# On a new type of recurrent sequences of primes - ACPOW chains

Marius Coman
Bucuresti, Romania
email: mariuscoman13@gmail.com

**Abstract.** An interesting type of recurrent sequences of primes which could eventually lead to longer chains of successive primes than known Cunningham chains or CPAP's . Few conjectures including a stronger version of Legendre's conjecture and one regarding the Fermat primes. A classification of the set of primes.

I was studying the numbers of the form b = 9 + 6*(10*a + 1), where 10*a + 1 prime, when I noticed that, for many of these numbers b and for many consecutive values of c, the number b – 2^c is a prime.
For instance, for a = 1 we get b = 75 and 75 – 2 = 73 (prime), 75 – 2^2 = 71 (prime), 75 – 2^3 = 67 (prime), 75 – 2^4 = 59 (prime), 75 – 2^5 = 43 (prime), 75 – 2^6 = 11 (prime), 75 – 2^7 = -53 (prime in absolute value), 75 – 2^8 = -181 (prime in absolute value).
This is how I discovered the potential of the recurrent sequences of primes of the type $p_0$ + 2^x = $p_1$, $p_1$ + 2^(x – 1) = $p_2$ ,..., $p_{x-1}$ + 2 = $p_x$, where $p_0$, $p_1$, ..., $p_x$ are integers, primes in absolute value.

**Descending ACPOW chains**

I so define a ***descending proper ACPOW chain*** of primes the recurrent sequence of primes that I expose it above.

Note: I name these chains of primes ACPOW-k chains, abbreviation from chains of k primes obtained "adding consecutive powers".

Example: the sequence -181, -53, 11, 43, 59, 67, 71, 73 is a descending ACPOW-8 chain of primes (in absolute value), because -181 + 2^7 = -53, -53 + 2^6 = 11, 11 + 2^5 = 43, 43 + 2^4 = 59, 59 + 2^3 = 67, 67 + 2^2 = 71, 71 + 2 = 73. It can be seen that the numbers of the terms from the sequence is equal to the power of 2 which we add to the first term plus 1.

I also define a ***descending improper ACPOW chain*** of primes the recurrent sequence of primes of the type $p_0$ + 2^x =

$p_1$, $p_1 + 2^{(x - 1)} = p_2$ ,..., $p_{x-y} + 2^y = p_{x-y+1}$, where $0 < y < x$.

Example: the sequence -1433, -409, 103, 359, 487 is a descending improper ACPOW-5 chain of primes (in absolute value), because -1433 + 2^10 = -409, -409 + 2^9 = 103, 103 + 2^8 = 359, 359 + 2^7 = 487.

Note: I name these chains "improper" because the last term is not equal to antepenultimate one plus 2, but plus a power of 2 bigger than 1 (indeed, in the example above, 487 + 2^6 is no longer prime).

I list below few of these chains that I discovered, with a minimum lengh 5 and maximum length 8 (because of the special nature of the number 1, I listed also the chains that include it):

Descending proper ACPOW chains:
: 1423, 1439, 1447, 1451, 1453;
: -181, -53, 11, 43, 59, 67, 71, 73.

Descending improper ACPOW chains:
: -769, -257, -1, 127, 191, 223, 239;
: -829, -317, -61, 67, 131, 163, 179;
: -1433, -409, 103, 359, 487.

Note: It can be seen the chains I exposed can't be extended neither before the first term nor further than the last one, so they are "complete" (in the same manner a Cunningham chain is "complete"); a descending proper ACPOW chain is complete "to the right" by definition.

Comment: I generalize these chains to the sequences as the ones I described above but allowing beside the powers of 2 the adding with the powers of any even number, in both proper/improper versions.

We note these chains with the abbreviation **ACPOW(m)-k**, meaning "a chain of k primes obtained adding consecutive powers of m". With this notation, an ACPOW-k is equivalent to an ACPOW(2)-k.

Example: The sequence 11, 1307, 1523, 1559 would be a descending improper ACPOW(6)-4 chain, cause 11 + 6^4 = 1307, 1307 + 6^3 = 1523, 1523 + 6^2 = 1559 (and 1559 + 6 is no longer a prime, neither 11 – 6^5 is not a prime in absolute value).

**Ascending ACPOW chains**

I define an ***ascending proper ACPOW chain*** of primes the recurrent sequence of primes of the type $p_0 + 2 = p_1$, $p_1 + 2^2 = p_2$ ,..., $p_{x-1} + 2^x = p_x$.

Note: An ascending proper ACPOW chain is complete (in the sence a Cunningham chain is "complete") "to the left" by definition.

It can be seen that an ascending proper ACPOW-k chain with k > 3 can start only with a prime having last digit 7 (if would be 3, the second term would be divizible by 5, it would be 9 the third term would be divisible by 5, if would be 1 the forth term would be divisible by 5). If the first term is 7, the last digit of the next terms would be 9, 3, 1 and then again 7, 9, 3, 1 repeatedly.

Example: the sequence 17, 19, 23, 31, 47, 79 is an ascending proper ACPOW-6 chain. It can be seen that the numbers of the terms from the sequence is equal to the power of 2 which we add to the last term plus 1.

I also define an ***ascending improper ACPOW chain*** of primes the recurrent sequence of primes of the type $p_0 + 2^x = p_1$, $p_1 + 2^{(x + 1)} = p_2$ ,..., $p_{y-1} + 2^{(x + y - 1)} = p_y$.

Note: I named these chains "improper" because the second term is not equal to first one plus 2, but plus a power of 2 bigger than 1. Such a chain is complete to the left if $p_0 - 2^{(x - 1)}$ is not a prime.

It can easily be proved that there are just the following posibilities for an ascending improper ACPOW chain to have the length bigger than 3:

: If the first term has the last digit 1, then the power of 2 added to it must be of the form 4*i; also first term can be only of the form 30*j + 1 (example of such a chain of length 4: 151, 167, 199, 263);

: If the first term has the last digit 3, then the power of 2 added to it must be of the form 4*i + 3; also first term can be only of the form 30*j + 23 (example of such a chain of length 7: 173, 181, 197, 229, 293, 421, 677);

: If the first term has the last digit 7, then the power of 2 added to it must be of the form 4*i + 1; also first term can be only of the form 30*j + 17 (example of such a chain of length 5: 617, 1129, 2153, 4201, 8297);

: If the first term has the last digit 9, then the power of 2 added to it must be of the form 4*i + 2; also first term can be only of the form 30*j + 19.

Comments:
: I also generalize these chains in the same manner: the sequence 11, 17, 53, 269 is an ascending proper ACPOW(6)-4 chain, because 11 + 6 = 17, 17 + 6^2 = 53, 53 + 6^3 = 269;
I used the terms "proper" and "ascending" to exhaustively define the concept, but it would be better that those two terms to be implied and only the terms "improper" and "descending" to be mentioned.

**Another generalization of ACPOW chains**

Note: I define further more ACPOW chains only for "proper" and "ascending" sences, the implied ones.

I define an ***ACPOW(2,n)-k chain*** of primes the recurrent sequence of k primes: $p_0$, $p_1 = p_0 + 2^n$, $p_2 = p_1 + 2^(2*n)$,..., $p_x = p_{x-1} + 2^(x*n)$.

I define an ***ACPOW(m,n)-k chain*** of primes the recurrent sequence of k primes: $p_0$, $p_1 = p_0 + m^n$, $p_2 = p_1 + m^(2*n)$,..., $p_x = p_{x-1} + m^(x*n)$.

Note: With this notation, an ACPOW-k is equivalent to an ACPOW(2,1)-k.

Examples of ACPOW(2,n)-k chains:

ACPOW(2,3)-4: 29, 37, 101, 613,
where 29 + 2^3 = 37, 37 + 2^(3*2) = 101, 101 + 2^(3*3) = 613;
ACPOW(2,3)-4: 59, 67, 131, 643.

**Quasi-ACPOW chains or QACPOW chains**

I define a ***quasi-ACPOW chain*** of primes or a ***QACPOW*** a proper or improper, ascending or descending ACPOW chain but with the difference that the terms of the series are not necessary primes but primes or squares of primes.

Examples:
107, 109, 113, 11^2, 137, 13^2, 233, 19^2, 617, 1129, 2153, 4201, 8297 is a QACPOW-13
(because: 107 + 2 = 109; 109 + 2^2 = 113; 113 + 2^3 = 121; 121 + 2^4 = 137; 137 + 2^5 = 169; 169 + 2^6 = 233; 233 + 2^7 = 361; 361 + 2^8 = 617; 617 + 2^9 = 1129; 1129 + 2^10 = 2153; 2153 + 2^11 = 4201; 4201 + 2^12 = 8297);

167, 13^2, 173, 181, 197, 229, 293, 421, 677 is QACPOW-9;

227, 229, 233, 241, 257, 17^2, 353 is a QACPOW-7.

*Observations about QACPOW chains and the difference between squares of consecutive primes:*

5^2, 41, 7^2
is a descending QACPOW because
25 + 2^4 = 41, 41 + 2^3 = 49
(incomplete chain, because -199, -71, -7, 5^2, 41, 7^2, 53 is the complete improper chain);

7^2, 131, 11^2
is a descending QACPOW(2,3) because
49 + 2^(3*2) = 113, 113 + 2^3 = 121
(incomplete chain, because -463, 7^2, 131, 11^2 is the complete improper chain);

11^2, 41, 13^2
is an ascending QACPOW because
121 + 2^4 = 137, 137 + 2^5 = 169
(incomplete chain, because 107, 109, 113, 11^2, 137, 13^2, 233, 19^2, 617, 1129, 2153, 4201, 8297 is the complete proper chain).

**Conjecture 1:** Between any two squares of consecutive odd primes $p_1$^2 and $p_2$^2 there are at least $p_2 - p_1$ prime numbers that can be written as $p_1$^2 + sum 2^x, where x from i to j, j ≥ i ≥ 1, where x, i, j positive integers.

Check for few first primes:

3^2 + 2 = 11 prime;
3^2 + 2 + 2^2 + 2^3 = 23 prime;
3^2 + 2^2 = 13 prime;
3^2 + 2^3 = 17 prime.

Note: between 3^2 and 5^2 there are 4 such primes.

5^2 + 2 + 2^2 = 31 prime;
5^2 + 2^2 = 29 prime;
5^2 + 2^2 + 2^3 = 37 prime;
5^2 + 2^4 = 41 prime.

Note: between 5^2 and 7^2 there are 4 such primes.

7^2 + 2 + 2^2 + 2^3 + 2^4 = 79 prime;
7^2 + 2^2 = 53 prime;
7^2 + 2^2 + 2^3 = 61 prime;
7^2 + 2^2 + 2^3 + 2^4 + 2^5 = 109 prime;
7^2 + 2^3 + 2^4 = 73 prime;
7^2 + 2^4 + 2^5 = 97 prime;
7^2 + 2^6 = 113 prime.

Note: between 7^2 and 11^2 there are 7 such primes.

11^2 + 2 + 2^2 = 127 prime;
11^2 + 2 + 2^2 + 2^3 + 2^4 = 151 prime;
11^2 + 2^2 + 2^3 + 2^4 = 149 prime;
11^2 + 2^4 = 137 prime.

Note: between 11^2 and 13^2 there are 4 such primes.

23^2 + 2^2 + 2^3 = 541 prime;
23^2 + 2^2 + 2^3 + 2^4 = 557 prime;
23^2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = 653 prime;
23^2 + 2^4 + 2^5 = 577 prime;
23^2 + 2^4 + 2^5 + 2^6 = 641 prime;
23^2 + 2^6 = 593 prime.

Note: between 23^2 and 29^2 there are 6 such primes.

Comment: it’s obviously that between two squares of consecutive primes, $p_1$^2 and $p_2$^2, there is not necessarily a prime of the form $p_1$^2 + 2 + 2^2 +...+ 2^h. Example: none of the numbers y = 23^2 + sum 2^x, where y < 29^2 (*i.e.* 531, 535, 543, 559, 591, 655, 783) is prime.

**Conjecture 2**: Between any two squares of odd primes p^2 and q^2, where p < q, there are at least (q – p)/2 + 4 prime numbers that can be written as p^2 + sum 2^x, where x from i to j, j ≥ i ≥ 1, where x, i, j positive integers.

Note:
: between 3^2 and 5^2 there are 4 such primes (*i.e.* 11, 13, 17, 23);
: between 3^2 and 7^2 there are 6 such primes (*i.e.* 11, 13, 17, 23, 37, 41);
: between 3^2 and 11^2 there are 8 such primes (*i.e.* 11, 13, 17, 23, 37, 41, 71, 73);
: between 3^2 and 13^2 there are 9 such primes (*i.e.* 11, 13, 17, 23, 37, 41, 71, 73, 137);
: between 3^2 and 17^2 there are 12 such primes (*i.e.* 11, 13, 17, 23, 37, 41, 71, 73, 137, 233, 257, 263);
: between 3^2 and 19^2 there are 12 such primes (*i.e.* 11, 13, 17, 23, 37, 41, 71, 73, 137, 233, 257, 263).

**Conjecture 3**: Between any two squares of consecutive numbers a^2 and (a + 1)^2 there are at least 2 prime numbers that can be written as a^2 + sum 2^x, where x from i to j, j ≥ i ≥ 1, where x, i, j positive integers, if a is odd or can be written as (a + 1)^2 – sum 2^x, where x from i to j, j ≥ i ≥ 1, where x, i, j positive integers, if a is even.

Comment: this is a stronger version of Legendre’s conjecture.

Check for few first consecutive numbers and few randomly chosen ones:

3^2 - 2 = 7 prime;
3^2 – 2^2 = 5 prime.

Note: between 2^2 and 3^2 there are 2 such primes.

3^2 + 2 = 11 prime;
3^2 + 2^2 = 13 prime.

Note: between 3^2 and 4^2 there are 2 such primes.

5^2 - 2 = 23 prime;
5^2 – (2 + 2^2) = 19 prime.

Note: between 4^2 and 5^2 there are 2 such primes.

5^2 + 2 + 2^2 = 31 prime;
5^2 + 2^2 = 29 prime.

Note: between 5^2 and 6^2 there are 2 such primes.

7^2 - 2 = 47 prime;
7^2 – (2 + 2^2) = 43 prime.

Note: between 6^2 and 7^2 there are 2 such primes.

23^2 + 2^2 + 2^3 = 541 prime;
23^2 + 2^2 + 2^3 + 2^4 = 557 prime.

Note: between 23^2 and 24^2 there are 2 such primes.

561^2 + 2 = 314723 prime;
561^2 + 2 + 2^2 +...+ 2^9 = 315743 prime.
561^2 + 2^3 + 2^4 + 2^5 = 314777 prime.
561^2 + 2^7 + 2^8 + 2^9 = 315617 prime.

Note: between 561^2 and 562^2 there are 4 such primes.

Comment: we could introduce the notion “general”, and note the new series with **GACPOW(m,f(x))**, for the series such that were already defined but where the exponents of 2 (or of a number m) are not the consecutive numbers 1, 2, 3, (...) neither the consecutive products 1*n, 2*n, 3*n, (...), but consecutive values of a bijective function f(x), where x and f(x) are positive integers.

**ACPOW pairs of primes**

I name an ACPOW pair of primes two primes of the form: [p + 2^m – 2, p + 2^n – 2], where p prime and n > m > 0.

Observations:

: for p = 3 the pair becomes [2^m + 1, 2^n + 1] so the only terms of the pair can be only Fermat primes; as the only known Fermat primes are 3, 5, 17, 257, 65537, we have the only known ACPOW pairs of primes having the first term 3: [3,5], [3,17], [3,257], [3,65537]; of course, [5,17], [5,257], [5,65537], [17,257], [17,65537], [257,65537] are also ACPOW pairs of primes;

: for p = 5 the pair becomes [2^m + 3, 2^n + 3] and the first few ACPOW pairs of primes having the first term 5, generated for p = 5, are: [5,7], [5,11], [5,19], [5,67], [5,131], [5,4099];

: for p = 7 the pair becomes [2^m + 5, 2^n + 5] and the first few ACPOW pairs of primes having the first term 7, generated for p = 7, are: [7,13], [7,37], [7,2053], [7,140737488355333], [7,9007199254740997];

: for p = 11 the pair becomes [2^m + 9, 2^n + 9] and the first few ACPOW pairs of primes having the first term 11, generated for p = 11, are: [11,13], [11,17], [11,41], [11,73], [11,137], [11,521], [11,1033], [11,262153], [11,8388617];

: for p = 13 the pair becomes [2^m + 11, 2^n + 11] and the first few ACPOW pairs of primes having the first term 13, generated for p = 13, are: [13,19], [13,43], [13,139], [13,523], [13,32779], [13,8388619].

: for p = 17 the pair becomes [2^m + 15, 2^n + 15] and the first few ACPOW pairs of primes having the first term 17, generated for p = 17, are: [17,19], [17,23], [17,31], [17,47], [17,79], [17,271], [17,1039], [17,2063], [17,4111].

*Properties of ACPOW pairs of primes:*

(1) First we notice that these pairs seems to be relatively rare for a given p as the value of n is growing (especially for p = 7, from the cases that I considered).

(2) The series of twin primes and the series of cousin primes are subsets of the series of ACPOW pairs of primes.

(3) At the numbers of the form N = 2^k + 9 we noticed something interesting: for k of the form k = 24*h – 1, N has frequently only few prime factors:

N = 2^191 + 9 (where 191 = 8*24 – 1) has just 3 prime factors;
N = 2^215 + 9 (where 215 = 9*24 – 1) has just 3 prime factors;
N = 2^239 + 9 (where 239 = 10*24 – 1) has just 3 prime factors;
N = 2^263 + 9 (where 263 = 11*24 – 1) is a prime number with 80 digits!

(4) If [q,r] and [q + 2,r + 2] are both ACPOW pairs of primes, then we have some interesting values for the numbers r – q – 2, but we didn't gather enough data to jump with conclusions.

(5) If [q,r] is an ACPOW pair of primes, and both q and r have the last digit 7, the numbers q*r – q - 1 and q*r – r – 1 are often primes or products of few primes; we name these numbers the pair affiliated to an ACPOW pair of such primes (having the last digit 7).

: for [17,257] the affiliated pair is [4111,19*229];
: for [17,65537] is [19*229*241,1114111];
: for [257,65537] is [3079*5449,16842751];
: for [7,67] is [401,461];
: for [7,37] is [13*17,251];
: for [7,9007199254740997] is
[11*13*17*22230849662051,23*2741321512312477];
: for [17,137] is [7*313,2311];
: for [17, 8388617] is [659*203669,7*61*333973].

**Conjecture 4:** I conjecture (against the heuristic arguments of Hardy and Wright) that are infinite many Fermat primes, that they are all, beside the first two terms, 3 and 5, of the form 10*k + 7, and that the all terms $F_n$, beginning with the term $F_4$ = 257, satisfy the relations:

(1) one of the numbers 17*$F_n$ – 18 and 17*$F_n$ – $F_n$ – 1 is a prime number having the last four digits 4111;
(2) the other one of the numbers 17*$F_n$ – 18 and 17*$F_n$ – $F_n$ – 1 is a product of primes as it follows: 19*229 for n = 4, 19*229*241 for n = 5, 19*229*241*p, where p prime, for n = 6 and so on (for $F_n$ corresponds a number of n - 2 primes).

**ACPOW chains of the second kind**

I define ***ACPOW chains of the second kind*** the recurrent sequences of primes of the type $p_n = p_{n-1} + 2^k$, where k has the smaller value for that $p_{n-1} + 2^k$ is a prime.

Examples:

: 3, 5, 7, 11, 13, 17, 19, 23, 31, 47, 79, 83, 211 (...);

: 29, 37, 41, 43, 59, 67, 71, 73, 89, 97, 101, 103 (...);

: 53, 61, 317 (...);

: 127, 131, 139, 1163 (...);

: 137, 139, 1163 (...).

We clasify the set of prime numbers in the following manner:
: a prime is of class ACPOW I if can be written as $p_n = p_{n-1} + 2^k$, where $p_n = 3$;
: a prime is of class ACPOW II if can be written as $p_n = p_{n-1} + 2^k$, where $p_n$ is the first prime that doesn't belong to the first class;
: a prime is of class ACPOW III if can be written as $p_n = p_{n-1} + 2^k$, where $p_n$ is the first prime that doesn't belong to the first and the second class,
and so on.

Comment: the clasification is not exclusive: a prime can belong to more than one class.

Note: to the end of the article an addenda will cover the division into classes of the first few primes from the first few classes.

We name the **ACPOW problems** the following two questions:
: is there or not an infinite number of such classes?
: is there such a class with a finite number of terms?

**Conclusion:** we end here for now this analysis, hoping the we highlighted some of the possible applications of ACPOW chains of primes and ACPOW pairs of primes.

## ADDENDA

*Primes from class ACPOW I*

3, 5, 7, 11, 13, 17, 19, 23, 31, 47, 79, 83, 211, 227, 229, 233, 241, 257, 769, 773 (...).

Note: we didn't find a prime of the form 773 + 2^n for n from 1 to 100 (773 + 2^100 is a number with 31 digits).

*Primes from class ACPOW II*

29, 37, 41, 43, 59, 67, 71, 73, 89, 97, 101, 103, 107, 109, 113, 241, 257 (...).

Note: we can see that from the number 241 class ACPOW II has the same terms with class ACPOW I.

*Primes from class ACPOW III*

53, 61, 317, 349, 353, 134218081 (...).

*Primes from class ACPOW IV*

127, 131, 139, 1163, 1171, 1187, 1699, 263843 (...).

*Primes from class ACPOW V*

137, 139, 1163 (...).

Note: we can see that from the number 139 class ACPOW V has the same terms with class ACPOW IV.

*Primes from class ACPOW VI*

149, 151, 167, 199, 263, 271, 1048847 (...).

*Primes from class ACPOW VII*

157, 173, 181, 197, 199, 263 (...).

Note: we can see that from the number 199 class ACPOW VII has the same terms with class ACPOW VI.

*Primes from class ACPOW VIII*

163, 167, 199, 263 (...).

Note: we can see that from the number 199 class ACPOW VIII has the same terms with class ACPOW VI.

*Primes from class ACPOW IX*

179, 181, 197 (...).

Note: we can see that from the number 181 class ACPOW IX has the same terms with class ACPOW VII.

*Primes from class ACPOW X*

191, 193, 197, 199, 263 (...).

Note: we can see that from the number 199 class ACPOW X has the same terms with class ACPOW VI.

*Primes from class ACPOW XI*

223, 227, 229 (...).

Note: we can see that from the number 227 class ACPOW XI has the same terms with class ACPOW I.

*Primes from class ACPOW XII*

239, 241, 257 (...).

Note: we can see that from the number 241 class ACPOW XII has the same terms with class ACPOW I.

*Primes from class ACPOW XIII*

251, 283, 347, 349, 353, 134218081 (...).

Note: we can see that from the number 353 class ACPOW XIII has the same terms with class ACPOW III.

*Primes from class ACPOW XIV*

269, 271, 1048847 (...).

Note: we can see that from the number 271 class ACPOW XIV has the same terms with class ACPOW VI.

*Primes from class ACPOW XV*

277, 281, 283, 347 (...).

Note: we can see that from the number 283 class ACPOW XV has the same terms with class ACPOW XIII.

*Primes from class ACPOW XVI*

293, 421, 677, 709, 773 (...).

Note: we can see that from the number 773 class ACPOW XVI has the same terms with class ACPOW I.

*Primes from class ACPOW XVII*

307, 311, 313, 317, 349, 353, 134218081 (...).

Note: we can see that from the number 353 class ACPOW XVII has the same terms with class ACPOW III.

*Primes from class ACPOW XVIII*

331, 347, 349 (...).

Note: we can see that from the number 349 class ACPOW XVIII has the same terms with class ACPOW XIII.

*Primes from class ACPOW XIX*

337, 353, 134218081 (...).

Note: we can see that from the number 349 class ACPOW XVIII has the same terms with class ACPOW XIII.

Observations:
: the classes I, II, XI, XII, XVI are converging to the same terms, also the classes III, XIII, XV, XVII and XIX, the classes IV and V, the classes VI, VII, VIII, IX, X and XIV;
: some primes seems to highlight as ***convergence primes for classes ACPOW***: five classes converge to the prime 353, four to the prime 199, three to the prime 241, two to the primes 139, 181, 227, 271, 283, 349, 773.