

On the Smarandache kn -digital subsequence

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Abstract For any positive integer n and any fixed positive integer $k \geq 2$, the Smarandache kn -digital subsequence $\{S_k(n)\}$ is defined as the numbers $S_k(n)$, which can be partitioned into two groups such that the second is k times bigger than the first. The main purpose of this paper is using the elementary method to study the convergent properties of the infinite series involving the Smarandache kn -digital subsequence $\{S_k(n)\}$, and obtain some interesting conclusions.

Keywords The Smarandache kn -digital sequence, infinite series, convergence.

§1. Introduction

For any positive integer n and any fixed positive integer $k \geq 2$, the Smarandache kn -digital subsequence $\{S_k(n)\}$ is defined as the numbers $S_k(n)$, which can be partitioned into two groups such that the second is k times bigger than the first. For example, the Smarandache $3n$ -digital subsequence are: $S_3(1) = 13$, $S_3(2) = 26$, $S_3(3) = 39$, $S_3(4) = 412$, $S_3(5) = 515$, $S_3(6) = 618$, $S_3(7) = 721$, $S_3(8) = 824$, $S_3(9) = 927$, $S_3(10) = 1030$, $S_3(11) = 1133$, $S_3(12) = 1236$, $S_3(13) = 1339$, $S_3(14) = 1442$, $S_3(15) = 1545$, $S_3(16) = 1648$, $S_3(17) = 1751$, $S_3(18) = 1854$, $S_3(19) = 1957$, $S_3(20) = 2060$, $S_3(21) = 2163$, $S_3(22) = 2266$, \dots .

The Smarandache $4n$ -digital subsequence are: $S_4(1) = 14$, $S_4(2) = 28$, $S_4(3) = 312$, $S_4(4) = 416$, $S_4(5) = 520$, $S_4(6) = 624$, $S_4(7) = 728$, $S_4(8) = 832$, $S_4(9) = 936$, $S_4(10) = 1040$, $S_4(11) = 1144$, $S_4(12) = 1248$, $S_4(13) = 1352$, $S_4(14) = 1456$, $S_4(15) = 1560$, \dots .

The Smarandache $5n$ -digital subsequence are: $S_5(1) = 15$, $S_5(2) = 210$, $S_5(3) = 315$, $S_5(4) = 420$, $S_5(5) = 525$, $S_5(6) = 630$, $S_5(7) = 735$, $S_5(8) = 840$, $S_5(9) = 945$, $S_5(10) = 1050$, $S_5(11) = 1155$, $S_5(12) = 1260$, $S_5(13) = 1365$, $S_5(14) = 1470$, $S_5(15) = 1575$, \dots .

These subsequences are proposed by Professor F.Smarandache, he also asked us to study the properties of these subsequences. About these problems, it seems that none had studied them, at least we have not seen any related papers before. The main purpose of this paper is using the elementary method to study the convergent properties of one kind infinite series involving the Smarandache kn -digital subsequence, and prove the following conclusion:

Theorem. Let z be a real number. If $z > \frac{1}{2}$, then the infinite series

$$f(z, k) = \sum_{n=1}^{+\infty} \frac{1}{S_k^z(n)} \quad (1)$$

is convergent; If $z \leq \frac{1}{2}$, then the infinite series (1) is divergent.

In these Smarandache kn -digital subsequences, it is very hard to find a complete square number. So we believe that the following conclusion is correct:

Conjecture. There does not exist any complete square number in the Smarandache kn -digital subsequence, where $k = 3, 4, 5$. That is, for any positive integer m , $m^2 \notin \{S_k(n)\}$, where $k = 3, 4, 5$.

§2. Proof of the theorem

In this section, we shall use the elementary method to complete the proof of our Theorem. We just prove that the theorem is holds for Smarandache $3n$ -digital subsequence. Similarly, we can deduce that the theorem is also holds for any other positive integer $k \geq 4$. For any element $S_3(a)$ in $\{S_3(n)\}$, let $3a = b_k b_{k-1} \cdots b_2 b_1$, where $1 \leq b_k \leq 9$, $0 \leq b_i \leq 9$, $i = 1, 2, \dots, k-1$. Then from the definition of the Smarandache $3n$ -digital subsequence we have

$$S_3(a) = a \cdot 10^k + 3 \cdot a = a \cdot (10^k + 3). \quad (2)$$

On the other hand, let $a = a_s a_{s-1} \cdots a_2 a_1$, where $1 \leq a_s \leq 9$, $0 \leq a_i \leq 9$, $i = 1, 2, \dots, s-1$. It is clear that if $a \leq \underbrace{33 \cdots 33}_s$, then $s = k$; If $a \geq \underbrace{33 \cdots 34}_s$, then $s = k-1$. So from the definition of $S_3(a)$ and the relationship of s and k we have

$$\begin{aligned} f(z, 3) &= \sum_{n=1}^{+\infty} \frac{1}{S_3^z(n)} = \sum_{i=1}^3 \frac{1}{S_3^z(i)} + \sum_{i=4}^{33} \frac{1}{S_3^z(i)} + \sum_{i=34}^{333} \frac{1}{S_3^z(i)} + \sum_{i=334}^{3333} \frac{1}{S_3^z(i)} + \sum_{i=3334}^{33333} \frac{1}{S_3^z(i)} + \cdots \\ &= \sum_{i=1}^3 \frac{1}{i^z \cdot 13^z} + \sum_{i=4}^{33} \frac{1}{i^z \cdot 103^z} + \sum_{i=34}^{333} \frac{1}{i^z \cdot 1003^z} + \sum_{i=3334}^{33333} \frac{1}{i^z \cdot 10003^z} + \cdots \\ &\leq \sum_{k=1}^{+\infty} \frac{3 \cdot 10^{k-1}}{10^{z(k-2)} \cdot 10^{zk}} \leq 3 \cdot \sum_{k=0}^{+\infty} \frac{10^k}{10^{z(k-1)} \cdot 10^{z(k+1)}} = 3 \cdot \sum_{k=0}^{+\infty} \frac{1}{10^{k \cdot (2z-1)}}. \end{aligned} \quad (3)$$

Now if $z > \frac{1}{2}$, then from (3) and the properties of the geometric progression we know that $f(z, 3)$ is convergent.

If $z \leq \frac{1}{2}$, then from (3) we also have

$$\begin{aligned} f(z, 3) &= \sum_{n=1}^{+\infty} \frac{1}{S_3^z(n)} = \sum_{i=1}^3 \frac{1}{S_3^z(i)} + \sum_{i=4}^{33} \frac{1}{S_3^z(i)} + \sum_{i=34}^{333} \frac{1}{S_3^z(i)} + \sum_{i=334}^{3333} \frac{1}{S_3^z(i)} + \sum_{i=3334}^{33333} \frac{1}{S_3^z(i)} + \cdots \\ &= \sum_{i=1}^3 \frac{1}{i^z \cdot 13^z} + \sum_{i=4}^{33} \frac{1}{i^z \cdot 103^z} + \sum_{i=34}^{333} \frac{1}{i^z \cdot 1003^z} + \sum_{i=3334}^{33333} \frac{1}{i^z \cdot 10003^z} + \cdots \\ &\geq \sum_{k=1}^{+\infty} \frac{3 \cdot 10^{k-1}}{10^{z(k-1)} \cdot 10^{z(k+1)}} \geq 3 \cdot \sum_{k=0}^{+\infty} \frac{10^k}{10^{zk} \cdot 10^{z(k+2)}} = 3 \cdot \sum_{k=0}^{+\infty} \frac{1}{10^{(2zk+2z-k)}}. \end{aligned} \quad (4)$$

Then from the properties of the geometric progression and (4) we know that the series $f(z, 3)$ is divergent if $z \leq \frac{1}{2}$. This proves our theorem for $k = 3$.

Similarly, we can deduce the other cases. For example, if $k = 4$, then we have

$$\begin{aligned}
 f(z, 4) &= \sum_{n=1}^{+\infty} \frac{1}{S_4^z(n)} = \sum_{i=1}^2 \frac{1}{S_4^z(i)} + \sum_{i=3}^{24} \frac{1}{S_4^z(i)} + \sum_{i=25}^{249} \frac{1}{S_4^z(i)} + \sum_{i=250}^{2499} \frac{1}{S_4^z(i)} + \sum_{i=2500}^{24999} \frac{1}{S_3^z(i)} + \cdots \\
 &= \sum_{i=1}^2 \frac{1}{i^z \cdot 14^z} + \sum_{i=3}^{24} \frac{1}{i^z \cdot 104^z} + \sum_{i=25}^{249} \frac{1}{i^z \cdot 1004^z} + \sum_{i=250}^{2499} \frac{1}{i^z \cdot 10004^z} + \cdots \\
 &\leq \sum_{k=1}^{+\infty} \frac{225 \cdot 10^{k-2}}{10^{z(k-2)} \cdot 10^{zk}} \leq 225 \cdot \sum_{k=0}^{+\infty} \frac{10^k}{10^{zk} \cdot 10^{z(k+2)}} = 225 \cdot \sum_{k=0}^{+\infty} \frac{1}{10^{k \cdot (2z-1) + 2z}} \quad (5)
 \end{aligned}$$

and

$$\begin{aligned}
 f(z, 4) &= \sum_{n=1}^{+\infty} \frac{1}{S_4^z(n)} = \sum_{i=1}^2 \frac{1}{S_4^z(i)} + \sum_{i=3}^{24} \frac{1}{S_4^z(i)} + \sum_{i=25}^{249} \frac{1}{S_4^z(i)} + \sum_{i=250}^{2499} \frac{1}{S_4^z(i)} + \sum_{i=2500}^{24999} \frac{1}{S_3^z(i)} + \cdots \\
 &= \sum_{i=1}^2 \frac{1}{i^z \cdot 14^z} + \sum_{i=3}^{24} \frac{1}{i^z \cdot 104^z} + \sum_{i=25}^{249} \frac{1}{i^z \cdot 1004^z} + \sum_{i=250}^{2499} \frac{1}{i^z \cdot 10004^z} + \cdots \\
 &\geq \sum_{k=1}^{+\infty} \frac{2 \cdot 10^k}{10^{z(k-1)} \cdot 10^{zk}} \geq 20 \cdot \sum_{k=0}^{+\infty} \frac{10^k}{10^{zk} \cdot 10^{z(k+1)}} = 20 \cdot \sum_{k=0}^{+\infty} \frac{1}{10^{k \cdot (2z-1) + z}}. \quad (6)
 \end{aligned}$$

From (5), (6) and the properties of the geometric progression we know that the theorem is holds for the Smarandache $4n$ -digital subsequence.

This completes the proof of Theorem.

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