A TREATY OF SYMMETRIC FUNCTION

Using Sums of Power for Arbitrary Arithmetic Progression to find an approximation for Sum of Power for non-integers R-th power and Expressing Riemann's Zeta Function into Symmetric Sums of Power Form.

Paper Part IV

Mohd Shukri Abd Shukor

In remembrance of my beloved father who passed away on the 23rd of June 2009 and my special thanks to my brother Mohd Yunus Abd Shukor for introducing me to Fermat's Last Theorem when I was a teenager. Although I didn't get the proof for this theorem, it enhanced my understanding towards developing the generalized equations for Symmetric Function for Sums of Powers, Fermat Last Theorem, Riemann Zeta Function and Sum of power for non-integer R-th power. Lastly, to my sister Nazirah Abd Shukor, thanks for all the supports and patience.

Mohd Shukri Abd Shukor

Snoc International Sdn Bhd
148 Jalan 2, Desa Mahkota,
43000, Bangi, Selangor,
Malaysia
Tel: 0060192084759
Fax:0060389263950
mshukri@mykocom.com

Abstract. A new approach in deriving Sum of Power series using reverse look up method, a ,method where a mathematical formulation is constructed from set of data. Faulhaber [1] derived a general equation for Power sums and calculated the terms up to p=17 (i.e. $\sum_{i=1}^{n} x_i^p$). However, these formulae only work for integers from $x_1 = 1$ to $x_n = n$. A depth study on Power series revealed a systematic general equation which applicable for all numbers with a condition that the series should be in an arithmetic progression without the power p (i.e. $\sum_{i=1}^{n} x_i$). The general formulation is given as follows

$$\sum_{i=1}^{n} x_i^p = \sum_{j=0}^{m} \left[\phi_j t^{2j} \frac{\left[\sum_{i=1}^{n} x_i \right]^{p-2j}}{n^{p-(2j+1)}} \right] \quad [1]$$

Where, $t = x_{i+1} - x_i$, ϕ_j is a coefficient and $\phi_0 = 1$

As can be read in the paper part II, the generalized equation works well with real p. The approximation value of $\sum_{i=1}^{n} x_i^p$ will converge to the real value when $m \to \infty$. Therefore it offers new way of expressing Riemann's Zeta Function [2] in form of symmetric function.

2010 AMS Subject Classification: 11B68, 11B30, 11S40, 32A05, 30F20

Keywords and phrases: Sum of power, Bernoulli Numbers, Arithmetic Combinatorics, Symmetric Function and Generalizations, Riemann Zeta Function, Euler

1 Introduction.

The detail of the coefficient for equation [1] is given as follows:

$$\phi_m = \frac{1}{2^{2m}}\frac{1}{(2m+1)}\binom{p}{2m}P_m$$

$$= \frac{1}{2^{2m}}\frac{1}{(2m+1)}\binom{p}{2m}\left[n^{2m} - 2\sum_{t=1}^{m}\left[(2t+1)(2^{2t-1}-1)\binom{m}{t}B_t n^{2(m-t)}\frac{\prod_{j=0}^{t-1}(2m-2j+1)}{\prod_{j=0}^{t}(2t-2j+1)}\right]\right]$$

By expanding equation [1] for non-integer *p*, yields:

$$\sum_{i=1}^{n} x_i^p = \frac{\left[\sum_{i=1}^{n} x_i\right]^p}{n^{p-1}} + \frac{p(p-1)(n^2-1)s^2\left[\sum_{i=1}^{n} x_i\right]^{p-2}}{24n^{p-3}} + \frac{p(p-1)(p-2)(p-3)(3n^2-7)(n^2-1)s^4\left[\sum_{i=1}^{n} x_i\right]^{p-4}}{5760n^{p-5}} + \ldots[2]$$

In this paper "*s*" is replaced with *t* in order not to be confused with *s* as in the Riemann's Zeta Function $\varsigma(s)$.

Rewriting the equation [2], yields:

$$\sum_{i=1}^{n} x_i^p = n\cdot\left[\frac{\left[\sum_{i=1}^{n} x_i\right]^p}{n^p}\right]\left[1+\phi_1 t^2\left(\frac{n}{\sum_{i=1}^{n} x_i}\right)^2 + \phi_2 t^4\left(\frac{n}{\sum_{i=1}^{n} x_i}\right)^4 + \phi_3 t^6\left(\frac{n}{\sum_{i=1}^{n} x_i}\right)^6 + \ldots\right] \qquad [3]$$

Let *p* as negative (i.e. $p = -q$) and rewriting equation [3] yields:

$$\sum_{i=1}^{n} x_i^{-q} = n\cdot\left[\frac{\left[\sum_{i=1}^{n} x_i\right]^{-q}}{n^{-q}}\right]\left[1+\phi_1 t^2\left(\frac{n}{\sum_{i=1}^{n} x_i}\right)^2 + \phi_2 t^4\left(\frac{n}{\sum_{i=1}^{n} x_i}\right)^4 + \phi_3 t^6\left(\frac{n}{\sum_{i=1}^{n} x_i}\right)^6 + \ldots\right]$$

$$= n\cdot\left[\frac{n^q}{\left[\sum_{i=1}^{n} x_i\right]^q}\right]\left[1+\phi_1 t^2\left(\frac{n}{\sum_{i=1}^{n} x_i}\right)^2 + \phi_2 t^4\left(\frac{n}{\sum_{i=1}^{n} x_i}\right)^4 + \phi_3 t^6\left(\frac{n}{\sum_{i=1}^{n} x_i}\right)^6 + \ldots\right] \qquad [4]$$

Or $\sum_{i=1}^{n} x_i^{-q} = n \cdot \left[\frac{\left[\sum_{i=1}^{n} x_i \right]^{-q}}{n^{-q}} \right] f\left(n, \left[\sum_{i=1}^{n} x_i \right], t \right)$

Expanding equation [3] with some of the coefficients ϕ_j yields:

$$\sum_{i=1}^{n} x_i^{-q} = n \cdot \left[\frac{n^q}{\left[\sum_{i=1}^{n} x_i \right]^q} \right] \left[1 + \frac{q(q+1)(n^2-1)t^2}{24} \left(\frac{n}{\sum_{i=1}^{n} x_i} \right)^2 + \frac{q(q+1)(q+2)(q+3)(3n^2-7)(n^2-1)t^4}{5760} \left(\frac{n}{\sum_{i=1}^{n} x_i} \right)^4 + \ldots \right] \quad [5]$$

Now, let $n = 2$ and $p = -1$ or $q = 1$

$$\sum_{i=1}^{2} x_i^{-1} = 2 \cdot \left[\frac{2}{\left[\sum_{i=1}^{2} x_i \right]} \right] \left[1 + \frac{(1)(1+1)(4-1)t^2}{24} \left(\frac{2}{\left[\sum_{i=1}^{2} x_i \right]} \right)^2 + \frac{(1)(1+1)(1+2)(1+3)(12-7)(4-1)t^4}{5760} \left(\frac{2}{\left[\sum_{i=1}^{2} x_i \right]} \right)^4 + \ldots \right]$$

$$= \left[\frac{4}{3} \right] \left[1 + \frac{1}{4} \left(\frac{2}{3} \right)^2 + \frac{1}{16} \left(\frac{2}{3} \right)^4 + \ldots \right]$$

$$= 1.497942387 \quad [6]$$

Yet,

$$\sum_{i=1}^{2} x_i^{-1} = \frac{1}{1} + \frac{1}{2} = 1.5$$

Therefore, more coefficients ϕ_j are needed for a better approximation. Considering more coefficients ϕ_j, yields:

$$\sum_{i=1}^{2} x_i^{-1} = \left[\frac{4}{3} \right] \left[1 + \frac{1}{4} \left(\frac{2}{3} \right)^2 + \frac{1}{16} \left(\frac{2}{3} \right)^4 + \frac{1}{64} \left(\frac{2}{3} \right)^6 + \frac{1}{256} \left(\frac{2}{3} \right)^8 + \frac{1}{1024} \left(\frac{2}{3} \right)^{10} + \frac{1}{4096} \left(\frac{2}{3} \right)^{12} + \frac{1}{16384} \left(\frac{2}{3} \right)^{14} \ldots \right]$$

$$= \left[\frac{4}{3} \right] \left[1 + \left(\frac{1}{3} \right)^2 + \left(\frac{1}{3} \right)^4 + \left(\frac{1}{3} \right)^6 + \left(\frac{1}{3} \right)^8 + \left(\frac{1}{3} \right)^{10} + \left(\frac{1}{3} \right)^{12} + \left(\frac{1}{3} \right)^{14} \ldots \right]$$

$$= \left[\frac{4}{3} \right] \left[\sum_{j=0}^{\infty} \left(\frac{1}{3} \right)^{2j} \right]$$

$$= \left[\frac{4}{3} \right] \left[\frac{9}{8} \right] = 1.5 \quad [7]$$

Now, let $n = 3$ and $p = -1$ or $q = 1$

$$\sum_{i=1}^{2} x_i^{-1} = 3 \cdot \left[\frac{3}{\left[\sum_{i=1}^{3} x_i\right]} \right] \left[1 + \frac{(1)(1+1)(9-1)t^2}{24} \left(\frac{3}{\left[\sum_{i=1}^{3} x_i\right]} \right)^2 + \frac{(1)(1+1)(1+2)(1+3)(27-7)(9-1)t^4}{5760} \left(\frac{3}{\left[\sum_{i=1}^{3} x_i\right]} \right)^4 \right]$$

$$= \left[\frac{9}{6}\right]\left[1 + \frac{2}{3}\left(\frac{1}{2}\right)^2 + \frac{2}{3}\left(\frac{1}{2}\right)^4\right]$$

$$= 1.75390625 \qquad [8]$$

Yet,

$$\sum_{i=1}^{2} x_i^{-1} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} = 1.8333333333$$

Therefore, more coefficients ϕ_j are needed for a better approximation. Considering more coefficients ϕ_j, yields:

$$\sum_{i=1}^{2} x_i^{-1} = \left[\frac{9}{6}\right]\left[1 + \frac{2}{3}\left(\frac{1}{2}\right)^2 + \frac{2}{3}\left(\frac{1}{2}\right)^4 + \frac{2}{3}\left(\frac{1}{2}\right)^6 + \frac{2}{3}\left(\frac{1}{2}\right)^8 + \frac{2}{3}\left(\frac{1}{2}\right)^{10} + \frac{2}{3}\left(\frac{1}{2}\right)^{12} + \frac{2}{3}\left(\frac{1}{2}\right)^{14} \ldots\right]$$

$$= \left[\frac{4}{3}\right]\left[1 + \frac{2}{3}\sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^{2j}\right]$$

$$= \left[\frac{9}{6}\right]\left[\frac{11}{9}\right] = \frac{11}{6} = 1.8333333333 \qquad [9]$$

Now, let $n = 100$ and $p = -1$ or $q = 1$

Using sum of arithmetic progression yields:

Let,

$$\left[\sum_{i=1}^{100} x_i\right] = \left[\sum_{i=1}^{100} i\right] = \frac{100(100+1)}{2} = 5050$$

$$\sum_{i=1}^{100} x_i^{-1} = \left[\frac{200}{101}\right]\left[1 + \frac{(1)(1+1)(100^2-1)t^2}{24}\left(\frac{2}{101}\right)^2 + \frac{(1)(1+1)(1+2)(1+3)(3(100)^2-7)(100^2-1)t^4}{5760}\left(\frac{2}{101}\right)^4 + \ldots\right]$$

$$\sum_{i=1}^{100} x_i^{-1} = \left[\frac{200}{101}\right]\left[1 + \frac{3333}{4}\left(\frac{2}{101}\right)^2 + \frac{(29993)(3333)}{80}\left(\frac{2}{101}\right)^4 + \cdots\right]$$

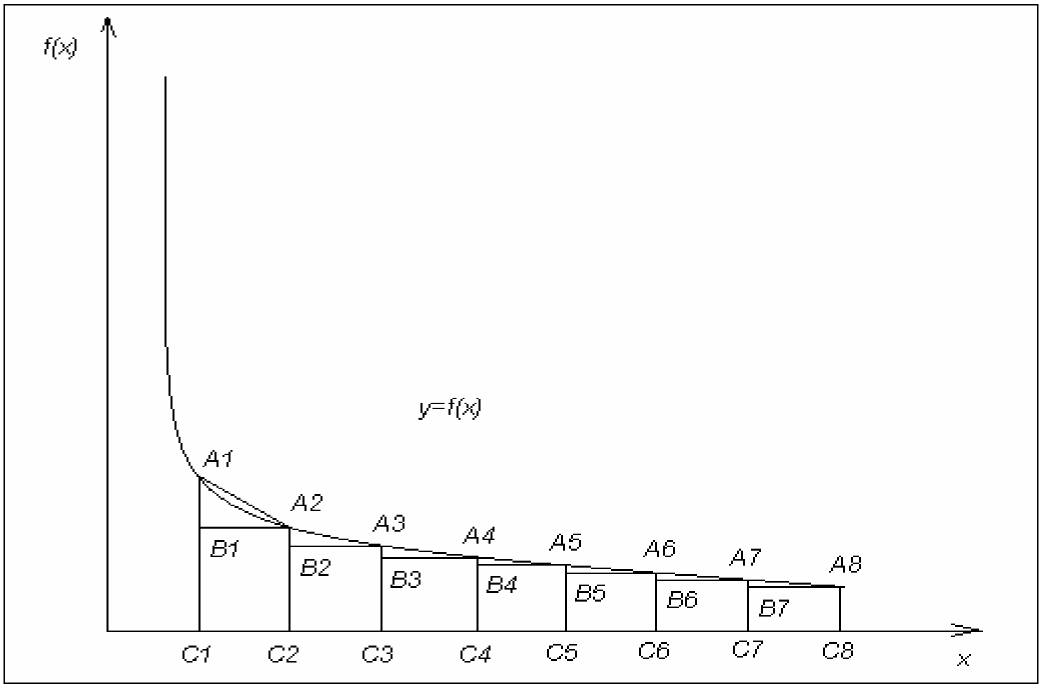
$$= \left[\frac{200}{101}\right]\left[1 + \frac{3333}{4}\left(\frac{2}{101}\right)^2 + \frac{(29993)(3333)}{80}\left(\frac{2}{101}\right)^4 + \cdots\right]$$

$$= (1.98019801980198)\left[1 + \frac{3333}{4}\left(\frac{2}{101}\right)^2 + \frac{(29993)(3333)}{80}\left(\frac{2}{101}\right)^4 + \cdots\right] \qquad [10]$$

The value of $\left[1 + \frac{3333}{4}\left(\frac{2}{101}\right)^2 + \frac{(29993)(3333)}{80}\left(\frac{2}{101}\right)^4 + \cdots\right]$ converges to a certain value as the summation terms approaches infinity.

However, when n is getting larger, the term of $f\left(\phi_j, t, \frac{n}{\sum_{i=1}^{n} x_i}\right) = \left[1+\phi_1 t^2\left(\frac{n}{\sum_{i=1}^{n} x_i}\right)^2 + \phi_2 t^4\left(\frac{n}{\sum_{i=1}^{n} x_i}\right)^4 + \phi_3 t^6\left(\frac{n}{\sum_{i=1}^{n} x_i}\right)^6 + \ldots\right]$ is getting difficult to calculate. Therefore, an approximation method should be developed. The method is given as follows:

Figure 1.0 Illustration of function $f(x)$.



Assuming total area under A_1A_2 is the area of triangle $A_1A_2B_1$ plus area of rectangle $B_1A_2C_2C_1$ is equal to a trapezium area $A_1A_2C_2C_1$.

Trapezium area of $A_1A_2C_2C_1 = \int_{C_1=1}^{C_2} f(x)dx$

Let trapezium area $A_1A_2C_2C_1 = A_{T1}$

$$A_{T1} = \int_{C_1=1}^{C_2} f(x)dx$$

$$= \frac{1}{2}(A_1 + A_2) = \int_{C_1=1}^{C_2} f(x)dx \qquad [11]$$

Also:

Trapezium area under $A_2A_3C_3C_2$ is equal to the area A_{T2}

$$A_{T2} = \frac{1}{2}(A_2 + A_3) = \int_{C_2}^{C_3} f(x)dx \qquad [12]$$

Repeating the same procedures yields:

$$A_{T3} = \frac{1}{2}(A_3 + A_4) = \int_{C_3}^{C_4} f(x)dx \quad [13]$$

Therefore, the total area under the curve from A_1 to A_n is given as follows:

$$\begin{aligned} A_{Tn} = A_{T1} + A_{T2} + \cdots A_{Tn} &= \int_{C_1=1}^{C_2} f(x)dx + \int_{C_2}^{C_3} f(x)dx + \cdots \int_{C_{n-1}}^{C_n} f(x)dx = \int_1^n f(x)dx \\ &= \frac{1}{2}(A_1 + A_2) + \frac{1}{2}(A_2 + A_3) + \cdots + \frac{1}{2}(A_{n-1} + A_n) = \int_1^n f(x)dx \\ &= \frac{1}{2}\left[A_1 + (2A_2 + 2A_3 + \cdots + 2A_{n-1}) + A_n\right] = \int_1^n f(x)dx \\ &= \left[A_1 + (2A_2 + 2A_3 + \cdots + 2A_{n-1}) + A_n\right] = 2\int_1^n f(x)dx \end{aligned} \quad [14]$$

Adding both sides with A_1 and A_n , yields:

$$A_1 + \left[A_1 + (2A_2 + 2A_3 + \cdots + 2A_{n-1}) + A_n\right] = 2\int_1^n f(x)dx + A_1 + A_n$$

$$2\left[(A_1 + A_2 + A_3 + \cdots + A_{n-1} + A_n\right] = 2\int_1^n f(x)dx + A_1 + A_n$$

$$\left[A_1 + A_2 + A_3 + \cdots + A_{n-1} + A_n\right] = \int_1^n f(x)dx + \frac{A_1}{2} + \frac{A_n}{2} \quad [15]$$

Further simplifying equation [16] yields:

$$\sum_1^n A_n = \int_1^n f(x)dx + \frac{A_1}{2} + \frac{A_n}{2} \quad [16]$$

Since A_1=1, thus:

$$\sum_1^n A_n = \int_1^n f(x)dx + \frac{1}{2} + \frac{A_n}{2} \quad [17]$$

For harmonic series, where s=1, yields:

$$\begin{aligned} \sum_{x=1}^n \frac{1}{x} &= \int_1^n \frac{1}{x}dx + \frac{1}{2} + \frac{A_n}{n} \\ &= \left[\ln(x)\right]_1^n + \frac{1}{2} = \ln(n) + \frac{1}{2} + \frac{A_n}{2} \end{aligned} \quad [18]$$

A better approximation could be achieved by adding the first p terms in which the radius of curvature is higher and contributing higher error due to the assumption that the area under the curve is a perfect trapezium. However, as $n \to \infty$, the radius of curvature of the curve is getting smaller and making the line a straight line. Thus giving a better trapezium shape, the corrective method is done as follows:

Lets consider the first summation of *p* as follows

$$\sum_{z=1}^{p} \frac{1}{x} = \alpha \qquad [19]$$

The remaining series is given as follows:

$$\sum_{x=p+1}^{n} \frac{1}{x} = \int_{p+1}^{n} \frac{1}{x} dx + \frac{A_{p+1}}{2} + \frac{A_n}{2} \qquad [20]$$

Therefore, the total summation can be calculated as follows:

$$\begin{aligned} \sum_{x=1}^{n} \frac{1}{x} &= \sum_{x=1}^{p} \frac{1}{x} + \int_{p+1}^{n} \frac{1}{x} dx + \frac{A_{p+1}}{2} + \frac{A_n}{2} \\ &= \alpha + [\ln(x)]_{p+1}^{n} + \frac{A_{p+1}}{2} + \frac{A_n}{2} \\ &= \alpha + \ln(n) - \ln(p+1) + \frac{A_{p+1}}{2} + \frac{A_n}{2} \\ &= \ln(n) + \left(\alpha - \ln(p+1) + \frac{A_{p+1}}{2} \right) + \frac{A_n}{2} \qquad [21] \end{aligned}$$

Or in the form of Euler's constant,

$$\gamma = \alpha - \ln(p+1) + \frac{A_{p+1}}{2} + \frac{A_n}{2} \qquad [22]$$

As $p \to \infty$, $\gamma \to$ Euler's Gamma Constant.

For example, consider $n = 10000$ and $p = 100$, thus

$$\sum_{x=1}^{p} \frac{1}{x} = \sum_{x=1}^{100} \frac{1}{x} = \alpha = 5.187377517639621 \qquad [23]$$

$$\gamma = 5.187377517639621 - \ln(101) + \frac{1}{2(101)} + \frac{1}{2(10000)} = 0.5772574958478659 \qquad [24]$$

It is found that the calculation done by the method formulized in this paper giving better approximation for smaller *n* and *p* than Euler-Mascheroni's method. The comparison of these two methods can be seen as follows:

Let the derived harmonic approximation equation as follows:

$$\sum_{x=1}^{n} \frac{1}{x} = \sum_{x=1}^{p} \frac{1}{x} + \ln(n) - \ln(p+1) + \frac{A_{p+1}}{2} + \frac{1}{2n} \qquad [25]$$

and Euler-Mascheroni's method as follows:

$$\sum_{x=1}^{n} \frac{1}{x} = \ln(n) + 0.5772156649015328606065120900824024310421593359 3992 \qquad [26]$$

Where 0.57721566490153286060651209008240243104215933593992 is the Euler's Gamma Constant.

Now let n=50

$$\sum_{x=1}^{50}\frac{1}{x}=\ln(50)+0.57721566490153286060651209008240243104215933593992$$

$$=4.489238670329679 \text{ (16 Decimal places)} \qquad [27]$$

Let p=2

$$\sum_{x=1}^{50}\frac{1}{x}=\sum_{1}^{2}\frac{1}{x}+\ln(50)-\ln(3)+\frac{1}{2(3)}+\frac{1}{2(50)}$$

$$=4.490077383426703 \text{ (16 Decimal places)} \qquad [28]$$

Direct calculation value of $\sum_{x=1}^{50}\frac{1}{x}$ calculated using MathCAD is given as follows:

$$\sum_{x=1}^{50}\frac{1}{x}=4.499205338329423 \qquad [29]$$

Let the error in calculation in Euler's method be E_e and it is given by direct calculation value minus calculated value using Euler's method, thus:

$$E_e = 4.499205338329423 - 4.489238670329679 = 0.00996666799974388 \qquad [30]$$

Let the error in calculation using derived method be E_a.

$$E_a = 4.499205338329423 - 4.490077383426703 = 0.00912795490271989 \qquad [31]$$

Error ratio between Euler's and derived method R is given as follows:

$$R=\frac{E_a}{E_e}=\frac{0.00912795490271989}{0.00996666799974388}=0.9158481955006885 \qquad [32]$$

As $p \to n$ the error $E_a \to 0$. Therefore, to increase the accuracy the value of p needs to be increased.

For s=2,

$$\sum_{x=1}^{n}\frac{1}{x^2}=\sum_{x=1}^{p}\frac{1}{x^2}+\int_{p+1}^{n}\frac{1}{x^2}dx+\frac{A_{p+1}}{2}+\frac{1}{2n}$$

$$=\sum_{x=1}^{p}\frac{1}{x^2}+\left[-\frac{1}{x}\right]_{p+1}^{n}+\frac{A_{p+1}}{2}+\frac{1}{2n}$$

$$=\sum_{x=1}^{p}\frac{1}{x^2}-\frac{1}{n}+\frac{1}{p+1}+\frac{1}{2(p+1)^2}+\frac{1}{2n} \qquad [33]$$

Lets p=20, yields:

$$\sum_{x=1}^{20}\frac{1}{x^2}=1.59616324 \qquad [34]$$

$$\sum_{x=1}^{n}\frac{1}{x^2}=1.59616324-\frac{1}{n}+\frac{1}{21}+\frac{1}{2(21)^2}+\frac{1}{2n^2}$$

$$\sum_{x=1}^{n}\frac{1}{x^2}=1.64491607-\frac{1}{n}+\frac{1}{2n^2} \qquad [35]$$

Therefore as $p,n\to\infty$, $\sum_{x=1}^{n}\frac{1}{x^2}\to\frac{\pi^2}{6}$

Now lets consider equation [10] again with $p = 20$.

$$f\left(\phi_j,1,\frac{100}{\sum_{i=1}^{100} i}\right)=\left[1+\frac{3333}{1}\left(\frac{2}{101}\right)^2+\frac{(29993)(3333)}{5}\left(\frac{2}{101}\right)^4+\cdots\right]$$
$$=\left[\frac{n+1}{2n}\right]\left[\sum_{i=1}^{p}\frac{1}{i}+\ln(n)-\ln(p+1)+\frac{A_{p+1}}{2}+\frac{1}{2n}\right]$$
$$=\left[\frac{101}{200}\right]\left[\sum_{i=1}^{20}\frac{1}{i}+\ln(100)-\ln(21)+\frac{1}{2(21)}+\frac{1}{2(100)}\right] \qquad [36]$$

Since,

$$\sum_{i=1}^{20}\frac{1}{i}=3.597739657143682$$

$$f\left(\phi_j,1,\frac{100}{\sum_{i=1}^{100} i}\right)\approx\left[\frac{101}{200}\right]\left[3.597739657143682+\ln(100)-\ln(21)+\frac{1}{2(21)}+\frac{1}{2(100)}\right]$$
$$\approx\left[\frac{101}{200}\right](5.187196929217875)$$
$$\approx 2.619534449255027 \qquad [37]$$

Therefore,

$$f\left(\phi_j,1,\frac{100}{\sum_{i=1}^{100} i}\right)=\left[1+\frac{3333}{4}\left(\frac{2}{101}\right)^2+\frac{(29993)(3333)}{80}\left(\frac{2}{101}\right)^4+\cdots\right]\approx 2.619534449255027 \qquad [38]$$

Direct calculation using MathCad, yields:

$$f\left(\phi_j,1,\frac{100}{\sum_{i=1}^{100} i}\right)=\left[\frac{101}{200}\right]\left[\sum_{i=1}^{100}\frac{1}{i}\right]\approx 2.6196256464080085 \qquad [39]$$

The error in [38] is given as follows:

$$\begin{aligned}\text{Error} &= \text{Direct Calculation Value} - \text{Approximation Method's Value}\\ &= 2.6196256464080085 - 2.619534449255027 = 0.00009119715298134\end{aligned} \quad [40]$$

2 The Riemann's Zeta Function for real and complex number *s* using Symmetric Function and Approximation Method.

Lets consider an integer progression as follows,

For integers, $\sum_{i=1}^{n} x_i = \frac{n(n+1)}{2}$ and t=1 [41]

and

$$\sum_{i=1}^{n} x_i^p = n\cdot\left[\frac{n+1}{2}\right]^p \left[1+\frac{4\phi_1}{(n+1)^2}+\frac{16\phi_2}{(n+1)^4}+\frac{64\phi_3}{(n+1)^6}+\frac{256\phi_4}{(n+1)^8}+\dots\right] \quad [42]$$

For Riemman's zeta function, p=-s

$$\begin{aligned}\varsigma(s) &= \sum_{n=1}^{n}\frac{1}{n^s} = \sum_{i=1}^{n} n^{-s} = n\cdot\left[\frac{n+1}{2}\right]^{-s}\left[1+\frac{4\phi_1}{(n+1)^2}+\frac{16\phi_2}{(n+1)^4}+\frac{64\phi_3}{(n+1)^6}+\frac{256\phi_4}{(n+1)^8}+\dots\right]\\ &= \frac{2^s n}{(n+1)^s}\cdot\left[1+\frac{4\phi_1}{(n+1)^2}+\frac{16\phi_2}{(n+1)^4}+\frac{64\phi_3}{(n+1)^6}+\frac{256\phi_4}{(n+1)^8}+\dots\right]\end{aligned} \quad [43]$$

Where:

$$\phi_1 = \frac{(n^2-1)}{24}s(s+1) \quad [44]$$

$$\phi_2 = \frac{(3n^2-7)(n^2-1)}{5760}s(s+1)(s+2)(s+3) \quad [45]$$

$$\phi_3 = \frac{(3n^4-18n^2+31)(n^2-1)}{967680}s(s+1)(s+2)(s+3)(s+4)(s+5) \quad [46]$$

$$\phi_4 = \frac{(5n^6-55n^4+239n^2-381)(n^2-1)}{464486400}s(s+1)(s+2)(s+3)(s+4)(s+5)(s+6)(s+7) \quad [47]$$

Taking limit to infinity or $n \to \infty$, yields:

$$\lim_{n\to\infty}\frac{4\phi_1}{(n+1)^2}s(s+1) = \lim_{n\to\infty}\frac{4(n^2-1)}{24(n+1)^2}s(s+1) = \lim_{n\to\infty}\frac{4(n-1)}{24(n+1)}s(s+1) = \lim_{n\to\infty}\frac{4\left(1-\frac{1}{n}\right)}{24\left(1+\frac{1}{n}\right)} = \frac{1}{6}s(s+1) \quad [48]$$

$$\lim_{n\to\infty}\frac{16\phi_2}{(n+1)^4}s(s+1)(s+2)(s+3)=\lim_{n\to\infty}\frac{16(3n^2-7)(n^2-1)}{5760(n+1)^4}s(s+1)(s+2)(s+3)=\frac{1}{120}s(s+1)(s+3) \quad [49]$$

$$\lim_{n\to\infty}\frac{64\phi_3}{(n+1)^6}s(s+1)(s+2)(s+3)(s+4)(s+5)=\frac{1}{5040}s(s+1)(s+2)(s+3)(s+4)(s+5) \quad [50]$$

$$\lim_{n\to\infty}\frac{256\phi_4}{(n+1)^8}s(s+1)(s+2)(s+3)(s+4)(s+5)(s+6)(s+7)$$
$$=\frac{1}{362880}s(s+1)(s+2)(s+3)(s+4)(s+5)(s+6)(s+7) \quad [51]$$

Therefore,

$$\varsigma(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\sum_{i=1}^{\infty}n^{-s}=\left[1+\frac{1}{6}s(s+1)+\frac{1}{120}s(s+1)(s+2)(s+3)+\ldots\right]\cdot\lim_{n\to\infty}\left[\frac{2^s n}{(n+1)^s}\right] \quad [52]$$

Rewritting equation [19], yields

$$\varsigma(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\sum_{i=1}^{\infty}n^{-s}=\left[\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\binom{s+(2m-1)}{2m}\right]\cdot\lim_{n\to\infty}\left[\frac{2^s n}{(n+1)^s}\right] \quad [53]$$

For harmonic series, $s=1$

$$\begin{aligned}\varsigma(1)&=\sum_{n=1}^{\infty}\frac{1}{n}=\sum_{i=1}^{\infty}n^{-1}=\left[\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\binom{1+(2m-1)}{2m}\right]\cdot\lim_{n\to\infty}\left[\frac{2n}{(n+1)}\right]\\&=\left[\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\binom{2m}{2m}\right]\cdot\lim_{n\to\infty}\left[\frac{2n}{(n+1)}\right]\\&=\left[\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\right]\cdot 2\\&=\left[2\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\right]\end{aligned} \quad [54]$$

Therefore, $\varsigma(1)$ is always twice the infinite sum of reciprocal of odd number and the sum diverges as $m\to\infty$. For special case, now consider when $s=2$

$$\begin{aligned}\varsigma(2)&=\sum_{n=1}^{\infty}\frac{1}{n^2}=\sum_{i=1}^{\infty}n^{-2}=\left[\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\binom{2+(2m-1)}{2m}\right]\cdot\lim_{n\to\infty}\left[\frac{2^2 n}{(n+1)^2}\right]\\&=\left[\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\binom{2m+1}{2m}\right]\cdot\lim_{n\to\infty}\left[\frac{4n}{(n+1)^2}\right]\\&=\left[\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\frac{(2m+1)(2m)!}{1!(2m)!}\right]\cdot\lim_{n\to\infty}\left[\frac{4n}{(n+1)^2}\right]\end{aligned}$$

$$=\left[\sum_{m=0}^{\infty}1\right]\cdot\lim_{n\to\infty}\left[\frac{4n}{(n+1)^2}\right] \quad [55]$$

Not to lose generalization, let *m*=*n* as both approaching infinity. Thus

$$\varsigma(2)=\left[\sum_{n=0}^{\infty}1\right]\lim_{n\to\infty}\left[\frac{4n}{(n+1)^2}\right]=\left(\left[\sum_{n=1}^{\infty}1\right]+1\right)\lim_{n\to\infty}\left[\frac{4n}{(n+1)^2}\right]=\lim_{n\to\infty}(n+1)\lim_{n\to\infty}\left[\frac{4n}{(n+1)^2}\right]\approx 4 \quad [56]$$

This shows that the value converges to a limit when *s*=2.

Let $s=3$, then

$$\varsigma(3)=\sum_{n=1}^{\infty}\frac{1}{n^3}=\sum_{i=1}^{\infty}n^{-3}=\left[\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\binom{3+(2m-1)}{2m}\right]\cdot\lim_{n\to\infty}\left[\frac{2^3 n}{(n+1)^3}\right]$$

$$=\left[\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\binom{2m+2}{2m}\right]\cdot\lim_{n\to\infty}\left[\frac{8n}{(n+1)^3}\right]$$

$$=\left[\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\frac{(2m+2)!}{2!(2m)!}\right]\cdot\lim_{n\to\infty}\left[\frac{8n}{(n+1)^3}\right]$$

$$=\left[\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\frac{(2m+2)(2m+1)(2m)!}{2!(2m)!}\right]\cdot\lim_{n\to\infty}\left[\frac{8n}{(n+1)^3}\right]$$

$$=\left[\sum_{m=0}^{\infty}\frac{(2m+2)}{2}\right]\cdot\lim_{n\to\infty}\left[\frac{8n}{(n+1)^3}\right]$$

$$=\left[\sum_{m=0}^{\infty}(m+1)\right]\cdot\lim_{n\to\infty}\left[\frac{8n}{(n+1)^3}\right]$$

$$=[1+2+3\ldots\infty]\cdot\lim_{n\to\infty}\left[\frac{8n}{(n+1)^3}\right] \quad [57]$$

Not to lose generalization, let *m*=*n* as both approaching infinity. Thus

$$\varsigma(3)=\cdot\lim_{n\to\infty}\frac{n(n+1)}{2}\lim_{n\to\infty}\left[\frac{8n}{(n+1)^3}\right]=\lim_{n\to\infty}\left[\frac{4n^2(n+1)}{(n+1)^3}\right]=\lim_{n\to\infty}\left[\frac{4n^2}{(n+1)^2}\right]\approx 4 \quad [58]$$

This shows that the value converges to a limit 4 when *s*=3.

Let $s=4$, then

$$\varsigma(4)=\sum_{n=1}^{\infty}\frac{1}{n^4}=\sum_{i=1}^{\infty}n^{-4}=\left[\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\binom{4+(2m-1)}{2m}\right]\cdot\lim_{n\to\infty}\left[\frac{2^4 n}{(n+1)^4}\right]$$

$$=\left[\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\binom{2m+3}{2m}\right]\cdot\lim_{n\to\infty}\left[\frac{16n}{(n+1)^4}\right]$$

$$=\left[\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\frac{(2m+3)!}{3!(2m)!}\right]\cdot\lim_{n\to\infty}\left[\frac{16n}{(n+1)^4}\right]$$

$$= \left[\sum_{m=0}^{\infty} \frac{1}{(2m+1)} \frac{(2m+3)(2m+2)(2m+1)(2m)!}{3!(2m)!} \right] \cdot \lim_{n\to\infty} \left[\frac{16n}{(n+1)^4} \right]$$

$$= \left[\sum_{m=0}^{\infty} \frac{(2m+3)(2m+2)}{6} \right] \cdot \lim_{n\to\infty} \left[\frac{16n}{(n+1)^4} \right]$$

$$= \left[\frac{1}{3} \sum_{m=0}^{\infty} (2m+3)(m+1) \right] \cdot \lim_{n\to\infty} \left[\frac{16n}{(n+1)^4} \right]$$

$$= \left[\frac{1}{3} \sum_{m=0}^{\infty} (2m^2+5m+3) \right] \cdot \lim_{n\to\infty} \left[\frac{16n}{(n+1)^4} \right]$$

$$= \left[\frac{1}{3} \left[\sum_{m=0}^{\infty} (2m^2) + \sum_{m=0}^{\infty} (5m) + \sum_{m=0}^{\infty} (3) \right] \right] \cdot \lim_{n\to\infty} \left[\frac{16n}{(n+1)^4} \right]$$

$$= \left[\frac{1}{3} \left[\lim_{m\to\infty} 2\left(\frac{2m^3+3m^2+m}{6} \right) + \lim_{m\to\infty} \frac{5m(m+1)}{2} + \lim_{m\to\infty} 3(m+1) \right] \right] \cdot \lim_{n\to\infty} \left[\frac{16n}{(n+1)^4} \right]$$

$$= \left[\frac{1}{3} \right] \lim_{n\to\infty} \left[\frac{2m^3}{3} + \frac{7m^2}{2} + \frac{35m}{6} + 3 \right] \lim_{n\to\infty} \left[\frac{16n}{(n+1)^4} \right]$$

$$= \lim_{m\to\infty} \left[\frac{2m^3}{9} + \frac{7m^2}{6} + \frac{35m}{18} + 1 \right] \lim_{n\to\infty} \left[\frac{16n}{(n+1)^4} \right] \qquad [59]$$

Not to lose generalization, let m=n as both approaching infinity. Thus

$$\varsigma(4) = \lim_{n\to\infty} \left[\frac{2n^3}{9} + \frac{7n^2}{6} + \frac{35n}{18} + 1 \right] \lim_{n\to\infty} \left[\frac{16n}{(n+1)^4} \right]$$

$$= \lim_{n\to\infty} \left[\frac{(4n+9)(n+2)(n+1)}{18} \right] \lim_{n\to\infty} \left[\frac{16n}{(n+1)^4} \right]$$

$$= \lim_{n\to\infty} \left[\frac{(4n+9)(n+2)(n+1)}{18} \frac{16n}{(n+1)^4} \right]$$

$$= \lim_{n\to\infty} \left[\frac{16n(4n+9)(n+2)(n+1)}{18(n+1)^4} \right] \approx \frac{16\cdot 4}{18} \approx \frac{32}{9} \approx 3.55555555 \qquad [60]$$

This shows that the value converges to a limit $\frac{32}{9}$ when s=4.

Repeating the same procedure for values of s up to 7, yields these limits:

$$\varsigma(5) = \lim_{n\to\infty} \left[\frac{n^4}{12} \frac{32n}{(n+1)^5} \right] \approx \frac{32}{12} \approx 2.6666666 \qquad [61]$$

$$\varsigma(6) = \lim_{n\to\infty} \left[\frac{2n^5}{75} \frac{64n}{(n+1)^6} \right] \approx \frac{128}{75} \approx 1.7066666 \qquad [62]$$

$$\varsigma(7) = \lim_{n\to\infty} \left[\frac{2^5 n^6}{6\cdot 6!} \frac{128n}{(n+1)^7} \right] \approx \frac{128}{135} \approx 0.9481481 \qquad [63]$$

These values show that the Riemann's Zeta Function converges for $s > 1 (s \in Z)$. The approximation values of Riemann's Zeta Function above can be calculated using the method as follows:

By referring to Figure 1.0 and equation [16], yields:

$$\sum_{1}^{n} A_n = \int_{1}^{n} f(x)dx + \frac{A_1}{2} + \frac{A_n}{2}$$

Now consider, $f(x) = \frac{1}{x^s}$ [64]

Therefore, the generalize equation for Riemann's Zeta Function with the modification part is given as follows:

$$\varsigma(s) = \sum_{x=1}^{\infty} f(x) = \sum_{x=1}^{\infty} \frac{1}{x^s} = \sum_{x=1}^{p} f(x) + \int_{p+1}^{\infty} f(x)dx + \frac{f(p+1)}{2} + \lim_{n \to \infty} \frac{f(n)}{2} \quad [65]$$

For $s > 1$,

$$\varsigma(s) = \sum_{x=1}^{p} \frac{1}{x^s} + \int_{p+1}^{\infty} \frac{1}{x^s} dx + \frac{1}{2(p+1)^s} + \lim_{n \to \infty} \frac{1}{2(n)^s} = \sum_{x=1}^{p} \frac{1}{x^s} + \left[\frac{x^{1-s}}{1-s} \right]_{p+1}^{\infty} + \frac{1}{2(p+1)^s} + \lim_{n \to \infty} \frac{1}{2(n)^s} \quad [66]$$

Let *x*=*n* and Riemann's Zeta Function of this form,

$$\varsigma(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{i=1}^{n} n^{-s} = n \cdot \left[\frac{n+1}{2} \right]^{-s} \left[1 + \frac{4\phi_1}{(n+1)^2} + \frac{16\phi_2}{(n+1)^4} + \frac{64\phi_3}{(n+1)^6} + \frac{256\phi_4}{(n+1)^8} + \ldots \right] \quad [67]$$

Not to lose generalization lets take the limit *n* to a certain value instead of to infinity. Therefore,

$$\varsigma(s) = \sum_{x=1}^{p} \frac{1}{x^s} + \int_{p+1}^{n} \frac{1}{x^s} dx + \frac{1}{2(p+1)^s} + \frac{1}{2(n)^s} = \sum_{x=1}^{p} \frac{1}{x^s} + \left[\frac{x^{1-s}}{1-s} \right]_{p+1}^{n} + \frac{1}{2(p+1)^s} + \frac{1}{2(n)^s} \quad [68]$$

Let *n*=1000, *s*=2, various value of *p* and the error compared to the direct value calculated by MathCAD.

$$\varsigma(2) = \sum_{x=1}^{p} \frac{1}{x^s} + \int_{p+1}^{1000} \frac{1}{x^s} dx + \frac{1}{2(p+1)^2} + \frac{1}{2(1000)^2} = \sum_{x=1}^{p} \frac{1}{x^s} + \left[\frac{x^{-1}}{-1} \right]_{p+1}^{1000} + \frac{1}{2(p+1)^2} + \frac{1}{2(1000)^2}$$

$$= \sum_{x=1}^{p} \frac{1}{x^s} + \int_{p+1}^{1000} \frac{1}{x^s} dx + \frac{1}{2(p+1)^2} + \frac{1}{2(1000)^2} = \sum_{x=1}^{p} \frac{1}{x^s} - \frac{1}{1000} + \frac{1}{p+1} + \frac{1}{2(p+1)^2} + \frac{1}{2(1000)^2} \quad [69]$$

Table 1.0 Tabulation of *p* and Error

p	$\sum_{x=1}^{p} \frac{1}{x^s}$	$\varsigma(2)$	$\sum_{x=1}^{1000} \frac{1}{x^s}$	Error
1	1.000000000	1.624000500	1.6439345667	0.0199340667
5	1.463611111	1.643167167	1.6439345667	0.0007674000
10	1.549767731	1.643809553	1.6439345667	0.0001250132
15	1.580440283	1.643893908	1.6439345667	0.0000406582
20	1.596163244	1.643916578	1.6439345667	0.0000179883
25	1.605723404	1.643925087	1.6439345667	0.0000094797
30	1.612150118	1.643928973	1.6439345667	0.0000055932

35	1.616766915	1.643930995	1.6439345667	0.0000035715
40	1.620243963	1.643932149	1.6439345667	0.0000024178
45	1.622956929	1.643932855	1.6439345667	0.0000017120
50	1.625132734	1.643933311	1.6439345667	0.0000012562
100	1.634983900	1.643934405	1.6439345667	0.0000001616
200	1.639946546	1.643934546	1.6439345667	0.0000000204
500	1.642936066	1.643934566	1.6439345667	0.0000000012
999	1.643933567	1.643934567	1.6439345667	0.0000000000

Let *n*=1000000000, *s*=2, various value of *p* and the error compared to the direct value calculated by MathCAD

$$\varsigma(2)=\sum_{x=1}^{p}\frac{1}{x^s}+\int_{p+1}^{1000000000}\frac{1}{x^s}dx+\frac{1}{2(p+1)^2}+\frac{1}{2(1000000000)^2}=\sum_{x=1}^{p}\frac{1}{x^s}+\left[\frac{x^{-1}}{-1}\right]_{p+1}^{1000000000}+\frac{1}{2(p+1)^2}+\frac{1}{2(1000000000)^2}$$
$$=\sum_{x=1}^{p}\frac{1}{x^s}-\frac{1}{1000000000}+\frac{1}{p+1}+\frac{1}{2(p+1)^2}+\frac{1}{2(1000000000)^2} \qquad [70]$$

Table 2.0 Tabulation of *p* and Error

p	$\sum_{x=1}^{p}\frac{1}{x^s}$	$\varsigma(2)$	$\sum_{x=1}^{1000000000}\frac{1}{x^s}$	Error
1	1.000000000	1.624999999	1.6449340578	0.0199340588
5	1.463611111	1.644166666	1.6449340578	0.0007673922
10	1.549767731	1.644809052	1.6449340578	0.0001250054
15	1.580440283	1.644893407	1.6449340578	0.0000406504
20	1.596163244	1.644916077	1.6449340578	0.0000179805
25	1.605723404	1.644924586	1.6449340578	0.0000094718
30	1.612150118	1.644928472	1.6449340578	0.0000055854
35	1.616766915	1.644930494	1.6449340578	0.0000035637
40	1.620243963	1.644931648	1.6449340578	0.0000024099
45	1.622956929	1.644932354	1.6449340578	0.0000017041
50	1.625132734	1.644932810	1.6449340578	0.0000012483
100	1.634983900	1.644933904	1.6449340578	0.0000001537
200	1.639946546	1.644934045	1.6449340578	0.0000000125
500	1.642936066	1.644934065	1.6449340578	-0.0000000067
1000	1.643933567	1.644934066	1.6449340578	-0.0000000078

Now consider equation [68] and *s*=i with *n*=1000000000, thus

$$\varsigma(i)=\sum_{x=1}^{p}\frac{1}{x^i}+\int_{p+1}^{1000000000}\frac{1}{x^i}dx+\frac{1}{2(p+1)^i}+\frac{1}{2(1000000000)^i}$$
$$=\sum_{x=1}^{p}\frac{1}{x^i}+\left[\frac{x^{1-i}}{1-i}\right]_{p+1}^{1000000000}+\frac{1}{2(p+1)^i}+\frac{1}{2(1000000000)^i} \qquad [71]$$

Table 3.0 Tabulation of *p* and Error

p	$\sum_{x=1}^{p}\frac{1}{x^{i}}$	$\varsigma(i)$	$\sum_{x=1}^{1000000000}\frac{1}{x^{i}}$	Error
1	1.000000000	328084502.43116593-626386909.43574643i	328084502.45810407-626386909.40598357i	0.02693814+0.029762864i
5	2.368896329-3.511819565i	328084502.44450605-626386909.4010824i	328084502.45810407-626386909.40598357i	0.013598025-0.004901171i
10	0.041897577-7.845484348i	328084502.45292091-626386909.39856672i	328084502.45810407-626386909.40598357i	0.00518316-0.007416844i
15	-4.108237633-10.579943699i	328084502.45616931-626386909.39928579i	328084502.45810407-626386909.40598357i	0.001934767-0.006697774i
20	-8.932370386-11.833954472i	328084502.45766264-626386909.40019417i	328084502.45810407-626386909.40598357i	0.000441432-0.005789399i
25	-13.922717292-11.87384003i	328084502.45841962-626386909.40096033i	328084502.45810407-626386909.40598357i	-0.000315547-0.005023241i
30	-18.827083314-10.934016183i	328084502.45882177-626386909.40156972i	328084502.45810407-626386909.40598357i	-0.0007177-0.004413843i
35	-23.512742729-9.202368208i	328084502.45903683-626386909.40205133i	328084502.45810407-626386909.40598357i	-0.000932753-0.003932238i
40	-27.908820911-6.827536308i	328084502.45914698-626386909.40243471i	328084502.45810407-626386909.40598357i	-0.001042902-0.003548861i
45	-31.978718765-3.927732355i	328084502.45919585-626386909.4027431i	328084502.45810407-626386909.40598357i	-0.001091778-0.003240466i
50	-35.70572324-0.597909691i	328084502.45920777-626386909.40299392i	328084502.45810407-626386909.40598357i	-0.001103699-0.00298965i
100	-55.11293062+44.441266802i	328084502.45886797-626386909.40406358i	328084502.45810407-626386909.40598357i	-0.000763893-0.001919985i
200	-27.742458174+138.615436023i	328084502.45839107-626386909.4043746i	328084502.45810407-626386909.40598357i	-0.000286996-0.001608968i
500	232.783644998+266.15918371i	328084502.45805782-626386909.40430951i	328084502.45810407-626386909.40598357i	0.000046253-0.001674056i
1000	698.390426163+112.522488207i	328084502.45799804-626386909.40421104i	328084502.45810407-626386909.40598357i	0.000106037-0.001772523i

Now consider when *s*=1+i with *n*=1000000000, thus

$$\varsigma(1+i)=\sum_{x=1}^{p}\frac{1}{x^{1+i}}+\int_{p+1}^{1000000000}\frac{1}{x^{1+i}}dx+\frac{1}{2(p+1)^{1+i}}+\frac{1}{2(1000000000)^{1+i}}$$

$$=\sum_{x=1}^{p}\frac{1}{x^{1+i}}-\left[\frac{x^{-i}}{i}\right]_{p+1}^{1000000000}+\frac{1}{2(p+1)^{1+i}}+\frac{1}{2(1000000000)^{1+i}} \qquad [72]$$

Table 4.0 Tabulation of *p* and Error

p	$\sum_{x=1}^{p}\frac{1}{x^{1+i}}$	$\varsigma(1+i)$	$\sum_{x=1}^{1000000000}\frac{1}{x^{1+i}}$	Error
1	1.000000000	1.50781986-1.227281627i	1.536629471-1.225150971i	0.028809611+0.002130656i
5	1.574368108-1.061947289i	1.534888588-1.222387479i	1.536629471-1.225150971i	0.001740883-0.002763492i
10	1.292666199-1.631073199i	1.536670713-1.224178538i	1.536629471-1.225150971i	-0.000041242-0.000972433i
15	0.972177396-1.847843363i	1.536815745-1.224730148i	1.536629471-1.225150971i	-0.000186274-0.000420823i
20	0.70294311-1.919627064i	1.536799238-1.224944653i	1.536629471-1.225150971i	-0.000169768-0.000206318i

25	0.485154721-1.922196975i	1.536766235-1.225042892i	1.536629471-1.225150971i	-0.000136764-0.000108079i
30	0.309464577-1.88899607i	1.536737496-1.225092946i	1.536629471-1.225150971i	-0.000108026-0.000058025i
35	0.167116773-1.836687474i	1.53671509-1.225120355i	1.536629471-1.225150971i	-0.00008562-0.000030615i
40	0.051183153-1.77426608i	1.536697985-1.225136126i	1.536629471-1.225150971i	-0.000068515-0.000014845i
45	-0.043641293-1.706858448i	1.536684895-1.225145504i	1.536629471-1.225150971i	-0.000055424-0.000005466i
50	-0.121415045-1.637494439i	1.536674779-1.225151191i	1.536629471-1.225150971i	-0.000045308+0.000000221i
100	-0.4126254-1.028898154i	1.536638394-1.225158308i	1.536629471-1.225150971i	-0.000008924+0.000007337i
200	-0.249657996-0.371795303i	1.536630034-1.225153833i	1.536629471-1.225150971i	-0.000000564+0.000002862i
500	0.514631929+0.070869108i	1.536629162-1.225151324i	1.536629471-1.225150971i	0.000000309+0.000000353i
1000	1.167312033-0.115926305i	1.536629355-1.225150989i	1.536629471-1.225150971i	0.000000116+0.000000019i

Now consider when $s = \frac{1}{2} + bi$,

$$\varsigma(\frac{1}{2}+bi) = \sum_{x=1}^{p} \frac{1}{x^{\frac{1}{2}+bi}} + \int_{p+1}^{n} \frac{1}{x^{\frac{1}{2}+bi}} dx + \frac{1}{2(p+1)^{\frac{1}{2}+bi}} + \frac{1}{2(n)^{\frac{1}{2}+bi}}$$

$$= \sum_{x=1}^{p} \frac{1}{x^{\frac{1}{2}+bi}} + \frac{1}{(\frac{1}{2}-bi)}\left[n^{\frac{1}{2}-bi} - (p+1)^{\frac{1}{2}-bi}\right] + \frac{1}{2(p+1)^{\frac{1}{2}+bi}} + \frac{1}{2(n)^{\frac{1}{2}+bi}} \quad [73]$$

Since,

$$\varsigma(s) = n \cdot \left[\frac{n+1}{2}\right]^{-s} \left[1 + \frac{4\phi_1}{(n+1)^2} + \frac{16\phi_2}{(n+1)^4} + \frac{64\phi_3}{(n+1)^6} + \frac{256\phi_4}{(n+1)^8} + \dots\right] \quad [74]$$

Let $s = \frac{1}{2} + bi$ and substitutes it into [74]

$$\varsigma(\frac{1}{2}+bi) = n \cdot \left[\frac{n+1}{2}\right]^{-\left(\frac{1}{2}+bi\right)} \left[1 + \frac{4\phi_1}{(n+1)^2} + \frac{16\phi_2}{(n+1)^4} + \frac{64\phi_3}{(n+1)^6} + \frac{256\phi_4}{(n+1)^8} + \dots\right] \quad [75]$$

From equation [53],

$$\varsigma(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{i=1}^{\infty} n^{-s} = \left[\sum_{m=0}^{\infty} \frac{1}{(2m+1)} \binom{s+(2m-1)}{2m}\right] \cdot \lim_{n\to\infty} \left[\frac{2^s n}{(n+1)^s}\right]$$

Thus,

$$\varsigma(\frac{1}{2}+bi) = \sum_{n=1}^{\infty} \frac{1}{n^{\left(\frac{1}{2}+bi\right)}} = \sum_{i=1}^{\infty} n^{-\left(\frac{1}{2}+bi\right)} = \left[\sum_{m=0}^{\infty} \frac{1}{(2m+1)} \binom{\frac{1}{2}+bi+(2m-1)}{2m}\right] \cdot \lim_{n\to\infty} \left[\frac{2^{\left(\frac{1}{2}+bi\right)} n}{(n+1)^{\left(\frac{1}{2}+bi\right)}}\right]$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{\left(\frac{1}{2}+bi\right)}} = \sum_{i=1}^{\infty} n^{-\left(\frac{1}{2}+bi\right)} = \left[\sum_{m=0}^{\infty} \frac{1}{(2m+1)} \binom{(2m+bi)-\frac{1}{2})}{2m} \right] \cdot \lim_{n \to \infty} \left[\frac{2^{\left(\frac{1}{2}+bi\right)} n}{(n+1)^{\left(\frac{1}{2}+bi\right)}} \right] \qquad [76]$$

Therefore,

$$\left[\sum_{m=0}^{\infty} \frac{1}{(2m+1)} \binom{(2m+bi)-\frac{1}{2})}{2m} \right] = \frac{\lim_{n \to \infty} \left[\sum_{x=1}^{p} \frac{1}{x^{\frac{1}{2}+bi}} + \frac{1}{(\frac{1}{2}-bi)} \left[n^{\frac{1}{2}-bi} - (p+1)^{\frac{1}{2}-bi} \right] + \frac{1}{2(p+1)^{\frac{1}{2}+bi}} + \frac{1}{2(n)^{\frac{1}{2}+bi}} \right]}{\lim_{n \to \infty} \left[\frac{2^{\left(\frac{1}{2}+bi\right)} n}{(n+1)^{\left(\frac{1}{2}+bi\right)}} \right]}$$

3 Discussion and Conclusion.

The finding of Sum of Power of symmetric function has a big impact on the field of mathematics as it contributes to a new frontier of research in many fields dealt with symmetric functions. The using of symmetric function through Sum of Power of real and complex *p-th* in Riemann's Zeta function could offer a new way of understanding Riemann's Zeta Function and offer an alternative studies in this field.

Reference

[1] Johann Faulhaber, Academia Algebræ, Darinnen die miraculosische Inventiones zu den hochsten Cossen weiters continuirt und profitiert werden. Augspurg, bey Johann Ul-rich Schonigs,1631. (Call number QA154.8 F3 1631a f MATH at Stanford University Libraries.)

[2] Bernhard Riemann, "On the Number of Prime Numbers less than a Given Quantity" . (Ueber die Anzahl der Primzahlen unter einer gegebenen Gr¨osse.) Bernhard Riemann Translated by David R.Wilkins Preliminary Version: December 1998.