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Two equations involving the Smarandache LCM dual function

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Abstract For any positive integer n, the Smarandache LCM dual function $SL^*(n)$ is defined as the greatest positive integer k such that $[1,\ 2,\ \cdots,\ k]$ divides n. The main purpose of this paper is using the elementary method to study the number of the solutions of two equations involving the Smarandache LCM dual function $SL^*(n)$, and give their all positive integer solutions.

Keywords Smarandache LCM dual function, equation, positive integer solution.

§1. Introduction and results

For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of all positive integers from 1 to k. That is,

$$SL(n) = \min\{k:\ k \in N,\ n \mid [1,\ 2,\ \cdots,\ k]\}.$$

About the elementary properties of $SL(n)$, many people had studied it, and obtained some interesting results, see references [1] and [2]. For example, Murthy [1] proved that if n is a prime, then $SL(n) = S(n)$, where $S(n) = \min\{m:\ n|m!,\ m \in N\}$ be the F.Smarandache function. Simultaneously, Murthy [1] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n\ ? \tag{1}$$

Le Maohua [2] solved this problem completely, and proved the following conclusion:

Every positive integer n satisfying (1) can be expressed as

$$n = 12 \ \ \texttt{or} \ \ n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where $p_1,\ p_2,\ \cdots,\ p_r,\ p$ are distinct primes and $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_r$ are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1,\ 2,\ \cdots,\ r$. Zhongtian Lv [3] proved that for any real number $x > 1$ and fixed positive integer k, we have the asymptotic formula

$$\sum_{n \le x} SL(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 2,\ 3,\ \cdots,\ k)$ are computable constants.

Now, we define the Smarandache LCM dual function $SL^*(n)$ as follows:

$$SL^*(n) = \max\{k :\ k \in N,\ [1,\ 2,\ \cdots,\ k] \mid n\}.$$

It is easy to calculate that $SL^*(1) = 1$, $SL^*(2) = 2$, $SL^*(3) = 1$, $SL^*(4) = 2$, $SL^*(5) = 1$, $SL^*(6) = 3$, $SL^*(7) = 1$, $SL^*(8) = 2$, $SL^*(9) = 1$, $SL^*(10) = 2$, $\cdots$. Obviously, if n is an odd number, then $SL^*(n) = 1$. If n is an even number, then $SL^*(n) \geq 2$. About the other elementary properties of $SL^*(n)$, it seems that none had studied it yet, at least we have not seen such a paper before. In this paper, we use the elementary method to study the number of the solutions of two equations involving the Smarandache LCM dual function $SL^*(n)$. For further, we obtain all the positive numbers n, such that

$$\sum_{d|n} SL^*(d) = n \tag{2}$$

or

$$\sum_{d|n} SL^*(d) = \phi(n), \tag{3}$$

where $\sum\limits_{d|n}$ denotes the summation over all positive divisors of n. That is, we shall prove the following two conclusions:

Theorem 1. The equation (2) has only one and only one solution $n = 1$, and $\sum\limits_{d|n} SL^*(d) > n$ is true if and only if $n = 2,\ 4,\ 6,\ 12$.

Theorem 2. The equation (3) is true if and only if $n = 1, 3, 14$.

§2. Some lemmas

To complete the proofs of the theorems, we need the following lemmas.

Lemma 1. (a) For any prime p and any real number $x \geq 1$, we have $p^x \geq x + 1$, and the equality is true if and only if $x = 1,\ p = 2$.

(b) For any odd prime p and any real number x, if $x \geq 2$, then we have $p^x > 2(x + 1)$; If $x \geq 3$, then we have $p^x > 4(x + 1)$.

(c) For any prime $p \geq 5$ and any real number $x \geq 2$, we have $p^x > 4(x + 1)$.

(d) For any prime $p \geq 11$ and any real number $x \geq 1$, we have $p^x > 4(x + 1)$.

Proof. We only prove case (a), others can be obtained similarly.

Let $f(x) = p^x - x - 1$, if $x \geq 1$, then

$$f^{'}(x) = p^x \ln p - 1 > p \ln e^{\frac{1}{2}} - 1 = \frac{p}{2} - 1 \geq 1.$$

That is to say, $f(x)$ is a monotone increasing function if $x \in [1,\ \infty)$. So $f(x) \geq f(1) \geq 0$, namely $p^x \geq x + 1$, and $p^x = 4(x + 1)$ is true if and only if $x = 1,\ p = 2$. This complete the proof of case (a).

Lemma 2. For all odd positive integer number n,

(a) the equation $d(n) = \phi(n)$ is true if and only if $n = 1, 3$;

(b) the inequality $8d(n) > \phi(n)$ is true if and only if $n = 1, 3, 5, 7, 9, 11, 13, 15, 21, 27, 33, 35, 39, 45, 63, 105$, where $d(n)$ is the divisor function of n, $\phi(n)$ is the Euler function.

Proof. Let $H(n) = \frac{\phi(n)}{d(n)}$, then the equation $d(n) = \phi(n)$ is equivalent to $H(n) = 1$ and $8d(n) > \phi(n)$ is equivalent to $H(n) < 8$. Because $d(n)$ and $\phi(n)$ are multiplicative functions, hence $H(n)$ is multiplicative. Assume that p, q are prime numbers and $p > q$, then $H(p) = \frac{p-1}{2} > \frac{q-1}{2} = H(q)$. On the other hand, for any given prime p and integer $k \geq 1$, we have $\frac{H(p^{k+1})}{H(p^k)} = \frac{p(1+k)}{2+k} > \frac{2k+2}{2+k} > 1$. Hence if $k \geq 1$, then $H(p^{1+k}) > H(p^k)$.

Because

$$\begin{aligned}
&H(1) = 1,\ H(3) = 1,\ H(5) = 2,\ H(7) = 3,\ H(11) = 5,\ H(13) = 6,\ H(17) = 8 \geq 8,\\
&H(3^2) = 2,\ H(5^2) = \frac{20}{3} \geq 8,\ H(7^2) = 14 \geq 8,\ H(11^2) = \frac{110}{3} \geq 8,\ H(13^2) = 52 \geq 8,\\
&H(3^3) = \frac{9}{2},\\
&H(3^4) = \frac{54}{5} \geq 8.
\end{aligned}$$

We have $H(1) = 1,\ H(3) = 1,\ H(5) = 2,\ H(7) = 3,\ H(9) = 2,\ H(11) = 5,\ H(13) = 6,\ H(15) = H(3)H(5) = 2,\ H(21) = H(3)H(7) = 3,\ H(27) = H(3^3) = \frac{9}{2}, H(33) = H(3)H(11) = 5,\ H(35) = H(5)H(7) = 6,\ H(39) = H(3)H(13) = 6,\ H(45) = H(3^2)H(5) = 4,\ H(63) = H(3^2)H(7) = 6,\ H(105) = H(3)H(5)H(7) = 6.$
Consequently, for all positive odd integer number n, $H(n) = 1$ is true if and only if $n = 1, 3$; the inequality $H(n) < 8$ is true if and only if $n = 1, 3, 5, 7, 9, 11, 13, 15, 21, 27, 33, 35, 39, 45, 63, 105$.

This completes the proof of Lemma 2.

§3. Proof of the theorems

In this section, we shall complete the proof of the theorems. First we prove Theorem 1. It is easy to see that $n = 1$ is one solution of the equation (2). In order to prove that the equation (2) has no other solutions except $n = 1$, we consider the following two cases:
(a) n is an odd number larger than 1.

Assume that $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, where p_i is an odd prime, $p_1 < p_2 < \cdots < p_s$, $\alpha_i \geq 1$, $i = 1,\ 2,\ \cdots,\ s$. In this case, for any $d|n$, d is an odd number, so $SL^*(d) = 1$. From Lemma 1 (a), we have

$$\sum_{d|n} SL^*(d) = \sum_{d|n} 1 = d(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1) < p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} = n.$$

(b) n is an even number.

Assume that $n = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} = 2^\alpha \cdot m$, where p_i is an odd prime, $p_1 < p_2 < \cdots < p_s$,

$\alpha_i \geq 1$, $i = 1, 2, \cdots, s$. In this case,

$$\begin{aligned}\sum_{d|n} SL^*(d) &= \sum_{i=0}^{\alpha}\sum_{d|m} SL^*(2^i d)\\ &< \sum_{i=0}^{\alpha} 2^{i+1}\sum_{d|m} 1 = (2+2^2+\cdots+2^{\alpha+1})d(m)\\ &= (2^{\alpha+2}-2)d(m) < 2^{\alpha}\cdot 4d(m). \qquad (4)\end{aligned}$$

(i) If $p_s \geq 11$, from Lemma 1 (a), we have

$$4d(m) = (\alpha_1+1)(\alpha_2+1)\cdots 4(\alpha_s+1) < p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s} = m.$$

Associated with (4) we have $\sum_{d|n} SL^*(d) < n$.

(ii) If there exists $i,\ j \in \{1,\ 2,\ \cdots,\ s\}$ and $i \neq j$ such that $\alpha_i \geq 2,\ \alpha_j \geq 2$, then from Lemma 1 (a), we have

$$4d(m) = (\alpha_1+1)\cdots 2(\alpha_i+1)\cdots 2(\alpha_j+1)\cdots(\alpha_s+1) < p_1^{\alpha_1}\cdots p_i^{\alpha_i}\cdots p_j^{\alpha_j}\cdots p_s^{\alpha_s} = m.$$

Associated with (4) we have $\sum_{d|n} SL^*(d) < n$.

(iii) If there exists $i \in \{1,\ 2,\ \cdots,\ s\}$ such that $\alpha_i \geq 3$, then from Lemma 1 (a), we have

$$4d(m) = (\alpha_1+1)\cdots 4(\alpha_i+1)\cdots(\alpha_s+1) < p_1^{\alpha_1}\cdots p_i^{\alpha_i}\cdots p_s^{\alpha_s} = m.$$

Associated with (4) we have $\sum_{d|n} SL^*(d) < n$.

(iv) If there exists $i \in \{1,\ 2,\ \cdots,\ s\}$ such that $p_i \geq 5,\ \alpha_i \geq 2$, then from Lemma 1 (a), we have

$$4d(m) = (\alpha_1+1)\cdots 4(\alpha_i+1)\cdots(\alpha_s+1) < p_1^{\alpha_1}\cdots p_i^{\alpha_i}\cdots p_s^{\alpha_s} = m.$$

Associated with (4) we also have $\sum_{d|n} SL^*(d) < n$.

From the discussion above we know that if n satisfies the equation (2), then m has only seven possible values. That is $m \in \{1,\ 3,\ 5,\ 7,\ 9,\ 15,\ 21\}$. We calculate the former three cases only, other cases can be discussed similarly.

If $m = 1$, namely $n = 2^{\alpha}(\alpha \geq 1)$, then

$$\begin{aligned}\sum_{d|n} SL^*(d) &= \sum_{i=0}^{\alpha}\sum_{d|1} SL^*(2^i d)\\ &= SL^*(1)+SL^*(2)+SL^*(2^2)+\cdots+SL^*(2^{\alpha})\\ &= 1+2+2+\cdots+2 = 2\alpha+1.\end{aligned}$$

$\alpha = 1,\ 2$, namely $n = 2,\ 4$. In this case, $2\alpha+1 > 2^{\alpha}$, so $\sum_{d|n} SL^*(d) > n$.

$\alpha \geq 3$. In this case, $2\alpha+1 < 2^{\alpha}$, so $\sum_{d|n} SL^*(d) < n$.

If $m = 3$, namely $n = 2^\alpha \cdot 3,\ (\alpha \geq 1)$, then

$$\begin{aligned}\sum_{d|n} SL^*(d) &= \sum_{i=0}^{\alpha}\sum_{d|3} SL^*(2^i d)\\ &= \sum_{d|3} SL^*(d) + \sum_{d|3} SL^*(2d) + \sum_{d|3} SL^*(2^2 d) + \cdots + \sum_{d|3} SL^*(2^\alpha d)\\ &= 2+5+6+\cdots+6 = 6\alpha+1.\end{aligned}$$

$\alpha = 1$, namely $n = 6$. In this case, $6\alpha + 1 = 7 > 2 \cdot 3$, so $\sum_{d|n} SL^*(d) > n$.

$\alpha = 2$, namely $n = 12$. In this case, $6\alpha + 1 = 13 > 2^2 \cdot 3$, so $\sum_{d|n} SL^*(d) > n$.

$\alpha \geq 3$. In this case, $2\alpha + 1 < 2^\alpha$, so $\sum_{d|n} SL^*(d) < n$.

If $m = 5$, namely $n = 2^\alpha \cdot 5,\ (\alpha \geq 1)$, then

$$\begin{aligned}\sum_{d|n} SL^*(d) &= \sum_{i=0}^{\alpha}\sum_{d|5} SL^*(2^i d)\\ &= \sum_{d|5} SL^*(d) + \sum_{d|5} SL^*(2d) + \sum_{d|5} SL^*(2^2 d) + \cdots + \sum_{d|5} SL^*(2^\alpha d)\\ &= 2+4+4+\cdots+4 = 4\alpha+2.\end{aligned}$$

For any $\alpha \geq 1$, we have $4\alpha + 2 < 2^\alpha \cdot 5$, so $\sum_{d|n} SL^*(d) < n$.

If $m = 7, 9, 15, 21$, using the similar method we can obtain that for any $\alpha \geq 1$, $\sum_{d|n} SL^*(d) < n$ is true.

Hence the equation (2) has no positive even integer number solutions, and $\sum_{d|n} SL^*(d) > n$ is true if and only if $n = 2,\ 4,\ 6,\ 12$.

Associated (a) and (b), we complete the proof of Theorem 1.

At last we prove Theorem 2. From Lemma 2, it is easy to versify that $n = 1,\ 3$ are the only positive odd number solutions of the equation (3). Following we consider the case that n is an even number.

Assume that $n = 2^\alpha \cdot m$, where $2 \nmid m$. In this case,

$$\begin{aligned}\sum_{d|n} SL^*(d) &= \sum_{i=0}^{\alpha}\sum_{d|m} SL^*(2^i d) < \sum_{i=0}^{\alpha} 2^{i+1} \sum_{d|m} 1\\ &= (2 + 2^2 + \cdots + 2^{\alpha+1})d(m) = (2^{\alpha+2} - 2)d(m) < 2^{\alpha-1} \cdot 8d(m),\end{aligned}$$

and $\phi(n) = \phi(2^\alpha m) = \phi(2^\alpha)\phi(m) = 2^{\alpha-1}\phi(m)$. Let

$$S = \{1,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,\ 15,\ 21,\ 27,\ 33,\ 35,\ 39,\ 45,\ 63,\ 105\}.$$

From Lemma 2, if $m \notin S$, then $\phi(m) \geq 8 \cdot d(m)$, consequently

$$\sum_{d|n} SL^*(d) < 2^{\alpha-1} \cdot 8d(m) \leq 2^{\alpha-1}\phi(m) = \phi(n).$$

Hence if n satisfies the equation (3), then $m \in S$. We only discuss the cases $m = 1$, 7, other cases can be discussed similarly.

If $m = 1$, namely $n = 2^\alpha (\alpha \geq 1)$, then

$$\begin{aligned} & \sum_{d|n} SL^*(d) = \sum_{i=0}^{\alpha} \sum_{d|1} SL^*(2^i d) \\ = & \ SL^*(1) + SL^*(2) + SL^*(2^2) + \cdots + SL^*(2^\alpha) \\ = & \ 1 + 2 + 2 + \cdots + 2 = 2\alpha + 1 \end{aligned}$$

and $\phi(n) = \phi(2^\alpha) = 2^{\alpha-1}$, $2 \dagger (2\alpha + 1)$, but $2|2^{\alpha-1}$, hence if $m = 1$, then the equation (2) has no solution.

If $m = 7$, namely $n = 2^\alpha \cdot 7,\ (\alpha \geq 1)$, then

$$\begin{aligned} & \sum_{d|n} SL^*(d) = \sum_{i=0}^{\alpha} \sum_{d|7} SL^*(2^i d) \\ = & \sum_{d|7} SL^*(d) + \sum_{d|7} SL^*(2d) + \sum_{d|7} SL^*(2^2 d) + \cdots + \sum_{d|7} SL^*(2^\alpha d) \\ = & \ 2 + 4 + 4 + \cdots + 4 = 4\alpha + 2, \end{aligned}$$

and $\phi(n) = \phi(2^\alpha \cdot 7) = 2^{\alpha-1} \cdot 6$, Solving the equation $4\alpha + 2 = 2^{\alpha-1} \cdot 6$, we have $\alpha = 1$. That is to say that $n = 14$ is one solution of the equation (3).

Discussing the other cases similarly, we have that if n is an even number, then the equation (3) has only one solution $n = 14$.

This completes the proof of Theorem 2.

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