Convergence of Quadratic Sequences

Roberto Luis Recalde Gutiérrez

rlrecalde@yahoo.com.ar

November 27, 2013

Abstract

In this paper, we intend to build Quadratic Sequences which converge to a number in the form $K\sqrt{2}$ or a similar form.

1 Introduction

Quadratic Sequences seem to be hard to solve. They have a chaotic characteristic in general. But in this paper, we are going to be able to find out how to build some of them, which converge to a specific number.

First of all, we'll start to think about the "limit definition of the derivative" and how can it help us to approximate values. More precisely, if we have a function $f_{(x)} = \sqrt{x}$, how can we compute the value of the square root of a number that is not a perfect square.

Once we have the approximation, we are going to study the "error" we made in it, given by the limit definition of the derivative.
Next, we'll try to find out the "error function" that makes this value an approximate value.

With the error function in our hands, we are going to build a Sequence - based on the first approach of the square root - which converges to the square root or a number in the form $k\sqrt{2}$.

2 The limit definition of the derivative

2.1 Building the Error Function for $f_{(x)} = \sqrt{x}$

Taking the Limit Definition of the Derivative:

$$f'_{(x)} = \lim_{h \to 0} \frac{f_{(x+h)} - f_{(x)}}{h}$$

We can say:

$$f_{(x+h)} \cong f_{(x)} + hf'_{(x)}$$

And the smaller h, the more precise the approximation.

Let:
$f_{(x)} = \sqrt{x}$

Let's suppose that we have a perfect square and, therefore, we know its square root.

$b = \sqrt{a}$ $\qquad$ $b \in \mathbb{N}; a \in \mathbb{N}$

But also we have an integer that is not a perfect square and we want to calculate the approximate value of its square root.

$d = \sqrt{c}$ $\qquad$ $d \in \mathbb{R}; c \in \mathbb{N}$

Let's assume: $c > a$
Then:

$c = a + h$ $\qquad$ $h \in \mathbb{N}$

This means: $d > b$
Then:

$d = b + m$ $\qquad$ $m \in \mathbb{R}$

Now we can do the following replacements:

$$\sqrt{c} = \sqrt{a} + m$$

$$c = \left(\sqrt{a} + m\right)^2 = a + 2m\sqrt{a} + m^2$$

But we said that:
$c = a + h$

Then:
$h = c - a$

$$h = a + 2m\sqrt{a} + m^2 - a$$

$$h = 2m\sqrt{a} + m^2$$

On the other hand, the derivative of $f_{(x)}$ is:

$$f'_{(x)} = \frac{d}{dx}\sqrt{x}$$

$$f'_{(x)} = \frac{1}{2\sqrt{x}}$$

$$f'_{(x)} = \frac{1}{2f_{(x)}}$$

Hence, we can say that:

$$f_{(c)} = f_{(a+h)} \cong f_{(a)} + hf'_{(a)}$$

$$f_{(c)} \cong f_{(a)} + \frac{2m\sqrt{a} + m^2}{2f_{(a)}}$$

But:

$$m = d - b$$

$$m = \sqrt{c} - \sqrt{a}$$

$$m = f_{(c)} - f_{(a)}$$

Then:

$$f_{(c)} \cong f_{(a)} + \frac{2\left(f_{(c)} - f_{(a)}\right)f_{(a)} + \left(f_{(c)} - f_{(a)}\right)^2}{2f_{(a)}}$$

$$f_{(c)} \cong f_{(a)} + f_{(c)} - f_{(a)} + \frac{\left(f_{(c)} - f_{(a)}\right)^2}{2f_{(a)}}$$

$$f_{(c)} \cong f_{(c)} + \frac{\left(f_{(c)} - f_{(a)}\right)^2}{2f_{(a)}} \qquad \textbf{\textit{Eq. 1}}$$

And here we could find the “error” that we made when we did the approximation with the formula of the Derivative.

$$e = \frac{\left(f_{(c)} - f_{(a)}\right)^2}{2f_{(a)}} \qquad \textbf{\textit{Eq. 2}}$$

2.2 The Error Function as a function of *c*.

Let’s suppose that $c = 2a$.

(This is a particular case of how to get the approximate value of $f_{(c)}$, but it will help us to build a nicely sequence. However, we can make as many assumptions as we like, and still get a sequence).

So, let’s rewrite the error function as a function of c.

$$f_{(a)} = f_{\left(\frac{c}{2}\right)}$$

But:

$$f_{(x)} = \sqrt{x}$$

So:

$$f_{(a)} = \frac{f_{(c)}}{\sqrt{2}}$$

Then:

$$e_{(c)} = \frac{\left(f_{(c)} - \frac{f_{(c)}}{\sqrt{2}}\right)^2}{\frac{2f_{(c)}}{\sqrt{2}}}$$

$$e_{(c)} = \frac{\left(\frac{\sqrt{2}f_{(c)} - f_{(c)}}{\sqrt{2}}\right)^2}{\sqrt{2}f_{(c)}}$$

$$e_{(c)} = \frac{\frac{\left(f_{(c)}\left(\sqrt{2}-1\right)\right)^2}{2}}{\sqrt{2}f_{(c)}}$$

$$e_{(c)} = \frac{f_{(c)}^2\left(\sqrt{2}-1\right)^2}{2\sqrt{2}f_{(c)}}$$

$$e_{(c)} = f_{(c)}\left(\frac{\left(\sqrt{2}-1\right)^2}{2\sqrt{2}}\right)$$

3 Making approximations to build a Sequence

3.1 The first approach

Now we have our error function as a function of c, the first approach of $f_{(c)}$ will be:

$$f_{(c)_0} = f_{(c)} + e_{(c)}$$

$$f_{(c)_0} = f_{(c)} + f_{(c)}\left(\frac{\left(\sqrt{2}-1\right)^2}{2\sqrt{2}}\right)$$

$$f_{(c)_0} = f_{(c)}\left(1 + \frac{\left(\sqrt{2}-1\right)^2}{2\sqrt{2}}\right)$$

$$f_{(c)_0} = f_{(c)}\frac{3}{2\sqrt{2}} \qquad \textbf{\textit{Eq. 3}}$$

So, using the definition of the Derivative, we found in ***Eq. 3*** the first approach of $f_{(c)}$.

3.2 The subsequent approaches

From now, we are going to find better approximations of $f_{(c)}$, replacing the previous approximation into the error function of the new approximation.
In other words, we are going to build a Sequence which converges to $f_{(c)}$.

To do this, we will take the first approach and subtract a new error value each time. (Think about the first approach: It has already got an "error". So, the idea is to subtract new error values to be able to approach better and better to $f_{(c)}$).
The subsequent values of $f_{(c)}$ will be:

$$f_{(c)_1} = f_{(c)}\frac{3}{2\sqrt{2}} - e_{(c)_0}$$

$$f_{(c)_2} = f_{(c)}\frac{3}{2\sqrt{2}} - e_{(c)_1}$$

$$f_{(c)_3} = f_{(c)}\frac{3}{2\sqrt{2}} - e_{(c)_2}$$

And so on…
Where $e_{(c)_n}$ is a function of $f_{(c)_n}$.

But this doesn't prove that the subsequent values will be better approaches! It's just a recurrent formula that we don't know if it is useful.

Let's work a little more.
Given $e_{(c)}$ is a function of $f_{(c)}$, we can say:

$$f_{(c)_n} = f_{(c)}k_n$$

Where k_n is a factor that produces each approximation of $f_{(c)}$.

Then, using ***Eq. 2*** for the error function:

$$f_{(c)_n} = f_{(c)}\frac{3}{2\sqrt{2}} - \frac{\left(f_{(c)_{n-1}} - f_{(a)}\right)^2}{2f_{(a)}}$$

$$f_{(c)_n} = f_{(c)}\frac{3}{2\sqrt{2}} - \frac{\left(f_{(c)_{n-1}} - \frac{f_{(c)}}{\sqrt{2}}\right)^2}{\frac{2f_{(c)}}{\sqrt{2}}}$$

$$f_{(c)_n} = f_{(c)}\frac{3}{2\sqrt{2}} - \frac{\left(\frac{\sqrt{2}f_{(c)_{n-1}} - f_{(c)}}{\sqrt{2}}\right)^2}{\sqrt{2}f_{(c)}}$$

$$f_{(c)_n} = f_{(c)}\frac{3}{2\sqrt{2}} - \frac{2f_{(c)_{n-1}}^{\ 2} - 2\sqrt{2}f_{(c)_{n-1}}f_{(c)} + f_{(c)}^2}{2\sqrt{2}f_{(c)}}$$

Now, we are going to introduce k_n in $f_{(c)_n}$ on the right side of the equation:

$$f_{(c)_n} = f_{(c)}\frac{3}{2\sqrt{2}} - \frac{2k_{n-1}^2 f_{(c)}^2}{2\sqrt{2}f_{(c)}} + k_{n-1}f_{(c)} - \frac{f_{(c)}}{2\sqrt{2}}$$

$$f_{(c)_n} = f_{(c)}\left(\frac{3}{2\sqrt{2}} - \frac{k_{n-1}^2}{\sqrt{2}} + k_{n-1} - \frac{1}{2\sqrt{2}}\right)$$

$$f_{(c)_n} = f_{(c)}\left(-\frac{k_{n-1}^2}{\sqrt{2}} + k_{n-1} + \frac{1}{\sqrt{2}}\right)$$

So, we can say:

$$k_n = -\frac{k_{n-1}^2}{\sqrt{2}} + k_{n-1} + \frac{1}{\sqrt{2}}$$

But what's the relationship between k_n and k_{n-1} ?

3.3 Proving the convergence of the Sequence

To answer our last question we have to analyze two scenarios:

1) $k_{n-1} > 1$
2) $k_{n-1} < 1$

So first, let $k_{n-1} > 1$.
Which means:

$$k_{n-1} = 1 + r$$

Then:

$$f_{(c)_n} = f_{(c)}\left(-\frac{(1+r)^2}{\sqrt{2}} + 1 + r + \frac{1}{\sqrt{2}}\right)$$

$$f_{(c)_n} = f_{(c)}\left(-\frac{1+2r+r^2}{\sqrt{2}} + 1 + r + \frac{1}{\sqrt{2}}\right)$$

$$f_{(c)_n} = f_{(c)}\left(-\frac{1}{\sqrt{2}} - \sqrt{2}\,r - \frac{r^2}{\sqrt{2}} + 1 + r + \frac{1}{\sqrt{2}}\right)$$

$$f_{(c)_n} = f_{(c)}\left(-\frac{r^2}{\sqrt{2}} + r\left(1-\sqrt{2}\right) + 1\right)$$

So, the term that represents k_n is a parabola.
Let's find its global extremum.

$$\frac{d}{dr}\left(-\frac{r^2}{\sqrt{2}} + r\left(1-\sqrt{2}\right) + 1\right) = -\sqrt{2}\,r + 1 - \sqrt{2}$$

$$-\sqrt{2}\,r + 1 - \sqrt{2} = 0$$

$$r = \frac{1-\sqrt{2}}{\sqrt{2}}$$

Is it a Maximum or a Minimum?
By taking the second derivative of k_n;

$$\frac{d}{dr}\left(-\sqrt{2}\,r + 1 - \sqrt{2}\right) = -\sqrt{2}$$

The parabola is concave down, so it's a Maximum.
And this Maximum is negative (for r). So, the graphic will be:

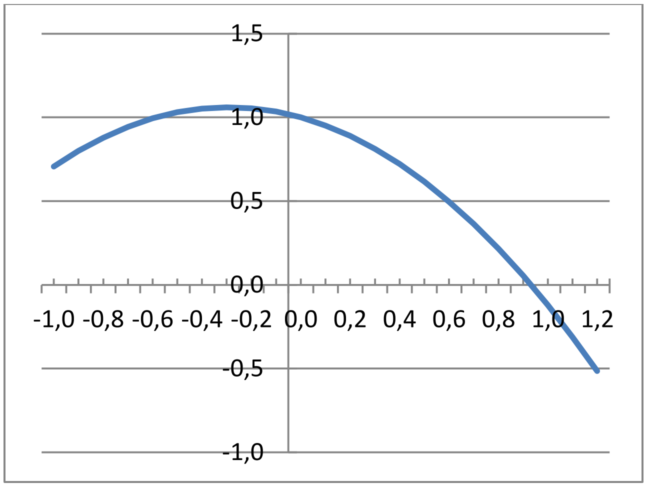


Figure 1

As $r > 0$ (Because we said that $k_{n-1} > 1$), we are going to analyze only the interval $(0, \infty)$ for r.

In this interval, the slope of the tangent lines of the curve is always negative. And as r increases, the absolute value of the slope increases too (because it's a concave down parabola).

But let's go further.
For which value of r, the slope of its tangent line is -1?

$$-\sqrt{2}\,r + 1 - \sqrt{2} = -1$$

$$r = \frac{2 - \sqrt{2}}{\sqrt{2}}$$

So, in the interval $\left(0, \frac{2-\sqrt{2}}{\sqrt{2}}\right)$ the variations in k_n are less than the variations in r.
And also we know that k_n is always less than 1 when r is greater than 0 (from the formula of k_n represented in the graphic).

Then, we can say:

$$k_n = 1 - s$$

And if:

$$0 < r < \frac{2-\sqrt{2}}{\sqrt{2}} \; then \; \boldsymbol{s} < \boldsymbol{r} \;\; because \; when \; r = \frac{2-\sqrt{2}}{\sqrt{2}}, s = \frac{\sqrt{2}-1}{\sqrt{2}} \; wich \; is \; less \; than \; r.$$

So, if s is always less than r (in the interval we mentioned), even though k_n is less than 1, $f_{(c)_n}$ will be a better approach than $f_{(c)_{n-1}}$.

Now, let $k_{n-1} < 1$.
Which means:

$$k_{n-1} = 1 - r$$

Then:

$$f_{(c)_n} = f_{(c)}\left(-\frac{(1-r)^2}{\sqrt{2}} + 1 - r + \frac{1}{\sqrt{2}}\right)$$

$$f_{(c)_n} = f_{(c)}\left(-\frac{1-2r+r^2}{\sqrt{2}} + 1 - r + \frac{1}{\sqrt{2}}\right)$$

$$f_{(c)_n} = f_{(c)}\left(-\frac{1}{\sqrt{2}} + \sqrt{2}\,r - \frac{r^2}{\sqrt{2}} + 1 - r + \frac{1}{\sqrt{2}}\right)$$

$$f_{(c)_n} = f_{(c)}\left(-\frac{r^2}{\sqrt{2}} - r\left(1-\sqrt{2}\right) + 1\right)$$

So, the term that represents k_n is a parabola.
Let's find its global extremum.

$$\frac{d}{dr}\left(-\frac{r^2}{\sqrt{2}} - r\left(1-\sqrt{2}\right) + 1\right) = -\sqrt{2}\,r - 1 + \sqrt{2}$$

$$-\sqrt{2}\,r - 1 + \sqrt{2} = 0$$

$$r = \frac{\sqrt{2}-1}{\sqrt{2}}$$

Is it a Maximum or a Minimum?
By taking the second derivative of k_n:

$$\frac{d}{dr}\left(-\sqrt{2}\,r - 1 + \sqrt{2}\right) = -\sqrt{2}$$

The parabola is concave down, so it's a Maximum.
And this Maximum is positive (for r). So, the graphic will be:

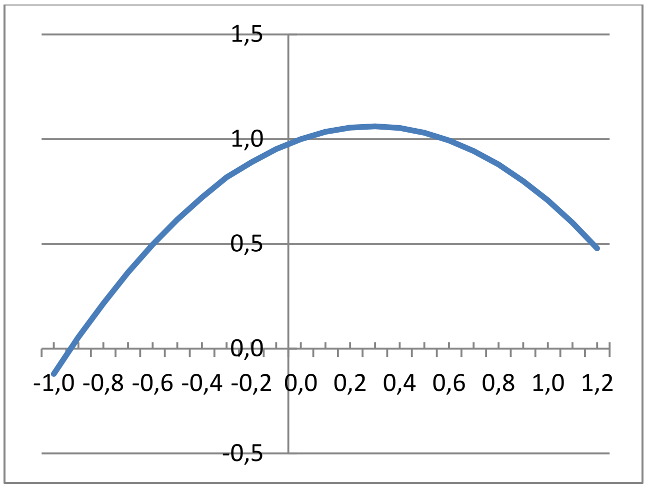


Figure 2

As $r > 0$ (Because we said that $k_{n-1} < 1$), we are going to analyze only the interval $(0, \infty)$ for r.

In this interval, the slope of the tangent lines of the curve is positive when r is less than the Maximum, and negative when it is greater. Plus, when $r = 0$, the slope is less than 1.

So, in the interval $\left(0, \frac{\sqrt{2}-1}{\sqrt{2}}\right)$ the variations in k_n are less than the variations in r. And k_n is greater than 1.

Then, we can say:

$$k_n = 1 + s$$

And if:

$$0 < r < \frac{\sqrt{2}-1}{\sqrt{2}} \text{ then } \boldsymbol{s} < \boldsymbol{r} \text{ because when } r = \frac{\sqrt{2}-1}{\sqrt{2}}, s = \frac{2\sqrt{2}-3}{2\sqrt{2}} \text{ wich is less than } r.$$

So, if s is always less than r (in the interval we mentioned), even though k_n is greater than 1, ${f_{(c)}}_n$ will be a better approach than ${f_{(c)}}_{n-1}$.

Given the intervals we mentioned, we can say that the Sequence we proposed converges to $f_{(c)}$.

4 Building the Sequence

4.1 First of all, the final prove

The first approach of $f_{(c)}$ is:

$$ {f_{(c)}}_0 = f_{(c)} \frac{3}{2\sqrt{2}} $$

So:

$$ k_0 = \frac{3}{2\sqrt{2}} $$

In this case, $k_0 > 1$.
Then:

$$ k_0 = 1 + r_0 \ \rightarrow r_0 = \frac{3 - 2\sqrt{2}}{2\sqrt{2}} \ \textit{which is less than} \ \frac{2-\sqrt{2}}{\sqrt{2}} $$

So, r_0 is in the interval that makes ${f_{(c)}}_1$ a better approach.

The second approach would be:

$$ {f_{(c)}}_1 = f_{(c)} \left(-\frac{k_0^2}{\sqrt{2}} + k_0 + \frac{1}{\sqrt{2}} \right) $$

$$ {f_{(c)}}_1 = f_{(c)} \left(-\frac{9}{8\sqrt{2}} + \frac{3}{2\sqrt{2}} + \frac{1}{\sqrt{2}} \right) $$

$$ {f_{(c)}}_1 = f_{(c)} \frac{11}{8\sqrt{2}} $$

In this case, $k_1 < 1$.
Then:

$$k_1 = 1 - r_1 \rightarrow r_1 = \frac{8\sqrt{2} - 11}{8\sqrt{2}} \ \textit{which is less than} \ \frac{\sqrt{2} - 1}{\sqrt{2}}$$

So, r_1 is in the interval that makes ${f_{(c)}}_2$ a better approach.

As ${f_{(c)}}_2$ will be greater than $f_{(c)}$ and a better approach than ${f_{(c)}}_0$, r_2 wil also be in the interval that makes ${f_{(c)}}_3$ a better approach.
And, as ${f_{(c)}}_3$ will be less than $f_{(c)}$ and a better approach than ${f_{(c)}}_1$, r_3 will also be in the interval that makes ${f_{(c)}}_4$ a better approach.

So, the Sequence we proposed really converges to $f_{(c)}$.

$$f_{(c)_n} = f_{(c)} \frac{3}{2\sqrt{2}} - \frac{\left(f_{(c)_{n-1}} - f_{(a)}\right)^2}{2f_{(a)}}$$

4.2 Now let's build the Sequence

Remember we said that a is a perfect square, and b is an integer wich is the square root of a.
Also we said that $c = 2a$.
So c is 2 times a perfect square.
Then:

$$f_{(c)} = b\sqrt{2}$$

$$f_{(c)_n} = \frac{3}{2}b - \frac{\left(f_{(c)_{n-1}} - f_{(a)}\right)^2}{2f_{(a)}}$$

$$f_{(c)_n} = \frac{3}{2}b - \frac{\left(f_{(c)_{n-1}} - b\right)^2}{2b} = \frac{3}{2}b - \frac{{f_{(c)}}_{n-1}^2 - 2b{f_{(c)}}_{n-1} + b^2}{2b}$$

$$f_{(c)_n} = \frac{3}{2}b - \frac{{f_{(c)}}_{n-1}^2}{2b} + {f_{(c)}}_{n-1} - \frac{1}{2}b$$

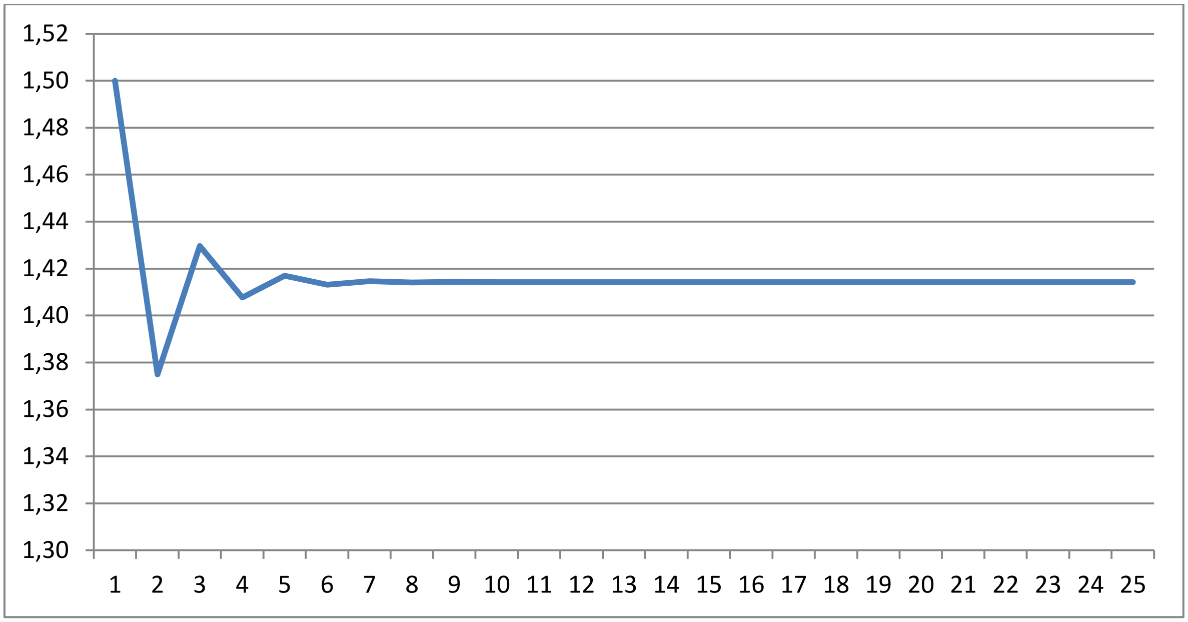
$$f_{(c)_n} = b - \frac{f_{(c)}{}_{n-1}^{2}}{2b} + f_{(c)_{n-1}}$$

In other words, we can say that the following Quadratic Sequence converges to $b\sqrt{2}$:

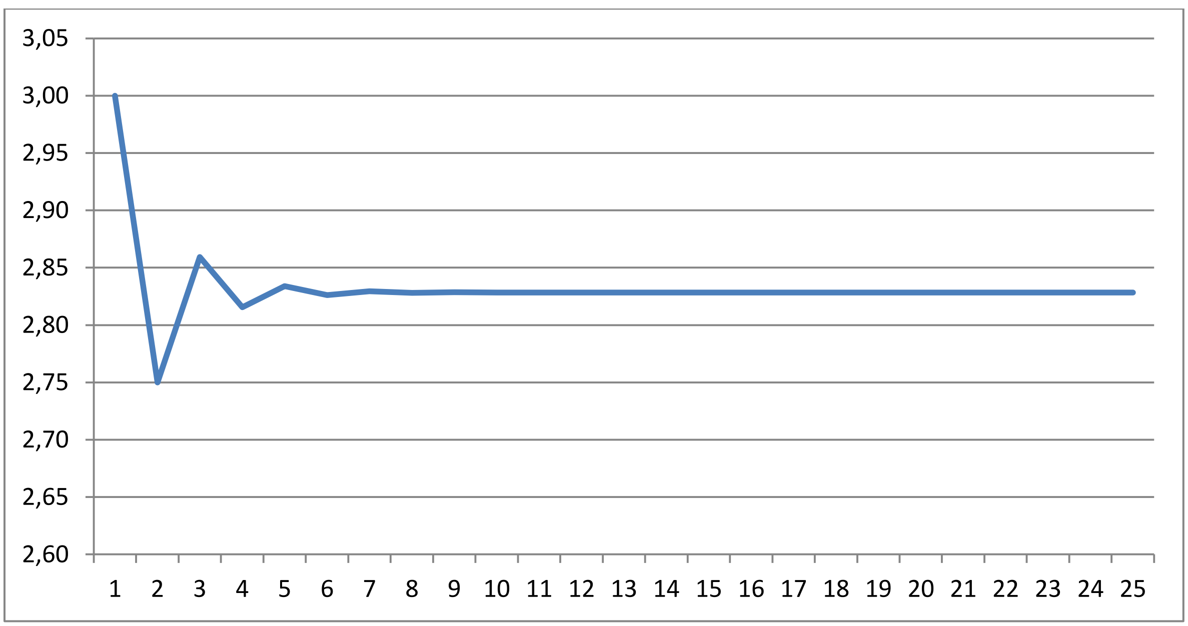
$$a_n = b - \frac{a_{n-1}^2}{2b} + a_{n-1} \qquad a_0 = \frac{3}{2}b;\ b \in \mathbb{N}$$

The following graphics show the shapes of these convergences.

Convergence to $\sqrt{2}$:



Convergence to $2\sqrt{2}$:



5 A nicer Quadratic Sequence

Let's go further to find a nicer Quadratic Sequence.

Let:

$$a_n = \frac{1}{d_n} + b \qquad\qquad d_0 = \frac{2}{b}$$

Then:

$$\frac{1}{d_n} + b = b - \frac{\left(\frac{1}{d_{n-1}} + b\right)^2}{2b} + \frac{1}{d_{n-1}} + b$$

$$\frac{1}{d_n} + b = b - \frac{\frac{1}{d_{n-1}^2} + \frac{2b}{d_{n-1}} + b^2}{2b} + \frac{1}{d_{n-1}} + b$$

$$\frac{1}{d_n} + b = b - \frac{1}{2bd_{n-1}^2} - \frac{1}{d_{n-1}} - \frac{b}{2} + \frac{1}{d_{n-1}} + b$$

$$\frac{1}{d_n} = \frac{b}{2} - \frac{1}{2bd_{n-1}^2}$$

$$\frac{1}{d_n} = \frac{b^2 d_{n-1}^2 - 1}{2bd_{n-1}^2}$$

$$d_n = \frac{2bd_{n-1}^2}{b^2 d_{n-1}^2 - 1}$$

Let:

$$d_n = \frac{1}{g_n} \qquad\qquad g_0 = \frac{1}{2}b$$

Then:

$$\frac{1}{g_n} = \frac{\frac{2b}{g_{n-1}^2}}{\frac{b^2}{g_{n-1}^2} - 1}$$

$$\frac{1}{g_n} = \frac{\frac{2b}{g_{n-1}^2}}{\frac{b^2 - g_{n-1}^2}{g_{n-1}^2}}$$

$$\frac{1}{g_n} = \frac{2b}{b^2 - g_{n-1}^2}$$

$$g_n = \frac{b^2 - g_{n-1}^2}{2b}$$

As $g_n = \frac{1}{d_n}$ and $d_n = \frac{1}{a_n - b}$:

We can say that the following Quadratic Sequence converges to $b\left(\sqrt{2} - 1\right)$:

$$g_n = \frac{1}{2}b - \frac{g_{n-1}^2}{2b} \qquad g_0 = \frac{1}{2}b;\ b \in \mathbb{N}$$

6 Final Conclusion

The number b doesn't have to be an integer. We can build the same sequences based, for example, on these values:

$c = \frac{18}{4}; a = \frac{9}{4}; b = \frac{3}{2}$ $\left(because\ c = 2a\ and\ b = \sqrt{a}\right)$

Furthermore, b doesn't have to be only an integer or a rational number. It could be a Real number. For example:

$$a_n = \frac{\sqrt{2}}{3} - \frac{3}{2\sqrt{2}}\ a_{n-1}^2 + a_{n-1} \quad converges\ to\ \frac{2}{3}$$