ON SMARANDACHE TRIPLE FACTORIAL FUNCTION

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Abstract For any positive integer n, the Smarandache triple factorial function $d3_f(n)$ is defined to be the smallest integer such that $d3_f(n)!!!$ is a multiple of n. In this paper, we study the hybrid mean value of the Smarandache triple factorial function and the Mangoldt function, and give a sharp asymptotic formula.

Keywords: Triple factorial numbers; Hybrid mean value; Asymptotic formula.

§1. Introduction

According to [1], for any positive integer n, the Smarandache triple factorial function $d3_f(n)$ is defined to be the smallest integer such that $d3_f(n)!!!$ is a multiple of n. About this problem, we know very little. The problem is interesting because it can help us to calculate the Smarandache function.

In this paper, we study the hybrid mean value of the Smarandache triple factorial function and the Mangoldt function, and give a sharp asymptotic formula. That is, we shall prove the following theorems.

Theorem 1. *If $x \geq 2$, then for any positive integer k we have*

$$\sum_{n\leq x} \Lambda_1(n) d3_f(n) = x^2 \left(\frac{1}{2} + \sum_{m=1}^{k-1} \frac{a_m}{\log^m x} \right) + O\left(\frac{x^2}{\log^k x} \right),$$

where

$$\Lambda_1(n) = \begin{cases} \log p, & \text{if } n \text{ is a prime } p\,; \\ 0, & \text{otherwise}, \end{cases}$$

and $a_m (m = 1, 2, \cdots, k-1)$ are computable constants.

Theorem 2. *If $x \geq 2$, then for any positive integer k we have*

$$\sum_{n\leq x} \Lambda(n) d3_f(n) = x^2 \left(\frac{1}{2} + \sum_{m=1}^{k-1} \frac{a_m}{\log^m x} \right) + O\left(\frac{x^2}{\log^k x} \right),$$

where $\Lambda(n)$ is the Mangoldt function.

§2. One lemma

To complete the proofs of the theorems, we need the following lemma.

Lemma. *For any positive integer α, if $p \geq (3\alpha - 2)$ we have*

$$d3_f(p^\alpha) = (3\alpha - 2)p.$$

Proof. Since

$$[(3\alpha - 2)p]!!! = (3\alpha - 2)p \cdots (3\alpha - 3)p \cdots p,$$

so $p^\alpha \mid [(3\alpha - 2)p]!!!$. On the other hand, if $p \geq (3\alpha - 2)$, then $(3\alpha - 2)p$ is the smallest integer such that $[(3\alpha - 2)p]!!!$ is a multiple of p^α. Therefore $d3_f(p^\alpha) = (3\alpha - 2)p$.

§3. Proof of the theorems

In this section, we complete the proofs of the theorems. Let

$$a(n) = \begin{cases} 1, & \text{if } n \text{ is prime;} \\ 0, & \text{otherwise.} \end{cases}$$

then for any positive integer k we have

$$\sum_{n \leq x} a(n) = \pi(x) = \frac{x}{\log x}\left(1 + \sum_{m=1}^{k-1} \frac{m!}{\log^m x}\right) + O\left(\frac{x}{\log^{k+1} x}\right).$$

By Abel's identity we have

$$\begin{aligned} & \sum_{n \leq x} \Lambda_1(n) d3_f(n) \\ = & \sum_{p \leq x} p \log p = \sum_{n \leq x} a(n) n \log n \\ = & \pi(x) \cdot x \log x - \int_2^x \pi(t)\left(\log t + 1\right) dt \\ = & x^2\left(1 + \sum_{m=1}^{k-1} \frac{m!}{\log^m x}\right) + O\left(\frac{x^2}{\log^k x}\right) \\ & - \int_2^x \left(t + \frac{t}{\log t} + t\sum_{m=1}^{k-1} \frac{m!}{\log^m t} + \frac{t}{\log t}\sum_{m=1}^{k-1} \frac{m!}{\log^m t}\right. \\ & \left. + O\left(\frac{t\left(\log t + 1\right)}{\log^{k+1} t}\right)\right) dt \\ = & x^2\left(\frac{1}{2} + \sum_{m=1}^{k-1} \frac{a_m}{\log^m x}\right) + O\left(\frac{x^2}{\log^k x}\right), \end{aligned}$$

where $a_m(m = 1, 2, \cdots, k-1)$ are computable constants. Therefore

$$\sum_{p\le x} p\log p = x^2\left(\frac{1}{2}+\sum_{m=1}^{k-1}\frac{a_m}{\log^m x}\right)+O\left(\frac{x^2}{\log^k x}\right).$$

So we have

$$\sum_{n\le x} \Lambda_1(n)d3_f(n) = x^2\left(\frac{1}{2}+\sum_{m=1}^{k-1}\frac{a_m}{\log^m x}\right)+O\left(\frac{x^2}{\log^k x}\right).$$

This proves Theorem 1.

It is obvious that $d3_f(p^\alpha) \le (3\alpha-2)p$. From Lemma 1 we have

$$\begin{aligned}&\sum_{n\le x}\Lambda(n)d3_f(n)\\ =\ &\sum_{p^\alpha\le x}\log p\left[(3\alpha-2)p\right]+\sum_{\substack{p^\alpha\le x\\ p<(3\alpha-2)}}\log p\left[d3_f(p^\alpha)-(3\alpha-2)p\right].\end{aligned}$$

Note that

$$\begin{aligned}&\sum_{p^\alpha\le x}(3\alpha-2)p\log p-\sum_{p\le x}p\log p\\ =\ &\sum_{\alpha\le\frac{\log x}{\log p}}\ \sum_{p\le x^{1/\alpha}}p\log p(2\alpha-1)-\sum_{p^\alpha\le x}p\log p\\ =\ &\sum_{2\le\alpha\le\frac{\log x}{\log p}}\ \sum_{p\le x^{1/\alpha}}p\log p(3\alpha-2)\\ \ll\ &\sum_{2\le\alpha\le\frac{\log x}{\log p}}\alpha x^{2/\alpha}\log x^{1/\alpha}\ll x\log^3 x\end{aligned}$$

and

$$\begin{aligned}\sum_{\substack{p^\alpha\le x\\ p<(3\alpha-2)}}\log p\left[d_f(p^\alpha)-(3\alpha-2)p\right]&\ll\sum_{\alpha\le\frac{\log x}{\log 2}}\ \sum_{p<(3\alpha-2)}\alpha p\log p\\ &\ll\sum_{\alpha\le\frac{\log x}{\log 2}}(3\alpha-2)^2\alpha\log(3\alpha-2)\ll\log^3 x,\end{aligned}$$

so we have

$$\sum_{n\le x} \Lambda(n)d3_f(n) = x^2\left(\frac{1}{2}+\sum_{m=1}^{k-1}\frac{a_m}{\log^m x}\right)+O\left(\frac{x^2}{\log^k x}\right).$$

This completes the proof of Theorem 2.

References

[1] "Smarandache k-factorial" at http://www. gallup. unm. edu/ smarandache/SKF.htm