	131	60	10										
5	0	0	0	0	0	5	23	13	-73	-133	-94	-35	-6							
6	0	0	0	0	0	0	0	-10	-60	-131	-135	-85	-45	-17	-3					
7	0	0	0	0	0	0	0	0	0	10	55	113	112	65	25	5				
8	0	0	0	0	0	0	0	0	0	0	0	-5	-17	-11	21	30	11	1		
9	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-3	-17	-23	-9	-1	
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	2	-1	-1
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

TABLE 9. The coefficients $\alpha_{6,i,j}$ (left) and $\beta_{6,i,j}$ (right) for $\hat{W}_6(x,y)$.

$i\backslash j$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1															
1	0	4	11	9	1											
2	0	0	6	24	28	9	1									
3	0	0	0	4	6	-73	-267	-358	-225	-67	-8					
4	0	0	0	0	1	-16	-158	-460	-573	-307	-62	-18	-12	-2		
5	0	0	0	0	0	0	-9	-30	229	1520	3688	4807	3764	1829	533	83
6	0	0	0	0	0	0	0	0	36	330	1052	1183	-703	-3095	-3094	-1296
7	0	0	0	0	0	0	0	0	0	0	-84	-960	-4463	-11297	-17787	-19399
8	0	0	0	0	0	0	0	0	0	0	0	0	126	1512	7428	19742
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-126	-1428

$i\backslash j$	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
5	5														
6	-124	51	6	-1											
7	-16215	-10700	-5109	-1537	-252	-18									
8	31632	32540	22694	11234	3935	917	145	18	1						
9	-6334	-13647	-12719	3584	21826	24858	15297	5703	1273	153	7				
10	84	780	2206	-746	-17672	-43935	-55310	-41091	-18837	-5458	-1047	-136	-8		
11	0	0	-36	-180	648	6788	20237	29004	18986	-252	-8959	-6130	-1936	-317	-26
12	0	0	0	0	9	-40	-838	-3242	-3736	5385	21112	27360	19106	7933	2039
13	0	0	0	0	0	0	-1	34	201	-189	-3615	-10332	-13694	-9039	-2229
14	0	0	0	0	0	0	0	0	0	-6	32	447	1337	1047	-2011
15	0	0	0	0	0	0	0	0	0	0	0	0	-15	-34	243

TABLE 10. The coefficients $\alpha_{7,i,j}$ for $\hat{W}_7(x,y)$, $j \le 30$.

$i\backslash j$	31	32	33	34	35	36	37	38	39	40	41	42	43	
12	325	27												
13	599	553	162	21										
14	-5195	-5051	-2773	-997	-238	-27								
15	1207	2211	2074	1120	421	140	33	3						
16	-20	-90	-94	150	439	451	270	100	19	2				
17	0	0	0	-15	-66	-117	-121	-106	-70	-21	-2			
18	0	0	0	0	0	0	-6	-16	-17	-13	-12	-4		
19	0	0	0	0	0	0	0	0	0	-1	2	5	3	
20	0	0	0	0	0	0	0	0	0	0	0	0	0	1

TABLE 11. The coefficients $\alpha_{7,i,j}$ for $\hat{W}_7(x,y)$, $j \geq 31$.

$i\backslash j$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1															
1	-1	-3	-4	-1												
2	0	-4	-26	-62	-66	-35	-9	-1								
3	0	0	-6	-41	-83	-15	128	149	65	12	1					
4	0	0	0	-4	-12	106	643	1426	1658	1148	515	151	26	2		
5	0	0	0	0	-1	16	197	682	804	-514	-2460	-2963	-1879	-690	-148	-18
6	0	0	0	0	0	0	9	30	-325	-2448	-7160	-11599	-11747	-8178	-4392	-1953
7	0	0	0	0	0	0	0	0	-36	-330	-968	-61	6072	16237	22168	19301
8	0	0	0	0	0	0	0	0	0	0	84	960	4547	11703	18509	20696
9	0	0	0	0	0	0	0	0	0	0	0	0	-126	-1512	-7722	-22206
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	126	1428

$i\backslash j$	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
5	-1														
6	-653	-131	-11												
7	12033	5751	2069	508	74	5									
8	20468	19966	16272	9433	3682	936	140	9							
9	-40934	-54086	-57528	-52223	-38655	-21566	-8639	-2413	-447	-48	-2				
10	6670	16275	21118	10397	-9680	-22234	-21700	-14314	-6786	-2192	-433	-45	-2		
11	-84	-780	-2410	-351	16939	51595	81330	82626	59650	32544	13698	4307	940	129	9
12	0	0	36	180	-582	-6835	-22911	-39563	-37397	-15591	4325	9392	5716	1976	420
13	0	0	0	0	-9	40	829	3428	4919	-3901	-24961	-41756	-39300	-23991	-10218
14	0	0	0	0	0	0	1	-34	-201	145	3671	11870	18638	15864	6516
15	0	0	0	0	0	0	0	0	0	6	-32	-447	-1422	-1357	2104
16	0	0	0	0	0	0	0	0	0	0	0	0	15	34	-243

TABLE 12. The coefficients $\beta_{7,i,j}$ for $\hat{W}_7(x,y)$, $j \le 30$.

$i\backslash j$	31	32	33	34	35	36	37	38	39	40	41	42	43	
12	52	3												
13	-3201	-754	-125	-11										
14	-60	-1379	-677	-153	-14									
15	7151	8821	6328	3046	1060	262	41	3						
16	-1287	-2601	-2763	-1656	-605	-187	-64	-12						
17	20	90	94	-185	-593	-716	-528	-280	-104	-23	-2			
18	0	0	0	15	66	117	117	102	84	39	6			
19	0	0	0	0	0	0	6	16	17	14	18	13	3	
20	0	0	0	0	0	0	0	0	0	1	-2	-5	-3	
21	0	0	0	0	0	0	0	0	0	0	0	0	0	-1

TABLE 13. The coefficients $\beta_{7,i,j}$ for $\hat{W}_7(x,y)$, $j \geq 31$.

The symmetry (1) leads to redundancy in the $W(r, c, w)$ tables.

Example 2. *Row $r = 3$ in Table 2 equals row $r = 2$ in Table 3. Row $r = 4$ in Table 2 equals row $r = 2$ in Table 4. Row $r = 4$ in Table 6 equals row $r = 5$ in Table 4. Row $r = 5$ in Table 8 equals row $r = 6$ in Table 6.*

In the data cube of the $W(r, c, w)$ one can also construct other slices of 2D datasets:

- On square boards, $W(r, r, w)$ is composed of the first row if Table 1, the second row if Table 2, the third row if Table 3, the fourth row if Table 4, the fifth row if Table 6, and so on [2, A232833].
- If w is constant and r, c are variable the symmetric table $W(r, c, 2)$ of (4) emerges:

$r\backslash c$	1	2	3	4	5	6	7	8
1	0	0	1	3	6	10	15	21
2	0	2	8	18	32	50	72	98
3	1	8	24	49	83	126	178	239
4	3	18	49	96	159	238	333	444
5	6	32	83	159	260	386	537	713
6	10	50	126	238	386	570	790	1046
7	15	72	178	333	537	790	1092	1443
8	21	98	239	444	713	1046	1443	1904

or the symmetric table $W(r, c, 3)$

$r\backslash c$	1	2	3	4	5	6	7	8
1	0	0	0	0	1	4	10	20
2	0	0	2	12	38	88	170	292
3	0	2	22	84	215	442	792	1292
4	0	12	84	276	652	1276	2212	3524
5	1	38	215	652	1474	2806	4773	7500
6	4	88	442	1276	2806	5248	8818	13732
7	10	170	792	2212	4773	8818	14690	22732
8	20	292	1292	3524	7500	13732	22732	35012

Three of the $W(r, c, 3)$ columns are registered in the Online Encyclopedia Of Integer Sequences [2, A000292,A035597,A172229].

4. Summary

We have fully qualified the bivariate generating function (6) counting configurations $W(r, c, w)$ with w non-attacking wazirs on $r \times c$ boards for widths $c \le 7$.

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