Relationship between irrational constants Phi and* e *(including new equations, possible implications)

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Phi

Recall the growth of rabbits thought experiment as posed by Fibonacci:

Given the following rules, how do the rabbits grow?

1) Begin with 1 baby rabbit (1B)
2) Each baby rabbit (1B) becomes an adult rabbit (1A) after 1 Generation
3) Each adult rabbit (1A) produces 1 new baby rabbit (1B) after 1 Generation
4) Rabbits never die.

Here is how the rabbits grow over the first 6 generations (for convenience sake we will start the numbering of the generations from 0; in any case, since the 1 baby rabbit is a condition given from the beginning, starting from Generation 0 is not a logical stretch of the imagination):

Generation (**FibN**)	Total Rabbits (**FibN TR**)	Composition of Rabbits (**FibN C**)	Summary
0	1	1 baby (1B)	1B
1	1	1 adult (1A)	1A
2	2	1 adult (1A), 1 baby (1B)	1A, 1B
3	3	1 adult, 1baby, 1 adult	2A, 1B
4	5	1adult 1 baby, 1adult, 1adult 1baby	3A, 2B
5	8	1A1B, 1A, 1A1B, 1A, 1A1B	5A, 3B
6	13	1A1B1A, 1A1B, 1A1B1A, 1A1B, 1A1B1A	8A, 5B

…

Table 1: First 6 Rabbit Generations in Total and in Composition

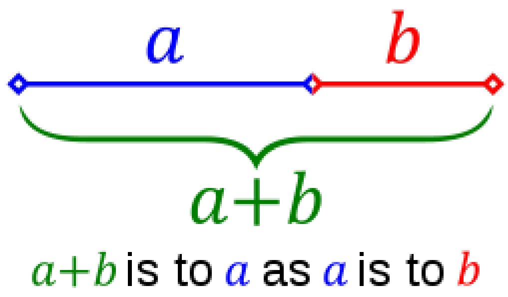
(Note: **FibN** is the generation, **FibN TR** is the total number of rabbits in that generation, **Fibn C** is the composition of rabbits in that generation.)

You can see the following:

1) Rules 1 through 4 are expressed precisely in FibN0 to FibN3 respectively.
2) The full Fibonacci sequence is a function of the Total Rabbits (FibN TR) in each generation (FibN0 -> FibN∞) given in series.

3) Beginning with FibN2, the Composition of Rabbits (FibN C) in each generation is equal to the FibN0 C as babies and FibN1 C as adults. This relationship continues in this way for each generation: the grandparent generation showing up as babies and the parent generation showing up as adults.

The Golden Ratio (denoted by the Greek letter φ or “Phi”) is that unique irrational constant such that when the ratio a:b is equal to the ratio a+b: a, then a and b are said to be in Golden Ratio.



Graphic 1: Line segment with sections a and b in Golden Ratio. (Source: Wikipedia)

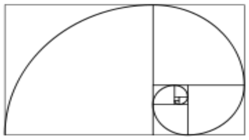
Returning to Fibonacci’s thought experiment, we can see that Phi is expressed both intra- and inter-generationally. *Inter*-generationally, the ratio of the Total Rabbits in any generation to Total Rabbits in the immediately previous generation (FibN TR : FibN-1 TR) tends toward the Golden Ratio (Phi) as FibN → infinity. We can also see Phi *intra*-generationally; within each FibN generation the ratio of existing adult rabbits (1A) to existing baby rabbits (1B) (1A:1B) also tends toward Phi as FibN → infinity. Though it seems logically redundant to say, without mentioning both inter- and intra-generational as *distinct*, the fact might be obscured that the rabbits are *multiplying or growing* themselves in time, *not adding themselves together from previous generations.* The additive property (inter-generational) mimicking the multiplicative property (intra-generational) of the Fibonacci sequence is directly related to time:

1) See Graphic 1 above. Each Fibonacci number could be considered to be forming Golden Ratios with the generation before it and generation after it, acting first as “a” (to the generation before as “b”) and then as “b” (to the generation after “a”).
2) Each Fibonacci number can *intra-generationally* be thought to be acting as a discrete approximation of the Golden Ratio with its ratio of Adult “a” : “b” Baby rabbits.

$$\frac{a+b}{a} = \frac{a}{b} \equiv \varphi\,.$$

Graphic 2: Mathematical definition of Phi. (Source: Wikipedia)

If thinking about this makes you feel dizzy, it’s probably because the Fibonacci sequence forms a spiral.



Graphic 3: Fibonacci Spiral. (Source: Wikipedia)

These inter and intra-generational relationships will be called upon again as we continue.

e

Recall the exponential function, e^x. This is known to expand mathematically in Taylor Series as follows:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ (Source: Wikipedia)

Using x = φ (Phi), we get the following Taylor Series expansion.

e^φ = 1 + φ/1! + φ^2/2! + φ^3/3! + φ^4/4! + φ^5/5! + φ^6/6! + ….

Reducing this further requires closer inspection of one of Phi's properties:

If you recall the discussion earlier regarding intra-generational composition of rabbits, Phi has the very interesting property such that φ to any whole integer power N is equal to B + A Phi (with N = to any FibN generation, B = total B rabbits in that FibN generation, and A = total A rabbits in that FibN generation).

Here are the first 7 examples of how Phi to the N power (FibN0 to FibN6) reduces (refer back to Table 1):

Phi^0 = 1 + 0Phi (FibN0 contains 1B rabbit and 0A rabbits).

Phi^1 = 0 + 1Phi (FibN1 contains 0B rabbits and 1A rabbit).

Phi^2 = 1 + 1Phi (FibN2 contains 1B rabbit and 1A rabbit).

Phi^3 = 1 + 2Phi (FibN3 contains 1B rabbit and 2A rabbits).

Phi^4 = 2 + 3Phi (FibN4 contains 2B and 3A).

Phi^5 = 3 + 5 Phi (FibN5 contains 3B and 5A).

Phi^6 = 5 + 8Phi (FibN6 contains 5B and 8A).

…

Table 2: First 6 Rabbit Generations Revisited, Reduction of Phi^N

Simply, whatever whole integer N power you raise Phi to, this is equal to the total B rabbits in that FibN Generation, plus the total A rabbits in that FibN generation *times Phi:*

φ^N = B + APhi.

(Source: Author)

Therefore, the Taylor Series expansion with x = φ in the following equation:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ (Source: Wikipedia)

= e^φ = 1 + φ/1! + φ^2/2! + φ^3/3! + φ^4/4! + φ^5/5! + φ^6/6! + ….

which based on the above equation (φ^N = B + APhi) further reduces to:

= 1 + Phi + (1 + 1Phi)/2! + (1 + 2Phi)/3! + (2 + 3Phi)/4! + (3 + 5Phi)/5! + (5 + 8Phi)/6! + …

which further reduces to:

$$e^{\varphi} = \left(\sum_{n=0}^{\infty} \frac{\text{FibN TR(B)}}{N!} \right) + \varphi \left(\sum_{n=1}^{\infty} \frac{\text{FibN TR(A)}}{N!} \right)$$

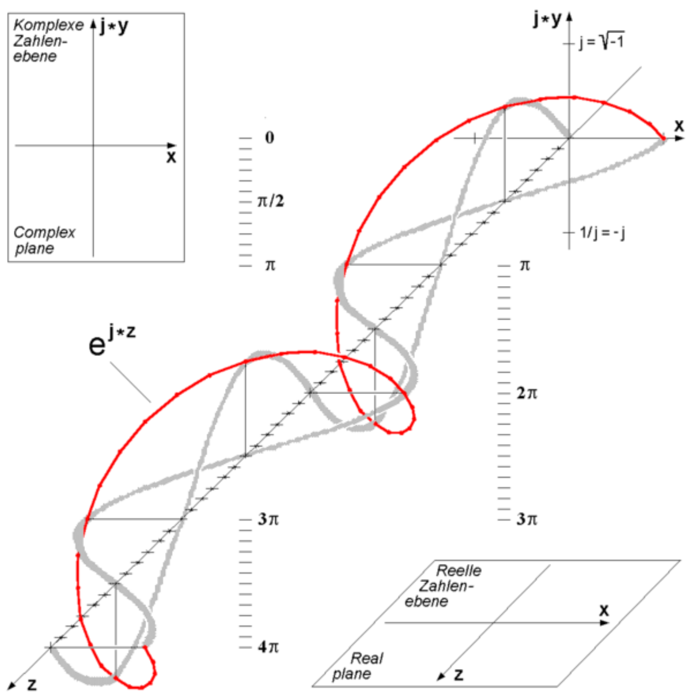
Note: FibN TR(B) is the total number of B rabbits in the FibN generation. FibN TR(A) is the total number of A rabbits in the FibN generation.

Graphic 4: Reduction of e^φ Taylor Series (Source: Author)

Numerically, this reduces to an irrational number ~5.04316564

Implications

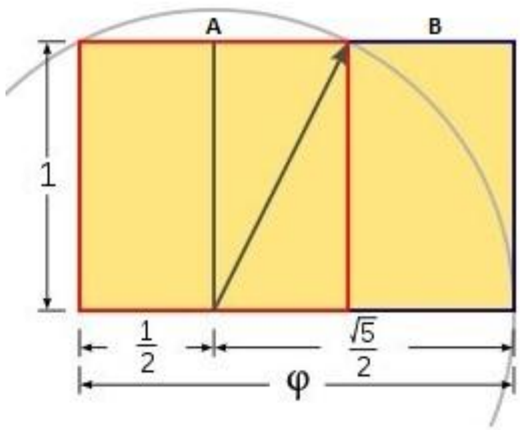
The implications of this are many. First and foremost the general property of *e* seems to be to split a complex function into its component, out-of-phase vectors. In this case, it does by factoring 1 and Phi into separate functions (notice that both integer (B) portion and Phi portion (A) are each multiplied by their own summations beginning with out-of-phase generations). Obviously *e* also does this in Euler's Formula between the imaginary number *i* and 1. This is made evident by the following graphic:



Graphic 5: Illustration of Orthogonal, Out-of-Phase Vectors i and 1. (Source: Wikipedia)

You can see that 1 and Phi also appear to have an orthogonal, out-of-phase relationship. At FibN0 you have the 1 plus 0Phi and at FibN1 you have 0 plus 1Phi. Furthermore if you look again at Table 2, both the whole number portion of the equation (B) and the Phi portion of the equation (APhi) perfectly form individual, Fibonacci sequences if you proceed from one FibN generation to the next (refer back to Table 2). The difference is that the sequence for B begins with 1 (FibN0) and the sequence for Phi begins with Phi at FibN1.

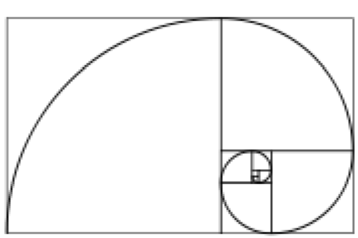
The polar relationship is perhaps the source of tension responsible for the growth of the rabbits at all. Within the first 4 generations, all four rules are expressed respectively. The remaining sequence carries from there. This is an out-of-phase, polar relationship, with 1 and Phi acting orthogonally. (In fact "out-of-phase" may be seen as the generalized definition for "orthogonal").



Graphic 6: Constructing Phi, Rectangles A + B with top sides in Golden Ratio (Source: Wikipedia)

Notice in order to establish Phi, the need to extend the vector orthogonally to the alignment of the original unit square. Notice 1 and Phi form an orthogonal relationship.

However, the difference between 1 and Phi from i and 1 as in Euler's formula is that neither Phi nor 1 are inert in this equation, but rather both grow in time with each cycle, this can be thought of with vectors as follows: B → A and A → B + A. Since it is forever growing, Fibonacci forms a logarithmic spiral, while Euler's cycle remains uniform (not-growing) and tends toward an identity (=1 when e^i2pi is used) rather than an irrationality as does this relationship.



Graphic 7: Expanding Fibonacci Spiral (approximation of Phi spiral (logarithmic)).

Among other implications, if this general implication for *e* is found true, it might be used to discover an equation splitting Energy into its component orthogonal out-of-phase vectors: Work and Entropy (i.e. when you have full work, you have no entropy, when you have full entropy, you have no work, and every combination in between). Though a relationship between energy, work, an entropy already exists, perhaps it could be expanded upon.

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(Thanks is given to Paul Mirsky for his contribution to the author's thinking with a complex interest rate experiment.)