Drude-Schwarzschild Metric and the Electrical Conductivity of Metals

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ABSTRACT

Starting from a string with a length equal to the electron mean free path and having a unit cell equal to the Compton length of the electron, we construct a Schwarzschild-like metric. We found that this metric has a surface horizon equal to that of the electron mean free path and its Bekenstein-like entropy is proportional to the number of squared unit cells contained in this spherical surface. The Hawking temperature goes with the inverse of the perimeter of the maximum circle of this sphere. Besides this, interesting analogies are traced out with some features of the particle physics.

1 – Introduction

Drude model of the electrical conductivity of metals [1,2], considers that in this medium the free electrons (the electrons of conduction) undergo Brownian motion with an average characteristic time τ between collisions. Due to the Pauli's exclusion principle, only the electrons with energies which are close to the Fermi energy participate of the conduction phenomena. These electrons travel freely on average by a distance called electron mean free path equal to $\ell = v_F \tau$, where v_F is the Fermi velocity.

Meanwhile, let us pay attention to the following feature of the black hole physics [3]: an observer sited at a distance greater than R_S (the Schwarzschild or the surface horizon radius) of the black hole, can not observe any process occurring inside the region delimited by this surface.

Going back to the phenomena of the electrical conductivity in metals let us as follow for instance in a copper crystal an electron of the conduction band which just suffered a collision. In the absence of an external electric field, al the directions in the space have equal probability to be chosen in a starting new free flight. Therefore if we take a sphere centered at the point where the electron have been scattered, with radius equal to the electron mean free path, the surface of this sphere may be considered as an event horizon for this process. Any electron starting from this center will be, on average, scattered when striking the event horizon, loosing the memory of its previous free flight. Besides this, all lattice sites of the metallic crystal are treated on equal footing, due to the translational symmetry of the system.

This analogy between these two branches of the physics, namely, general relativity (GR) and the electrical conduction in metals (ECM), will be considered in the present work. As we will see, we are going to use the GR tools as a means to evaluate some basic quantities related to the ECM. But not only the GR tools will be here considered. We are also using some concepts related to the study of the particle lifetimes of the particle physics (PP).

2 – The electron mean free path as a Schwarzschild radius

Let us consider a string of length ℓ (coinciding with the electron mean free path), composed by N unit cells of size equal to the Compton wavelength of the electron (λ_C). Associating a relativistic energy pc to each of these cells, we have an overall kinetic energy K given by

$$K = N\, p\, c = (\ell / \lambda_C)\, p\, c = (\ell\, m\, c^2 / h)\, p. \qquad (1)$$

In a paper entitled: "Is the universe a vacuum fluctuation?", E. P. Tryon [4], considers a universe created from nothing, where half of the mass-energy of a created particle just cancels its gravitational interaction with the rest of matter in the universe. Inspired in the Tryon proposal we can write

$$K + U = 0, \qquad (2)$$

implying in

$$U = - K = - (\ell\, m\, c^2 / h)\, p. \qquad (3)$$

However we seek for a potential energy which depends on the radial coordinate r, and by using the uncertainty relation $p = h / r$, we get

$$U = - (m\, c^2\, \ell) / r. \qquad (4)$$

Next we deduce a metric, in the curved space, which is governed by the potential energy defined in (4). We follow the procedure established in reference [5]. A form of equivalence principle was propose by Derek Paul [6], and when it is applied to the potential energy (4) reads

$$\hbar\, d\omega = dU = (m\, c^2\, \ell)\, r^{-2}\, dr. \qquad (5)$$

Now we consider the de Broglie relation

$$\hbar\omega = 2\ m\ c^2. \tag{6}$$

Dividing (5) by (6) yields

$$d\omega / \omega = ½\ \ell\ r^{-2}\ dr. \tag{7}$$

Performing the integration of (7) between the limits ω_0 and ω, and between R and r, we get

$$\omega = \omega_0 \exp[\ -\ ½\ \ell\ (1/r - 1/R)], \tag{8}$$

and

$$\omega^2 = \omega_0^2 \exp[\ -\ \ell\ (1/r - 1/R)]. \tag{9}$$

Making the choice R = ℓ, leads to

$$\omega^2 = \omega_0^2 \exp(1 - \ell/R). \tag{10}$$

Then we construct the auxiliary metric

$$d\sigma^2 = \omega^2\ dt^2 - k^2\ dr^2 - r^2(\ d\theta^2 + \sin^2\theta\ d\varphi^2\ ). \tag{11}$$

In (11) we take k^2, such that

$$k^2/k_0^2 = \omega_0^2/\omega^2. \tag{12}$$

Relation (12) is a reminiscence of the time dilation and space contraction of the special relativity.
Now we seek for a metric which becomes flat in the limit $r \to \infty$. This can be accomplished by defining [7]

$$w^2 = \ln(\ \omega^2/\omega_0^2\ ), \qquad \text{and} \qquad \kappa^2 = 1/w^2. \tag{13}$$

Making the above choices we can write

$$ds^2 = (\ 1 - \ell/r)\ dt^2\ -\ (\ 1 - \ell/r)^{-1}\ dr^2\ -\ r^2(\ d\theta^2 + \sin^2\theta\ d\varphi^2). \tag{14}$$

We observe that (14) express just the Schwarzschild metric, where ℓ is just the Schwarzschild radius of the system.

3 – A Schwarzschild-like metric

In the last section we deduced a metric where the Schwarzschild radius is just the conduction's electron mean free path. But that construction seems not to be totally satisfactory, once the viscous character of the fluid embedding the charge carriers has not yet been considered. By taking separately in account the effect of the viscous force, we can write

$$m\, dv/dt = - p/\tau^*. \tag{15}$$

In (15) τ^* is a second characteristic time, which differs from the first one τ that was defined in the previous section. Pursuing further we write

$$v\, dt = dr, \qquad \text{and} \qquad p = h/r. \tag{16}$$

Upon inserting (16) into (15), and multiplying (15) by v and integrating, we get the decreasing in the kinetic energy of the conduction's electron

$$\Delta K_{qt} = - (h/\tau^*) \ln(r/R), \tag{17}$$

where R is some radius of reference.
Next, by defining $\Delta U_{qt} = - \Delta K_{qt}$, we have the total potential energy U_t, namely

$$U_t = U + \Delta U_{qt} = - (m c^2 \ell)/r + (h/\tau^*) \ln(r/R). \tag{18}$$

In the next step, we consider the equivalence principle [6] and the de Broglie frequency to a particle pair, writing

$$dU/(2 m c^2) = d\omega/\omega = (\ell/2)(dr/r^2) + ½ (dr/r). \tag{19}$$

Upon integrating we get

$$\omega = \omega_0 \exp[- \ell/(2 r) + ½ \ln(e r/\ell)]. \tag{20}$$

In obtaining (20), we have also made the choices

$$m c^2 \tau^* = h, \qquad \text{and} \qquad r/ R = e r/\ell. \tag{21}$$

Squaring (20), yields

$$\omega^2 = \omega_0^2 \exp[- \ell/r + \ln(e r/\ell)]. \tag{22}$$

Defining

$$w^2 = \ln(\omega^2 / \omega_0^2), \qquad \text{and} \qquad \kappa^2 = 1 / w^2, \tag{23}$$

we finally get

$$ds^2 = [\ln(e\, r / \ell) - \ell / r]\, dt^2 - [\ln(e\, r / \ell) - \ell / r]^{-1} dr^2 - r^2\, d\Omega^2. \tag{24}$$

Relation (24) is just a Schwarzschild-like metric [8], which displays the same qualitative behavior than that describing the Schwarzschild geometry. We also have used in (24) a compact form of writing the solid angle differential, namely $d\Omega$ (please compare with the last term of eq. (11)).

4 – Average collision time as a particle lifetime

There are two characteristics linear momenta that we can associate to the free electrons responsible for the electrical conductivity of metals. They are the Fermi momentum mv_F and the Compton momentum mc. By taking into account the fermionic character of the electron, we will write a non-linear Dirac-like equation describing the "motion" of this particle. We have [8]

$$\partial\Psi / \partial x - (1/c)\, \partial\Psi / \partial t = [(m\, v_F) / \hbar]\, \Psi - [(mc) / \hbar]\, |\Psi^*\Psi| \Psi. \tag{25}$$

We see that eq. (25) contains only first order derivatives of the field Ψ. Besides this, the field Ψ exhibits not a spinorial character. Taking the zero of (25) and solving for $|\Psi^*\Psi|$, we get

$$|\Psi^*\Psi| = v_F / c. \tag{26}$$

On the other hand in the collision process, the conduction's electron loss its memory. We may think that this feature looks similar to the annihilation of a particle-anti particle pair, each of mass-energy equal to E_F. Putting this in a form of the uncertainty principle yields

$$2\, E_F\, \Delta t = h / 2 \qquad \text{or} \qquad h\, \upsilon / 2 = 2\, E_F. \tag{27}$$

Solving equation (27) for υ, we get

$$\upsilon = 1 / \Delta t = 4\, E_F / h. \tag{28}$$

By combining the results of (28) and (26) we obtain the line width Γ tied to the "particle" decay

$$\Gamma = \upsilon\, |\Psi^*\Psi| = 4\, E_F\, v_F / (h\, c). \tag{29}$$

The averaged time between collisions τ is then given by

$$\tau = 1/\Gamma = (\, h\, c\,)/(\, 4\, E_F\, v_F\,). \qquad (30)$$

Now, let us compare the two characteristics times appearing in this work. By considering (21) and (30), we get

$$\tau/\tau^* = ½\, (\, c/v_F\,)^3, \qquad (31)$$

and the electron mean free path

$$\ell = v_F\, \tau = ½\, (\, c/v_F\,)^2\, [h/(mc)]. \qquad (32)$$

It is also the place of evaluating the number of unit cells in the string of size ℓ. We have

$$N = \ell/\lambda_C = m\, c^2/(\, 4\, E_F\,). \qquad (33)$$

It is also possible to define an effective gravitational constant G_W as

$$\ell = 2\, G_W\, (Nm)/c^2 = G_W\, m^2/(\, 2\, E_F\,). \qquad (34)$$

Taking M = Nm, we can write

$$2\, G_W\;\; M\, /c^2\;\; = \ell = G_W\, m\, /v_F^2, \qquad (35)$$

which leads to

$$M = \; ½\, (\, c/v_F\,)^2\, m. \qquad (36)$$

In order to better numerically evaluate the quantities we described in this work, let us take

$$E_F = ¼\, \alpha^2\, m\, c^2. \qquad (37)$$

This value for E_F [eq.(37)], is representative of the Fermi energy of metals, namely it is close to the Fermi energy of the copper crystal. Using (37) as a typical value of E_F, we get

$$(c/v_F)^2 = 2/\; \alpha^2. \qquad (38)$$

Inserting (38) into the respective quantities we want to evaluate, we have

$$\ell = h/(\alpha^2 m c); \quad \tau = \sqrt{2}\, h/(\alpha^3 m c^2); \quad M = m/\alpha^2. \tag{39}$$

Putting numbers in (39) yields

$$\ell = 453 \text{ Å}; \quad \tau = 2.93 \times 10^{-14} \text{ s}; \quad M = 9590 \text{ MeV}/c^2. \tag{40}$$

It would be worth to evaluate the strength of G_W. We have

$$G_W M^2 \sim 10^{-8}\, \hbar c. \tag{41}$$

We notice that M is approximately equal to ten times the proton mass.

5 – The event horizon temperature and entropy

To obtain the Hawking [9,11,12] temperature of this model, we proceed following the same steps outlined in reference [5]. First, by setting $t \rightarrow i\tau$, we perform Wick rotation on the metric given by (24). We write

$$ds^2 = -(y\, d\tau^2 + y^{-1} dr^2 + r^2 d\Omega^2), \tag{42}$$

where y is given by

$$y = \ln(e r/\ell) - \ell/r. \tag{43}$$

Now, let us make the approximation

$$y^{1/2} \approx \ell^{-1/2} [r \ln(e r/\ell) - \ell]^{1/2} = \ell^{-1/2} u^{1/2}. \tag{44}$$

In the next step we make the change of coordinates

$$R\, d\alpha = \ell^{-1/2} u^{1/2} d\tau, \quad \text{and} \quad dR = \ell^{1/2} u^{-1/2} dr. \tag{45}$$

Upon integrating, taking the limits between 0 and 2π for α, from 0 to β for τ, and from ℓ to r for r, we get

$$R = \ell^{1/2} u^{1/2}, \quad \text{and} \quad R\, 2\pi = \ell^{-1/2} u^{1/2} \beta. \tag{46}$$

Finally from (46), we find the temperature T of the horizon of events, namely

$$T \equiv 1/\beta = 1/(2\pi\, \ell). \tag{47}$$

Once we are talking about event's horizon, it would be worth to evaluate the Bekenstein [10,11,12] entropy of the model. Let us write

$$\Delta F = \Delta U - T\, \Delta S. \tag{48}$$

In (48), we have the variations of the free energy F, the internal energy U, and the entropy S. In a isothermal process, setting $\Delta F = 0$, and taking $\Delta U = N\, m\, c^2$, and inserting T given by (47), we have

$$\Delta F = (\ell / \lambda_C)\, m\, c^2 - h\, c / (2\pi\, \ell)\, \Delta S = 0, \tag{49}$$

which leads to

$$\Delta S = 2\pi\, (\ell / \lambda_C)^2. \tag{50}$$

The entropy of the event's horizon is then (putting $S_0 = 0$)

$$S = S_0 + \Delta S = 2\pi\, (\ell / \lambda_C)^2. \tag{51}$$

Therefore the analogy developed in this work between the black hole physics and the electrical conductivity of metals seems to make sense. This feature was discussed in a previous paper [8] where the connection with the cosmological constant problem [13] has also been considered.

REFERENCES

[1] C. Kittel, Introduction to Solid State Physics, Fifth Edition, Wiley, 1976 - Ch. 6

[2] P. R. Silva, M. Sampaio, C. Nassif, M. C. Nemes, "Quantum conductance and electrical resistivity", Phys. Lett. A **358**, 358-362 (2006)

[3] T. Damour, arXiv:hep-th/0401160

[4] E. P. Tryon, Nature **246,** 396(1973)

[5] P. R. Silva, arXiv:1302.1491v1

[6] D. Paul, Amer. Jour. Phyics **48**, 283(1980)

[7] Ted Jacobson, arXiv:0707.3222v3[gr-qc]

[8] P. R. Silva, viXra:1209.0071

[9] S. W. Hawking, Commun. Math. Phys. **43**,199(1975).

[10] J. D. Bekenstein, Phys. Rev. D **7**, 2333(1973).

[11] A. Zee, Quantum Field Theory in a Nutshell, Princeton University Press, 2003.

[12] P. R. Silva, arXiv:gr-qc/0605051v1

[13] S. Hsu and A. Zee, arXiv:hep-th/0406142