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Through-Thickness Shearing Effects on Geometric Non-linear Behavior of Thin and Thick Functionally Graded Plates under Pressure Loads

M. Hajikazemi[1], M. H. Sadr[1], M. Ramezani-Oliaee[1]

[1] Aerospace Engineering Department and Centre of Excellence in Computational Aerospace Engineering, Amirkabir University of Technology, Tehran, Iran
Hajikazemi@aut.ac.ir, Sadr@aut.ac.ir, Mona.ramezani@aut.ac.ir

Abstract. The effects of through-the-thickness shearing strain energy on the geometric non-linear behavior of thin and relatively thick rectangular functionally graded plates are studied in this paper. It is assumed that the mechanical properties of the plates, graded through the thickness, are described by a simple power law distribution in terms of the volume fractions of constituents. The plates are assumed to be under lateral pressure loads. The fundamental equations for rectangular plates of FGM are obtained using the classical laminated plate theory (CLPT), first order shear deformation theory (FSDT) and higher order shear deformation theory (HSDT) for large deflection and the solution is obtained by minimization of the total potential energy.

Keywords: Functionally graded material; Large deflection; Power law; Through-the-thickness shearing effect.

1 Introduction

The concept of functionally graded materials (FGM), as ultra-high temperature resistant materials for aircraft, space vehicles and other engineering applications, was first introduced in 1984 by a group of material scientists in Japan [1].

FGMs are heterogeneous composite materials usually made from a mixture of metals and ceramics. By gradually varying the volume fraction of constituent materials, the material properties of FGMs exhibit a smooth and continuous change from one surface to another, thus eliminating interface problems. With the escalating application of functionally graded materials, the attention of some researchers in the field has been attracted to the investigation of the non-linear behavior of the structures made up of these materials. Woo and Meguid [2] have given an analytical solution for large deflection of FGM plates and shallow shells. Chakraborty *et al* [3] have developed a new beam element to study the thermoelastic behavior of functionally graded beam structures. GhannadPour *et al* [4, 5] have given an analytical solution for large deflection behavior of functionally graded plates under pressure load using classical laminated plate theory (CLPT) and first order shear deformation theory

(FSDT). Yang and Shen [6] have studied the large defection and post-buckling responses of functionally graded rectangular plates under transverse and in-plane loads by using a perturbation technique in conjunction with one-dimensional differential quadrature approximation and Galerkin procedure. Sadr and Hajikazemi [7] have analyzed the large deflection behavior of functionally graded plates using higher order shear deformation theory. In their studies, the material properties of the functionally graded plate are assumed to vary continuously through the thickness of the plate, according to the simple power law distribution in terms of the volume fractions of constituents.

In the current paper, the large deflection behavior of square plates made of functionally graded material concerning the effects of through-the thickness shear strain energy are studied. The material properties of the functionally graded plates, except for the Poisson's ratio, are assumed to vary continuously through the thickness of the plate, according to the simple power law distribution. The plates are assumed to be simply supported along all edges and the Classical Plate Theory (CPT), First-order Shear Deformation Theory (FSDT) and higher order shear deformation theory (HSDT) are applied throughout this work. The solution is obtained by minimization of the total potential energy of the plate. The effects of material properties on through the thickness stress fields and central deflection of the plates are determined and discussed. The through-the-thickness shearing effects and the accuracy of various plate theories on large deflection behavior of functionally graded plates are examined in detail. It is noted that the current paper is an updated and revised version of the conference papers [5, 7]. Whilst the same formulations as those used in the conference papers are implemented in the current paper, the scope and application of the current paper are strengthened by studying more previous works in the introduction section as well as considering significantly more results and discussions about the effects of through-the-thickness shear strain energy on the large deflection behaviour of thin and relatively thick FGM plates.

2 Theoretical Formulation

A FGM square plate of side A and thickness h, made from a mixture of ceramics and metals, as shown in Fig. 1, is considered. It is assumed that the composition is so varied that the top surface of the plate ($z=h/2$) is ceramic-rich, whereas the bottom surface ($z=-h/2$) is metal-rich. Thus, the material properties of the FGM plate (ϑ), such as the Young's modulus (E) and the shear modulus (G) are functions of depth (z). In this study the simple power law is used [8]. The functional relationship between ϑ and z for ceramic and metal FGM plates is given by [9] as:

$$\vartheta(z) = \vartheta_{tb}\left(\frac{z}{h}+\frac{1}{2}\right)^n + \vartheta_b \tag{1}$$

Where $\vartheta_{tb} = \vartheta_t - \vartheta_b \,.\, \vartheta(z)$ denotes a typical material property (*E*, *G*). ϑ_t and ϑ_b denote values of the variables at top and bottom surfaces of plates, respectively. It is evident that when $z=-h/2$, $\vartheta=\vartheta_b$ and when $z=h/2$, $\vartheta=\vartheta_t$.

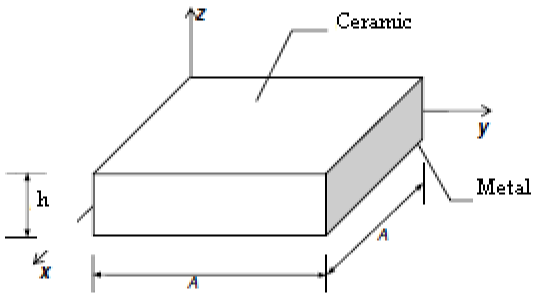

Fig. 1. Typical FGM square plate.

In the following, the fundamental equations of the large deflection analysis of pressure-loaded functionally graded plates are briefly outlined. The plates are assumed to be simply supported along all edges and the CLPT, FSDT and HSDT are applied throughout this work. As a result of the HSDT assumption:

$$\bar{u}(x,y,z) = u(x,y) + z\varphi_x(x,y) - z^3 C_1(\varphi_x(x,y) + \frac{\partial w(x,y)}{\partial x})$$
$$\bar{\upsilon}(x,y,z) = \upsilon(x,y) + z\varphi_y(x,y) - z^3 C_1(\varphi_y(x,y) + \frac{\partial w(x,y)}{\partial y}) \quad (2)$$
$$\bar{w}(x,y,z) = w(x,y) \qquad Where \quad C_1 = 4/3h^2$$

where $\bar{u}, \bar{\upsilon}$ and $\bar{w}$ are components of displacement at a general point, whilst u, υ and w are similar components at the middle surfaces (z=0) and φ_x, φ_y are rotations respect to y, x axis respectively. Using Eq.2 the non-linear in-plane strains and through the thickness shear strains at a general point are:

$$\bar{\varepsilon} = \left\{\bar{\varepsilon}_x, \bar{\varepsilon}_y, \bar{\varepsilon}_{xy}\right\}^T = \varepsilon_0 + z\varepsilon_1 + z^3\varepsilon_3 \quad , \quad \bar{\gamma} = \left\{\bar{\gamma}_{yz}, \bar{\gamma}_{xz}\right\}^T = \gamma_0 + z^2\gamma_2$$

$$Where:\ \varepsilon_0{}^T = \left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \frac{\partial \upsilon}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial x}\right),$$

$$\varepsilon_1{}^T = \left(\frac{\partial \varphi_x}{\partial x}, \frac{\partial \varphi_y}{\partial y}, \frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}\right), \tag{3}$$

$$\varepsilon_3{}^T = -C_1\left(\frac{\partial \varphi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}, \frac{\partial \varphi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}, \frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y}\right),$$

$$\gamma_0{}^T = \left(\varphi_y + \frac{\partial w}{\partial y}, \varphi_x + \frac{\partial w}{\partial x}\right), \quad \gamma_2{}^T = -3C_1\left(\varphi_y + \frac{\partial w}{\partial y}, \varphi_x + \frac{\partial w}{\partial x}\right)$$

The stress-strain relationship at a general point for the plate becomes:

$$\left\{\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xy}\right\}^T = \left[Q_{ij}\right]\left\{\bar{\varepsilon}_x, \bar{\varepsilon}_y, \bar{\gamma}_{xy}\right\}^T, \qquad where \qquad i,j = 1,2,3$$

$$\left\{\bar{\tau}_{yz}, \bar{\tau}_{xz}\right\}^T = \left[Q_{ij}\right]\left\{\bar{\gamma}_{yz}, \bar{\gamma}_{xz}\right\}^T, \qquad where \qquad i,j = 4,5 \tag{4}$$

$$Q_{11} = Q_{22} = \frac{E(z)}{1-v^2}; Q_{12} = Q_{21} = \frac{vE(z)}{1-v^2}; Q_{33} = Q_{55} = Q_{44} = \frac{E(z)}{2(1+v)};$$

$$Q_{13} = Q_{45} = Q_{23} = 0$$

The constitutive equations for a plate can be obtained by the use of Eqs.3 and 4 and appropriate integration through the uniform thickness as Eq.5.

In the Eq.5, $\{N\},\{M\},\{P\},\{Q\},\{R\}$ are the stress resultants per unit length and A,B,E,D,F,H,A_s,D_s,F_s are the plate stiffness matrices.

$$\begin{Bmatrix} \{N\} \\ \{M\} \\ \{P\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] & [E] \\ [B] & [D] & [F] \\ [E] & [F] & [H] \end{bmatrix} \begin{Bmatrix} \{\varepsilon_0\} \\ \{\varepsilon_1\} \\ \{\varepsilon_3\} \end{Bmatrix}$$

$$\left(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}\right) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} Q_{ij}\left(1, z, z^2, z^3, z^4, z^6\right) dz \quad (i, j = 1,2,3) \tag{5}$$

$$\begin{Bmatrix} \{Q\} \\ \{R\} \end{Bmatrix} = \begin{bmatrix} [A_s] & [D_s] \\ [D_s] & [F_s] \end{bmatrix} \begin{Bmatrix} \{\gamma_0\} \\ \{\gamma_2\} \end{Bmatrix}$$

$$\left(A_{ij_s}, D_{ij_s}, F_{ij_s}\right) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} Q_{ij}\left(1, z^2, z^4\right) dz, \quad (i, j = 4,5)$$

The strain energy per unit volume is $\frac{1}{2}\bar{\sigma}^T\bar{\varepsilon}^T + \frac{1}{2}\bar{\tau}^T\bar{\gamma}^T$. Using Eqs.4 and 5 to form the strain energy, and integrating through the thickness with respect to z gives an expression for the strain energy of the plate which can be put into the form:

$$U_s = \frac{1}{2}\iint \begin{pmatrix} \varepsilon_0^T [A]\varepsilon_0 + 2\varepsilon_0^T [B]\varepsilon_1 + 2\varepsilon_0^T [E]\varepsilon_3 + \varepsilon_1^T [D]\varepsilon_1 + 2\varepsilon_1^T [F]\varepsilon_3 + \\ \varepsilon_3^T [H]\varepsilon_3 + \gamma_0^T [A_s]\gamma_0 + 2\gamma_0^T [D_s]\gamma_2 + \gamma_2^T [F_s]\gamma_2 \end{pmatrix} dxdy \tag{6}$$

As previously indicated, the present concern is the evaluation of the response of plates under uniform pressure. Thus the external force exists in the current problem. As a result, the total potential energy is equal to the summation of strain energy and potential energy of uniform pressure, i.e.

$$V = U_s + V_P \tag{7}$$

where V_P is the potential energy of uniform pressure and is equal to :

$$V_P = \iint qwdxdy \tag{8}$$

Where q is the uniform pressure. Solution of the non-linear problem is sought through the application of the principle of Minimum Potential Energy. This, of course, requires the assumption of a displacement field to represent the variations of u, υ, w and φ_x, φ_y over the middle surfaces. It is assumed that all four edges of plates are simply supported with no in-plane displacements, i.e., they are prevented from moving in the x and y directions. These following displacement fields are same for both CLPT, FSDT [5], HSDT analyses.

$$u=\sum_{i=1}^{m}\sum_{j=1}^{m}u_{i,j}\sin\left(\frac{2i\pi x}{A}\right)\sin\left(\frac{(2j-1)y}{A}\right),$$
$$v=\sum_{i=1}^{m}\sum_{j=1}^{m}v_{i,j}\sin\left(\frac{2i\pi y}{A}\right)\sin\left(\frac{(2j-1)x}{A}\right),\qquad \textbf{(9)}$$
$$w=\sum_{i=1}^{n}\sum_{j=1}^{n}w_{i,j}\sin\left(\frac{(2i-1)x}{A}\right)\sin\left(\frac{(2j-1)y}{A}\right)$$

And for φ_x,φ_y in FSDT, HSDT analysis:

$$\varphi_x=\sum_{i=1}^{n}\sum_{j=1}^{n}\varphi_{xi,j}\cos\left(\frac{(2i-1)\pi x}{A}\right)\sin\left(\frac{(2j-1)\pi y}{A}\right)$$
$$\varphi_y=\sum_{i=1}^{n}\sum_{j=1}^{n}\varphi_{y_{i,j}}\cos\left(\frac{(2i-1)\pi y}{A}\right)\sin\left(\frac{(2j-1)\pi x}{A}\right)\qquad \textbf{(10)}$$

Where $u_{i,j},v_{i,j},w_{i,j}$ *and* $\varphi_{xi,j},\varphi_{y_{i,j}}$ are undetermined displacement coefficients. The displacement functions could be in a variety of forms, but in the present work they are Harmonic functions. With the establishment of the plate displacement field, according to the above mentioned equations, the potential energy of a plate can ultimately be calculated in terms of $u_{i,j},v_{i,j},w_{i,j},\varphi_{xi,j},\varphi_{yi,j}$. The plate equilibrium equations are obtained by applying the principle of minimum potential energy. That is to say the partial differentiation of the plate potential energy with respect to $u_{i,j},v_{i,j},w_{i,j},\varphi_{xi,j},\varphi_{yi,j}$. In turn gives a set of non-linear equilibrium equations. The latter set of equations must be solved in term of undetermined displacement coefficients. After finding the undetermined displacement coefficients, it is possible to calculate the displacements u, υ ,w and φ_x,φ_y at any point in plate using Eqs. 9 and 10, to determine stress quantities through use of Eqs. 3 and 4.

3 Numerical Results and Discussion

As previously indicated, the analysis of FGM plate is conducted for type of ceramic and metal combination. The set of materials considered is alumina and aluminum.

Young's modulus and Poisson's ratio were selected as being 70 GPa and 0.3 for aluminum, and 380 GPa and 0.3 for alumina, respectively. In all cases, the lower surface of the plate is assumed to be metal (aluminum) rich and the upper surface is assumed to be pure ceramic (alumina).
The analytic results are presented in terms of dimensionless deflection and stress. The dimensionless parameters used are as follows:

center deflection $W = w/h$
load parameter $Q = qA^4/(E_b h^4)$
axial stress $\sigma = \sigma_x A^2/(E_b h^4)$
thickness coordinate $Z = z/h$

where E_b is Young's modulus of metal (bottom surface) used in the functionally graded material, q is an uniformly distributed pressure load, A is a projected length of the plate in the xy plane, and h is a thickness of that plate. The analysis is performed on square plates with $A/h=20$ and $A/h=5$.

The number of terms needed for convergence is carefully examined. For generating the converged results, the essential number of terms in the displacement fields for u, v is 9 and for w, φ_x, φ_y is 16.

Fig. 2 show the central deflection due to mechanically applied load Q for four simply supported square aluminum-alumina plates with $A/h=20$ using HSDT ,FSDT and CLPT assumptions. Under the same load, the pure aluminum plate has the largest central deflection. This is due to the fact that it has the lowest Young's modulus E_b. Even though the FGM plate (n=2) contains a small volume fraction of alumina, it is much stiffer than the pure aluminum plate. This figure shows a negligible difference between three theories due to the high length to thickness ratio and the high level of in-plane restraints imposed at the edges, thus causing relatively small transverse deflections at the centre [5, 10]. In other words, through-the thickness shearing effects for relatively thin plates with high level of in-plane restraints imposed at the edges are negligible.

Figs. 3 and 4 show the variation of the dimensionless axial stress σ across the thickness Z at the center of the four plates investigated due to a load parameter Q=-400 for $A/h=20$ and $A/h=5$ respectively. The stress distribution in the aluminum and alumina plates is linear; whereas, for functionally graded material, the behavior is nonlinear and is governed by the variation of the properties in the thickness direction. It is clearly seen in Fig.4, there are noticeable difference between the results obtained from various theories. It is noted that difference between the results obtained from various theories are due to consideration of through-the-thickness shearing terms in HSDT and FSDT and elimination of them in CLPT. Therefore, the effects of through-the-thickness strain energy in the large deflection analysis of relatively thick functionally grade plates should be taken into account.

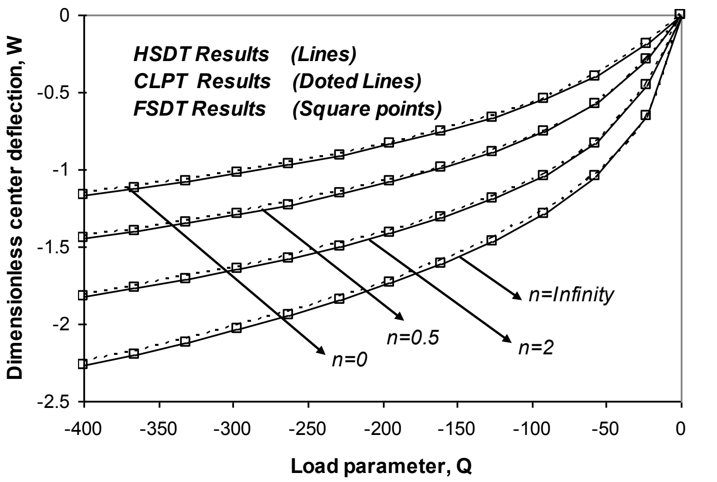

Fig. 2. Dimensionless central deflection versus uniform pressure load for A/h=20.

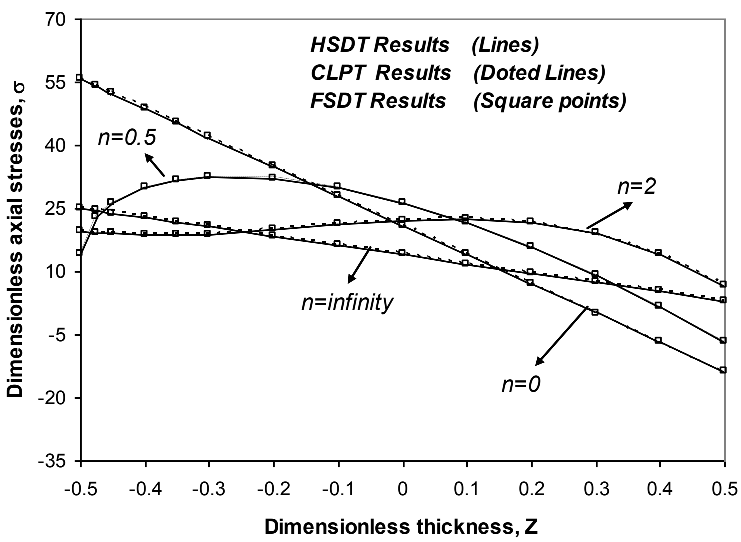

Fig. 3. Dimensionless axial stress σ along the thickness Z at the center of the plate under load Q=-400 for A/h=20.

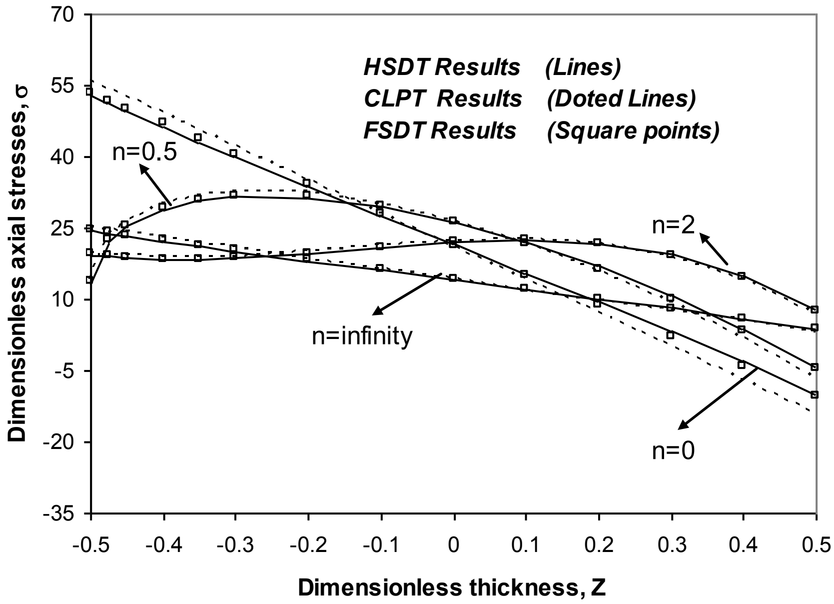

Fig. 4. Dimensionless axial stress σ along the thickness Z at the center of the plate under load Q=-400 for A/h=5.

In Fig. 5 the effect of constituent volume fraction on the center deflection of FGM plate for *A/h=20* and *Q*=-400 is presented by varying the volume fraction exponent *n* using higher order shear deformation theory and first order shear deformation theory. It is clearly seen that for *n* greater than 60, the behavior of FG plate is not very affected with more increasing of *n*. It is due to this fact that the behavior of FG plates in these cases is governed by metal phase.

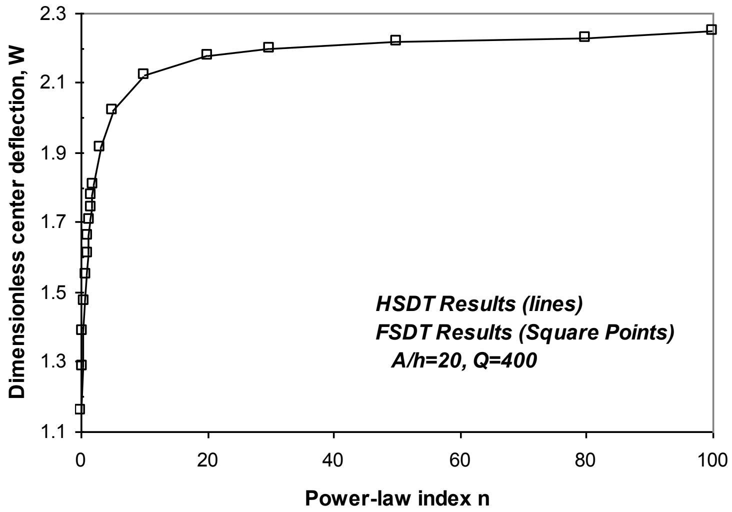

Fig. 5. Dimensionless central deflection versus power-law index n of the plate under load Q=400 for A/h=20.

4 Conclusion

The large deflection of plates made of FGMs under pressure load concerning the effects of through-the-thickness strain energy is studied. The material properties of FGM plate are assumed to vary continuously through the thickness of the plate, and were graded according to a power law distribution of a volume fraction of the constituents. The solution is obtained by minimization of the total potential energy. Dimensionless deflection and stresses were computed for metal-ceramic combination of functionally graded plates with various lengths to thickness ratio. The effects of material properties on the stress field through the thickness are determined and discussed. It is shown that through-the-thickness shearing effects in the case of relatively thick plates should be taken into account; however, for thin plates the mentioned effects are negligible.

References

1. Koizumi. M.: FGM activities in Japan, *J. Compos*. 28 (1997) 1–4.
2. Woo. J., Meguid. S. A.: Nonlinear analysis of functionally graded plates and shallow shells, *Int. J. Solids Struct.* 38 (2001) 7409–7421.
3. Chakraborty. A., Gopalakrishnan S., Reddy. J. N.: A new beam finite element for the analysis of functionally graded materials, *Int. J. Mech. Sci.* 45 (2003) 519–539.
4. GhannadPour. S. A. M., Alinia. M. M.: Large deflection behavior of functionally graded plates under pressure loads, Compos. Struct. 75(1–4) (2006) 67–71.
5. GhannadPour. S. A. M., Hajikazemi. M., Ovesy. H. R.: Large deflection behavior of functionally graded plates under pressure loads, proceeding of the First international conference on composites (CCFA-1),Kish island, IRAN , (15-18 Dec. 2008).
6. Yang. J., Shen. H. S.: Non-linear analysis of functionally graded plates under transverse and in-plane loads. *Int. J. Non-Linear Mech*. 38 (2003) 467–482.
7. Sadr. M. H., Hajikazemi. M., Ramezani-Oliaee. M.: Large deflection behavior of relatively thick functionally graded plates under pressure loads using higher order shear deformation theory. Key Engineering Materials Vols. 471-472 (2011) pp 709-714.
8. Markworth. A. J., Ramesh. K. S., Parks. Jr. W.P: Modeling studies applied to functionally graded materials. Journal of Material Science 1995; 30: 2183–93.
9. Ma. L. S, Wang. T.J.: Relationships between axisymmetric bending and buckling solutions of FGM circular plates based on third-order plate theory and classical plate theory. International Journal of Solids and Structures 2004; 41: 85-101.
10. Reddy. J.N.: Mechanics of laminated composite plates and shells theory and analysis, CRC press, 2004.