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Smarandache $\nu-$Connected spaces

S. Balasubramanian [†], C. Sandhya [‡] and P. Aruna Swathi Vyjayanthi [♯]

† Department of Mathematics, Government Arts College (Autonomous), Karur(T.N.), India
‡ Department of Mathematics, C.S.R. Sharma College, Ongole (A.P.), India
♯ Department of Mathematics, C.R. College, Chilakaluripet(A.P.), India
E-mail: mani55682@rediffmail.com sandhya_karavadi@yahoo.co.uk.
vyju_9285@rediffmail.com

Abstract In this paper Smarandache $\nu-$connectedness and Smarandache locally $\nu-$connectedness in topological space are introduced, obtained some of its basic properties and interrelations are verified with other types of connectedness.

Keywords Smarandache $\nu-$connected and Smarandache locally $\nu-$connected spaces

§1. Introduction

After the introduction of semi open sets by Norman Levine various authors have turned their attentions to this concept and it becomes the primary aim of many mathematicians to examine and explore how far the basic concepts and theorems remain true if one replaces open set by semi open set. The concept of semi connectedness and locally semi connectedness are introduced by Das and J. P. Sarkar and H. Dasgupta in their papers. Keeping this in mind we here introduce the concepts of connectedness using $\nu-$open sets in topological spaces. Throughout the paper a space X means a topological space (X, τ). The class of $\nu-$open sets is denoted by $\nu - O(X, \tau)$ respectively. The interior, closure, $\nu-$interior, $\nu-$closure are defined by A^o, A^-, νA^o, νA^- respectively. In section 2 we discuss the basic definitions and results used in this paper. In section 3 we discuss about Smarandache $\nu-$connectedness and $\nu-$components and in section 4 we discuss locally Smarandache $\nu-$connectedness in the topological space and obtain their basic properties.

§2. Preliminaries

A subset A of a topological space (X, τ) is said to be regularly open if $A = ((A)^-)^o$, semi open(regularly semi open or $\nu-$open) if there exists an open(regularly open) set O such that $O \subset A \subset (O)^-$ and $\nu-$closed if its complement is $\nu-$open. The intersection of all $\nu-$closed sets containing A is called $\nu-$closure of A, denoted by $\nu(A)^-$. The class of all $\nu-$closed sets are denoted by $\nu-$CL(X, τ). The union of all $\nu-$open sets contained in A is called the $\nu-$interior of A, denoted by $\nu(A)^o$. A function f: $(X, \tau) \to (Y, \sigma)$ is said to be $\nu-$continuous if the inverse image of any open set in Y is a $\nu-$open set in X; said to be $\nu-$irresolute if the inverse image of any $\nu-$open set in Y is a $\nu-$open set in X and is said to be $\nu-$open if the image of every $\nu-$open set is $\nu-$open. f is said to be $\nu-$homeomorphism if f is bijective, $\nu-$irresolute and $\nu-$open. Let x be a point of (X, τ) and V be a subset of X, then V is said to be $\nu-$neighbourhood of x if there exists a $\nu-$open set U of X such that $x \in U \subset V$. $x \in X$ is said to be $\nu-$limit point of U iff for each $\nu-$open set V containing x, $V \cap (U - \{x\}) \neq \phi$. The set of all $\nu-$limit points of U is called $\nu-$derived set of U and is denoted by $D_\nu(U)$. union and intersection of $\nu-$open sets is not open whereas union of regular and $\nu-$open set is $\nu-$open.

Note 1. Clearly every regularly open set is $\nu-$open and every $\nu-$open set is semi-open but the reverse implications do not holds good. that is, RO(X)$\subset \nu - O(X) \subset$SO(X).

Theorem 2.1. (i)If $B \subset$X such that $A \subset B \subset (A)^-$ then B is $\nu-$open iff A is $\nu-$open.
(ii)If A and R are regularly open and S is $\nu-$open such that $R \subset S \subset (R)^-$. Then $A\cap$R $= \phi$ $\Rightarrow A \cap S = \phi$.

Theorem 2.2. (i) Let $A \subseteq Y \subseteq X$ and Y is regularly open subspace of X then A is $\nu-$open in X iff A is $\nu-$open in $\tau_{/Y}$.
(ii)Let $Y \subseteq X$ and $A \in \nu - O(Y, \tau_{/Y})$ then $A \in \nu - O(X, \tau)$ iff Y is $\nu-$open in X.

Theorem 2.3. An almost continuous and almost open map is $\nu-$irresolute.

Example 1. Identity map is $\nu-$irresolute.

§3.$\nu-$Connectedness.

Definition 3.01. A topological space is said to be Smarandache $\nu-$connected if it cannot be represented by the union of two non-empty disjoint $\nu-$open sets.

Note 2. Every Smarandache $\nu-$connected space is connected but the converse is not true in general is shown by the following example.

Example 2. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, X\}$; then (X, τ) is connected but not $\nu-$connected

Note 3. Every Smarandache $\nu-$connected space is r-connected but the converse is not true in general is shown by the following example.

Example 3. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ then (X, τ) is r-connected but not Smarandache $\nu-$connected

similary one can show that every semi connected space is Smarandache $\nu-$connected but the converse is not true in general.

Theorem 3.01. Let (X, τ) and (Y, σ) be two topological spaces. If $f : (X, \tau) \to (Y, \sigma)$ is a $\nu-$open and $\nu-$continuous mapping, then the inverse image of each $\nu-$open set in Y is $\nu-$open in X

Corollary 3. Let (X, τ) and (Y, σ) be two topological spaces. If $f : (X, \tau) \to (Y, \sigma)$ is an r-open and r-continuous mapping, then the inverse image of each $\nu-$open set in Y is $\nu-$open in X

Theorem 3.02. If $f : (X, \tau) \to (Y, \sigma)$ is a $\nu-$continuous mapping, and (X, τ) is Smarandache $\nu-$connected space, then (Y, σ) is also $\nu-$connected.

Corollary 4. If $f : (X, \tau) \to (Y, \sigma)$ is a r-continuous mapping, and (X, τ) is Smarandache $\nu-$connected space, then (Y, σ) is also Smarandache $\nu-$connected.

Theorem 3.03. Let (X, τ) be a topological space and
(i) A be $\nu-$open. Then A is Smarandache $\nu-$connected if and only if $(A, \tau/A)$ is Smarandache $\nu-$connected
(ii) A be r-open. Then A is Smarandache $\nu-$connected if and only if $(A, \tau/A)$ is Smarandache $\nu-$connected

Lemma 3.01. If A and B are two subsets of a topological space (X, τ) such that $A \subset B$ then $\nu(A)^- \subset \nu(B)^-$

Lemma 3.02. If A is $\nu-$connected and $A \subset C \cup D$ where C and D are $\nu-$separated, then either $A \subset C$ or $A \subset D$.

Proof. Write $A = (A \cap C) \cup (A \cap D)$. Then by lemma 3.01, we have $(A \cap C) \cap (\nu(A)^- \cap \nu(D)^-) \subset C \cap \nu(D)^-$. Since C and D are $\nu-$separated, $C \cap \nu(D)^- = \phi$ and so $(A \cap C) \cap (\nu(A)^- \cap \nu(D)^-) = \phi$. Similar argument shows that $(\nu(A)^- \cap \nu(C)^-) \cap (A \cap D) = \phi$. So if both $(A \cap C) \neq \phi$ and $(A \cap D) \neq \phi$, then A is not Smarandache $\nu-$connected, which is a contradiction for A is Smarandache $\nu-$connected. Therefore either $(A \cap C) = \phi$ or $(A \cap D) = \phi$, which in turn implies that either $A \subset C$ or $A \subset D$.

Lemma 3.03. The union of any family of Smarandache $\nu-$connected sets having non-empty intersection is a Smarandache $\nu-$connected set.

Proof. If $E = \cup E_\alpha$ is not $\nu-$connected where each E_α is Smarandache $\nu-$connected, then $E = A \cup B$, where A and B are $\nu-$separated sets. Let $x \in \cap E_\alpha$ be any point, then $x \in E_\alpha$ for each E_α and so $x \in E$ which implies that $x \in A \cup B$ in turn implies that either $x \in A$ or $x \in B$.

Without loss of generality assume $x \in A$. Since $x \in E_\alpha$, $A \cap E_\alpha \neq \phi$ for every α. By lemma 3.02, either each $E_\alpha \subset A$ or each $E_\alpha \subset B$. Since A and B are disjoint we must have each $E_\alpha \subset A$ and hence each $E \subset A$ which gives that $B = \phi$.

Lemma 3.4. If A is Smarandache $\nu-$connected and $A \subset B \subset \nu(A)^-$, then B is Smarandache $\nu-$connected set.

Proof. If E is not $\nu-$connected, then $E = A \cup B$, where A and B are $\nu-$separated sets. By lemma 3.02 either $E \subset A$ or $E \subset B$. If $E \subset A$, then $\nu(E)^- \subset \nu(A)^-$ and so $\nu(E)^- \cap B \subset \nu(A)^- \cap B = \phi$. On the other hand $B \subset E \subset \nu(E)^-$ and so $\nu(E)^- \cap B$. Thus we have $B = \phi$, which is a contradiction. Hence the Lemma.

Corollary 5. If A is Smarandache $\nu-$connected and $A \subset B \subset (A)^-$, then B is Smarandache $\nu-$connected set.

Lemma 3.5. If $f\colon (X,\tau) \to (Y,\sigma)$ is $\nu-$open and $\nu-$continuous, $A \subset X$ is $\nu-$open. Then if A is $\nu-$connected, f(A) is also Smarandache $\nu-$connected.

Proof. Let $f : (X,\tau) \to (Y,\sigma)$ is open and $\nu-$continuous, $A \subset X$ be open. Since A is $\nu-$connected and open in (X,τ), then $(A, \tau_{/A})$ is also $\nu-$connected (by Th. 3.03). Now $f_{/A} : (A, \tau_{/A}) \to (f(A), \sigma_{f(A)})$ is onto and $\nu-$continuous and so by theorem 3.02 $f(A)$ is also $\nu-$connected in $(f(A), \sigma_{f(A)})$. Now for f is open, f(A) is open in (Y,σ) and so by theorem 3.03, f(A) is Smarandache $\nu-$connected in (Y,σ)

We have the following corollaries from the above theorem

Corollary 6. If $f\colon (X,\tau) \to (Y,\sigma)$ is r-open and r-continuous, $A \subset X$ is r-open. Then if A is $\nu-$connected, then f(A) is also Smarandache $\nu-$connected.

Corollary 7. If $f\colon (X,\tau) \to (Y,\sigma)$ is $\nu-$open and $\nu-$continuous, $A \subset X$ is r-open. Then if A is connected, then f(A) is also Smarandache $\nu-$connected.

Definition 3.02. Let (X,τ) be a topological space and $x \in X$. The $\nu-$component of x, denoted by ν C(x), is the union of Smarandache $\nu-$connected subsets of X containing x.

Further if $E \subset X$ and if $x \in E$, then the union of all $\nu-$connected set containing x and contained in E is called the $\nu-$component of E corresponding to x. By the term that C is a $\nu-$component of E, we mean that C is $\nu-$component of E corresponding to some point of E.

Lemma 3.06. Show that $\nu C(x)$ is Smarandache $\nu-$connected for any $x \in X$

Proof. As the union of any family of Smarandache $\nu-$connected sets having a non-empty intersection is a Smarandache $\nu-$connected set, it follows that $\nu C(x)$ is Smarandache $\nu-$connected

Theorem 3.04. In a Topological space (X,τ),
(i) Each $\nu-$component ν(x) is a maximal Smarandache $\nu-$connected set in X.
(ii) The set of all distinct $\nu-$components of points of X form a partition of X and (iii) Each $\nu(x)$ is $\nu-$closed in X

Proof. (i) follows from the definition 3.02

(ii) Let x and y be any two distinct points and $\nu C(x)$ and $\nu C(y)$ be two r-components of x and y respectively. If $\nu C(x) \cap \nu C(y) \neq \phi$, then by lemma 3.03, $\nu C(x) \cup \nu C(y)$ is Smarandache $\nu-$connected. But $\nu C(x) \subset \nu C(x) \cup \nu C(y)$ which contradicts the maximality of $\nu C(x)$.

Let $x \in X$ be any point, then $x \in \nu C(x)$ implies $\cup\{x\} \subset \cup \nu C(x)$ for all $x \in X$ which implies $X \subset \cup \nu C(x) \subset X$. Therefore $\cup \nu C(x) = X$

(iii) Let $x \in X$ be any point, then $(\nu C(x))^-$ is a $\nu-$connected set containing x. But νC(x) is the maximal Smarandache $\nu-$connected set containing x, therefore $(\nu C(x))^- \subset \nu C(x)$. Hence $\nu C(x)$ is $\nu-$closed in X.

§4. Locally $\nu-$connectedness

Definition 4.01. A topological space (X, τ) is called
(i) Smarandache locally $\nu-$connected at $x \in X$ iff for every $\nu-$open set U containing x, there exists a Smarandache $\nu-$connected open set C such that $x \in C \subset U$.
(ii) Smarandache locally $\nu-$connected iff it is Smarandache locally $\nu-$connected at each $x \in X$.

Remark 3. Every Smarandache locally $\nu-$connected topological space is Smarandache locally connected but converse is not true in general.

Remark 4. Smarandache local $\nu-$connectedness does not imply Smarandache $\nu-$connectedness as shown by the following example.

Example 4. X = {a, b, c} and $\tau = \{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$ then (X, τ) is Smarandache locally $\nu-$connected but not Smarandache $\nu-$connected.

Remark 5. Smarandache $\nu-$connectedness does not imply Smarandache local $\nu-$connectedness in general.

Theorem 4.01. A topological space (X, τ) is Smarandache locally $\nu-$connected iff the $\nu-$components of $\nu-$open sets are open sets.

Theorem 4.02. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a $\nu-$continuous open and onto mapping, and (X, τ) is Smarandache locally $\nu-$connected space, then (Y, σ) is also locally Smarandache $\nu-$connected.

Proof. Let U be any $\nu-$open subset of Y and C be any $\nu-$component of U, then $f^{-1}(U)$ is $\nu-$open in X. Let A be any $\nu-$component of $f^{-1}(U)$. Since X is locally $\nu-$connected and $f^{-1}(U)$ is $\nu-$open, A is open by theorem 4.01. Also f(A) is $\nu-$connected subset of Y and since C is $\nu-$component of U, it follows that either $f(A) \subset C$ or $f(U) \cap C = \phi$. Thus $f^{-1}(C)$ is the union of collection of $\nu-$components of $f^{-1}(U)$ and so $f^{-1}(C)$ is open. As f is open and onto, $C = f \circ f^{-1}(C)$ is open in Y. Thus any $\nu-$component of $\nu-$open set in Y is open in Y and hence by above theorem Y is Smarandache locally $\nu-$connected.

Corollary 8. If $f:(X,\tau)\to(Y,\sigma)$ is a r-continuous r-open and onto mapping, and (X,τ) is Smarandache locally $\nu-$connected space, then (Y,σ) is also Smarandache locally $\nu-$connected.

Proof. Immediate consequence of the above theorem.

Note 4. semi connectedness need not imply and implied by locally semi connectedness. Similarly a Smarandache $\nu-$connected space need not imply and implied by Smarandache locally $\nu-$connected in general.

Theorem 4.03. A topological space (X,τ) is Smarandache locally $\nu-$connected iff given any $x\in X$ and a $\nu-$open set U containing x, there exists an open set C containing x such that C is contained in a single $\nu-$component of U.

Proof. Let X be Smarandache locally $\nu-$connected, $x\in X$ and U be a $\nu-$open set containing x. Let A be a $\nu-$ component of U containing x. Since X is Smarandache locally $\nu-$connected and U is $\nu-$open, there is a Smarandache $\nu-$connected set C such that $x\in C\subset U$. By theorem 3.01, A is the maximal Smarandache $\nu-$connected set containing x and so $x\in C\subset A\subset U$. Since $\nu-$components are disjoint sets, it follows that C is not contained in any other $\nu-$component of U.

Conversely, suppose that given any point $x\in X$ and any $\nu-$open set U containing x, there exists an open set C containing x which is contained in a single $\nu-$component F of U. Then $x\in C\subset F\subset U$. Let $y\in F$, then $y\in U$. Thus there is an open set O such that $y\in O$ and O is contained in a single $\nu-$component of U. As the $\nu-$components are disjoint sets and $y\in F, y\in O\subset F$. Thus F is open. Thus for every $x\in X$ and for every $\nu-$open set U containing x, there exists a Smarandache $\nu-$connected open set F such that $x\in F\subset U$. Thus (X,τ) is Smarandache locally $\nu-$connected at x. Since $x\in X$ is arbitrary, (X,τ) is Smarandache locally $\nu-$connected.

Remark 6.

$$\begin{array}{ccc} \text{Connected} \Leftarrow & & \text{semi-connected} \\ \Downarrow & & \Downarrow \\ \text{r-Connected} \Leftarrow & & \nu-\text{Connected.} \end{array}$$

none is reversible

Example 5. FOR X = {a, b, c, d}; $\tau_1=\{\phi,\{b\},\{a,b\},\{b,c\},\{a,b,c\},X\}$
$\tau_2=\{\phi,\{a\},\{b\},\{a,b\},X\}$ and $\tau_3=\{\phi,\{a\},\{b\},\{d\},\{a,b\},\{a,d\},\{b,d\},\{a,b,c\},\{a,b,d\},X\}$

(X,τ_1) is both r-connected and Smarandache $\nu-$connected; (X,τ_2) is r-connected but not Smarandache $\nu-$connected and (X,τ_3) is neither r-connected and nor Smarandache $\nu-$connected

Conclusion.

In this paper we defined new type of connectedness using $\nu-$open sets and studied their interrelations with other connectedness.

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Smarandache #-rpp semigroups whose idempotents satisfy permutation identities

Chengxia Lei† and Xiaomin Zhang‡

† ‡ Departmrnt of Mathematics, Linyi Normal University, Linyi 276005, Shandong, P.R.China.
E-mail: leichx001@163.com lygxxm@tom.com

Abstract The aim of this paper is to study Smarandache #-rpp semigroups whose idempotents satisfy permutation identities. After some properties are obtained, the weak spined product structure of such semigroups is established.

Keywords Smarandache #-rpp semigroups, normal band, weak spined product.

§1. Introduction

Similar to rpp rings, a semigroup S is called an rpp semigroup if for any $a \in S, aS^1$ regarded as a right S^1 system is projective. In the study of the structure of rpp semigroups, Fountain[1] considered a Green-like right congruence relation $\mathcal{L}^*$ on a semigroup S defined by $(a, b \in S)a\mathcal{L}^*b$ if and only if $ax = ay \Leftrightarrow bx = by$ for all $x, y \in S^1$. Dually, we can define the left congruence relation $\mathcal{R}^*$ on a semigroup S. It can be observed that for $a, b \in S, a\mathcal{L}^*b$ if and only if $a\mathcal{L}b$ when S is a regular semigroup. Also, we can easily see that a semigroup S is an rpp semigroup if and only if each $\mathcal{L}^*$-class of S contains at least one idempotent. Later on, Fountain[2]called a semigroup S an abundant semigroup if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class of contain at least one idempotent. An important subclass of the class of rpp semigroups is the class of C-rpp semigroups. We call an rpp semigroup S a C-rpp semigroup if the idempotents of S are central. It is well known that a semigroup S is a C-rpp semigroup if and only if S is a strong semilattice of left cancellative monoids (see [1]). Because a Clifford semigroup can always be expressed as a strong semilattice of groups, we see immediately that the concept of C-rpp semigroups is a proper generalization of Clifford semigroups. Guo-Shum-Zhu [3] called an rpp semigroup S a strongly rpp semigroup if every L_a^* contains a unique idempotent $a^+ \in L_a^* \cap E(S)$ such that $a^+a = a$ holds, where $E(S)$ is the set of all idempotents of S. They then called a strongly rpp semigroup S a left C-rpp semigroup if $\mathcal{L}^*$ is a congruence on S and $eS \subseteq Se$ holds for any $e \in E(S$. It is noticed that the set $E(S)$ of idempotents of a left C-rpp semigroup S forms a left regular band, that is,$ef = efe$ for $e, f \in E(S)$. Because of this crucial observation, we can describe the left C-rpp semigroup by using the left regular band and the C-rpp semigroup.The structure of left C- rpp semigroups and abundant semigroups has been investigated by many authors (see[4-12], In particular, it was proved in [3] that if S is a strongly rpp semigroup whose set of idempotents $E(S)$ forms a left regular band, then S is a left C-rpp semigroup if and only if S is a semilattice of direct products of a left zero band and a left cancellative monoid, that is, the left C-rpp semigroup S is expressible as a semilattice of left cancellative strips.

Let S be a semigroup, A a subset of S and let

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

a non-identity permutation on n objects. Then a semigroup A is said to satisfy the permutation identity determined by σ (in short, to satisfy a permutation identity if there is no ambiguity) If $(\forall x_1, x_2, \cdots, x_n \in A)x_1x_2\cdots x_n = x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}$, Where $x_1x_2\cdots x_n$ is the product of $x_1, x_2, \cdots, x_n$ in S. If $A = S$, then S is called a PI- semigruop. Guo[13] investigated abundant semigroups whose idempotents satisfy permutation identities,and the quasi-spined product structure of such semigroups was established. In particular, the structure of PI-abundant semigroups was obtained. Later,Guo[14] again discussed strongly rpp semigroups whose idempotents satisfy permutation identities,Du-He[15]obtained the structure of eventually strongly rpp semigroups whose idempotents satisfy permutation identities.

By modifying Green's star relations, Kong-Shum[16] have introduced a new set of Green's #-relations on a semigroup and by using these new Green's relations, they were able to give a description for a wider class of abundant semigroups, namely, the class of #- abundant semigroups(see[16]). As a generalization of rpp semigroups whose idempotents satisfy permutation identities, the aim of this paper is to investigate Smarandache #-rpp semigroups whose idempotents satisfy permutation identities,that is,PI- #-rpp.

For terminology and notations not give in this paper, the reader is referred to references[17,18].

§2.Preliminaries

We first recall that the Green's #- relations defined in [16].

$$a\mathcal{L}^{\#}b \text{ if and only if for all } e, f \in E(S^1), ae = af \Leftrightarrow be = bf,$$
$$a\mathcal{R}^{\#}b \text{ if and only if for all } e, f \in E(S^1), ea = fa \Leftrightarrow eb = fb.$$

We easily check that the relations $\mathcal{L}^{\#}$ and $\mathcal{R}^{\#}$ are equivalent relation. However, $\mathcal{L}^{\#}$ is not a right compatible (that is,right congruence),$\mathcal{R}^{\#}$ is not a left compatible (that is,left congruence), and $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \mathcal{L}^{\#}, \mathcal{R} \subseteq \mathcal{R}^* \subseteq \mathcal{R}^{\#}$. A semigroup S is right #-abundant if each $\mathcal{L}^{\#}$- class of S contains at least one idempotent,write as #- rpp. we can define left # -abundant semigroups dually, write as #-lpp. A semigroup S is called #-abundant if it is both right # -abundant and left #- abundant. Abundant semigroups S is proper subclass of #-abundant semigroups(see[16]), and if a, b are regular elements of S , then $a\mathcal{L}^{\#}b$ if and only if $a\mathcal{L}b$ (see[16]).

If there is no special indication of $\mathcal{L}^{\#}$ relation on S, we always suppose $\mathcal{L}^{\#}$ is a right congruence on S, and always suppose that S is a Smarandache #-rpp semigroup whose idempotents satisfy permutation identities,that is,PI- #-rpp.

Lemma 2.1. **[16]** For any $e \in E(S), a \in S$, the following conditions are equivalent:
(1)$(e, a) \in \mathcal{L}^{\#}$;
(2)$a = ae$ and $ag = ah \Leftrightarrow eg = eh(\forall g, h \in E(S^1))$.

A band B is that a semigroup in which every element is an idempotent. We call a band B a [left,right]normal band if B satisfies the identity $(abc = acb, abc = bac)abcd = acbd$.

Lemma 2.2. **[14]** The following statements are equivalent for a band B:

(1)B is normal;

(2)B is a strong semilattice of rectangular bands;

(3)$\mathcal{L}$ and $\mathcal{R}$ are a left normal band congruence and a right normal band congruence on, respectively.

It is well known that any band is a semilattice of rectangular bands. If $B = \cup_{\alpha\in Y} B_\alpha$ is the semilattice decomposition of a band B into rectangular bands B_α with $\alpha \in Y$, then we shall write $B_\alpha = E(e)$ for $e \in B_\alpha$ and $B_\alpha \geq B_\beta$ when $\alpha \geq \beta$ on the indexed semilattice Y. Next, we always assume that S is a Smarandache #-rpp semigroup satisfying the permutation identity: $x_1x_2\cdots x_n = x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}$. We denote idempotents in the $\mathcal{L}^{\#}$- class of a by $a^{\#}$ for every $a \in S$.

Lemma 2.3. **[17]** $E(S)$ is a normal band.

Lemma 2.4. Let $a \in S, e, f \in E(S)$. Then

(1)$efa = efa^{\#}$;

(2)$eaf = eaef$.

Proof. Suppose that i is the minimum positive number such that $\sigma(i) \neq i$. Obviously $i < \sigma(i)$.

(1)Take $x_j = e$ when $1 \leq j < i, x_j = f$, if $1 \leq j < i, x_j = f, i \leq j < \sigma(i), \sigma(i) = a$ otherwise $\sigma(i) = a^{\#}$. Then $e(x_1x_2\cdots x_n)a^{\#} = efa$. On the other hand, by Lemma 2.4, $e(x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}) = eafa^{\#}$, and hence, $efa = eafa^{\#}$.

(2)Now, assume that $x_j = e$ when $1 \leq j < \sigma(i), x_{\sigma(i)} = a$ otherwise $x_j = f$, then $e(x_1x_2\cdots x_n)f = eaf$, and $ee(x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)})f = eaef$ or $eafef$. Hence, by Lemma 2.3, we can infer that $eafef = eaa^{\#}fef = eaa^{\#}eff = eaef$. Thus, we get $eaf = eaef$

Now suppose that $E(S) = [Y, E_\alpha, \Psi_{\alpha,\beta}]$ is the strong semilattice of rectangular bands E_α. If $e \in E_\alpha$, we will write the rectangular band E_α by $E(e)$. Also if $E_\alpha E_\beta \subseteq E_\beta$, then we can write $E_\beta \leq E_\alpha$.

Lemma 2.5. For every $a, b \in S$ and $f \in E(S)$. If $a = bf$,then $E(a^{\#}) \leq E(f)$.

Proof. If $a = bf$, then $a = af$. By the definition of $\mathcal{L}^{\#}$, we have $a^{\#} = a^{\#}f$, and so $E(a^{\#}) \leq E(f)$.

§3.The structure of Smarandache #-rpp semigroups satisfying permutation identities

In this section, we will give the concept of weak spined product of semigroups, and the structure of Smarandache #-rpp semigroups satisfying permutation identities is obtained.

We now define a relation ε on S as follows:

$$a\varepsilon b \text{ if and only if for some } f \in E(b^{\#}), a = bf,$$

where $a, b \in S$.

Lemma 3.1.(1) ε is a congruence on S preserving $\mathcal{L}^{\#}$-class;
(2)$\varepsilon \cap \mathcal{L}^{\#} = \imath_S$ (the identical mapping on S).

Proof. First of all,we prove that ε is an equivalence relation. Obviously, $a\varepsilon a$ since $a = aa^{\#}$ for all $a \in S$. Hence ε is reflexive.

Let $a, b \in S$ with $a\varepsilon b$. Then for some $f \in E(b^{\#}), a = bf$. By Lemma 2.5,$E(a^{\#}) < E(f) = E(b^{\#})$. It follows that $a^{\#}b^{\#} \in E(a^{\#})$, and $a\varepsilon b$ means that $E(b^{\#}) < E(b^{\#})$. Since

$$a(b^{\#}b^{\#}) = ab^{\#} = bfb^{\#} = bb^{\#}fb^{\#} = bb^{\#} = b.$$

We have $b\varepsilon a$, and hence ε satisfies the symmetric relation. On the other hand, by the above proof, $b\varepsilon a$ means that $E(b^{\#}) \leq E(a^{\#})$. Therefore,$E(b^{\#}) = E(a^{\#})$.

Next,we show that ε is transitive. We let $a, b, c \in S$ with $a\varepsilon b, b\varepsilon c$. Then we have $E(a^{\#}) = E(b^{\#}) = E(c^{\#})$. By the definition of ε, there exist $e, f \in E(c^{\#})$ such that $a = be, b = cf$. Thus, $a = efe$. Notice that $fe \in E(c^{\#})$,we get $a\varepsilon c$. So ε is indeed an equivalence relation on S.

Following we show that ε is both left and right compatible. Now let $a, b, c \in S$ and $a\varepsilon b$. Then there exists $f \in E(b^{\#})$ such that $a = bf$ .Obviously,$ca = cbf = cb(cb)^{\#}f$. By Lemma2.5, $E(ca^{\#}) \leq E((ca)^{\#})$ and $E((ca)^{\#}) \leq E(f)$, and hence $(ca)^{\#}b^{\#} \in E((ca)^{\#})$ but $a = bf$, this implies that $b = ab^{\#}$. We have $cb = cab^{\#} = ca(ca)^{\#}b^{\#}$. Hence $ca\varepsilon cb$. By lemma 2.4, we have

$$ac = bfc = bb^{\#}fc = bb^{\#}cfc = bc(fc^{\#}) = bc(bc)^{\#}fc^{\#}.$$

We can deduce that$E((ac)^{\#}) \leq E((bc)^{\#})$ and $E((ac)^{\#}) \leq E(fc^{\#})$ by lemma 2.5. A similar argument for $b = ab^{\#}$, we can infer that $E((bc)^{\#}) \leq E((ac)^{\#})$. Hence $E((ac)^{\#}) = E((bc)^{\#})$. This means that $(bc)^{\#}fc^{\#} \in E((bc)^{\#})$. Therefore,$ac\varepsilon bc$, and hence ε is a congruence on S.

Finally, we prove that ε preserves $\mathcal{L}^{\#}$- class. We need to prove that if $a\mathcal{L}^{\#}b$ then $(ag)\varepsilon = (ah)\varepsilon$ implies $(bg)\varepsilon = (bh)\varepsilon$, where $g, h \in E(S^1)$ and $g\varepsilon, h\varepsilon \in (S/\varepsilon)^1$. Since $(ag)\varepsilon(ah)$, by the definition of ε, we have $ah = agf$ for some $f \in E((ag)^{\#})$. Therefore, we obtain $bh = bgf$ since $a\mathcal{L}^{\#}b$. Since $\mathcal{L}^{\#}$ is a right congruence on S, we have $ag\mathcal{L}^{\#}bg$ for $g \in E(S^1)$ so that $E((ag)^{\#}) = E((bg)^{\#})$ and thereby $f \in E((ag)^{\#}) = E((bg)^{\#})$. Thus, by the definition of ε, $bh = bgf$ allows $(bg, bh) \in \varepsilon$, that is, $(bg)\varepsilon = (bh)\varepsilon$. According to this result and its dual and the definition of $\mathcal{L}^{\#}$, we conclude that $a\varepsilon\mathcal{L}^{\#(S/\varepsilon)}b\varepsilon$.

(2) Let $(a, b) \in \varepsilon \cap \mathcal{L}^{\#}$. Then $a = bf$ for some $f \in E(b^{\#})$. Since $a\mathcal{L}^{\#}b$,we get $a^{\#}\mathcal{L}^{\#}b^{\#}$. So $a = ab^{\#} = bfb^{\#} = bb^{\#}fb^{\#} = bb^{\#} = b$. Hence $\varepsilon \cap \mathcal{L}^{\#} = \imath_S$.

Lemma 3.2. $E(S/\varepsilon)$ is a left normal band.

Proof. Since S is a PI-semigroup, we easily check that S/ε is a PI-semigroup. Notice that $a\varepsilon \in E(S/\varepsilon)$ implies $a \in E(S)$ and $\varepsilon \cap (E(S) \times E(S)) = \mathcal{R}$. So $E(S/\varepsilon) = E/\mathcal{R}$. Thus $E(S/\varepsilon)$ is a left normal band.

Lemma 3.3. If $E(S)$ is a left normal band, then S satisfies the identity $abc = acb$.

Proof. Let i have the same neaning as that in the proof of lemma 2.4. For every $a, b, c \in S$, take $x_i = b, x_{\sigma(i)} = c$ and $x_j = a^{\#}$ otherwise then $a^{\#}(x_1x_2 \cdots x_n)c^{\#} = a^{\#}ba^{\#}ca^{\#}c^{\#}, a^{\#}bca^{\#}c^{\#}$ $a^{\#}ba^{\#}c$ or $a^{\#}bc$. By hypothesis and Lemma 2.4,we have

$$a^{\#}ba^{\#}ca^{\#}c^{\#} = a^{\#}ba^{\#}c = a^{\#}bb^{\#}a^{\#}c = a^{\#}bb^{\#}a^{\#}b^{\#}c = a^{\#}ba^{\#}b^{\#}c = a^{\#}bb^{\#}c = a^{\#}bcc^{\#} = a^{\#}bc.$$

Thus, we obtain that $a^{\#}(x_1x_2\cdots x_n)c^{\#} = a^{\#}ca^{\#}ba^{\#}c^{\#}, a^{\#}cba^{\#}c^{\#}$ or $a^{\#}cbc^{\#}$. In other word, we have

$$\begin{aligned} a^{\#}ca^{\#}bc^{\#} &= a^{\#}ca^{\#}ba^{\#}c^{\#} = a^{\#}cc^{\#}a^{\#}ba^{\#}c^{\#} = a^{\#}cc^{\#}a^{\#}cba^{\#}c^{\#} \\ &= a^{\#}c^{\#}ac^{\#}ba^{\#}c^{\#} = a^{\#}cc^{\#}ba^{\#}c^{\#} = a^{\#}cba^{\#}c^{\#} \\ &= a^{\#}cbc^{\#} = a^{\#}cc^{\#}b(b^{\#}c^{\#}b^{\#}) = a^{\#}cc^{\#}bc^{\#}b^{\#} \\ &= a^{\#}cc^{\#}bb^{\#} = a^{\#}cb. \end{aligned}$$

Hence $a^{\#}(x_{\sigma(1)})x_{\sigma(2)}\cdots x_{\sigma(n)} = a^{\#}ca^{\#}, ba^{\#}c^{\#}, a^{\#}cba^{\#}c^{\#}$ or $a^{\#}cbc^{\#}$. So we obtain $a^{\#}bc = a^{\#}cb$, and hence we have $abc = aa^{\#}bc = aa^{\#}cb = acb$.

Theorem 3.4. Let S be a Smarandache #-rpp semigroup with set $E(S)$ of idempotents. Denote by λ_a the inner left translation of determined by $a \in S$. Then the following statements are equivalent:

(1) S satisfies permutation identities;

(2) S satisfies the identity ;

(3) For all $e \in E(S)$,eSe is a commutative semigroup and λ_e is a homomorphism;

(4) For all $e \in E(S)$,eS satisfies the identity:$abc = bac$ and λ_e is a homomorphism.

Proof. (1) $\Rightarrow$ (2). Let S be a PI-Smarandache #-rpp semigroup. For every $a, b, c, d \in S$, by Lemma 3.1-3.3,$(adcd)\varepsilon = (acbd)\varepsilon$. Then for some $f \in E((acbd)^{\#}), abcd = acbdf$. Furthermore, we also have

$$\begin{aligned} abcd = abcdd^{\#}acbdfd^{\#} &= acbd(acbd)^{\#}fd^{\#} \\ &= acbd(acbd)^{\#}f(acbd)^{\#}d^{\#} \\ &= acbd(acbd)^{\#}d^{\#} = acbd. \end{aligned}$$

by Lemma 2.3.

(2) $\Rightarrow$ (3). Assume that (2) holds. Let $e \in E(S)$. Then for every $a, b \in eSe$. we have $a = ea = ae$ and $b = eb = be$. Hence $ab = eabe = ebae = eab$. This means that eSe is a commutative semigroup. On the other hand, since $\lambda_e(a)\lambda_e(b) = eaeb = eeab = eab = \lambda_e(ab)$ is a homomorphism of into itself. Therefore (3) holds.

(3) $\Rightarrow$ (4). Suppose that (3) holds. It remains to prove the first part. For all $a, b, c \in eS$, we get $a = ea, b = eb$ and $c = ec$. Since λ_e is a homomorphism, We have

$$abc = eaebec = (eae)(ebe)c = (ebe)(eae)(ebe)c = (eb)(ea)(ec) = bac.$$

(4) $\Rightarrow$ (2). Let $a, b, c, d \in S$. Then

$$abcd = a(a^{\#}bcd) = a(a^{\#}b)(a^{\#}c)(a^{\#}d) = a(a^{\#}c)(a^{\#}b)(a^{\#}d) = a(a^{\#}cbd) = acbd.$$

(2) $\Rightarrow$ (1). This part is trivial.

Let S be a Smarandache #-rpp semigroup whose idempotents form a subsemigroup $E(S)$. Let Y be the structure semilattice of $E(S)$ such that $E(S) = \cup_{\alpha\in Y}E_\alpha$ is structure decomposition of $E(S)$. Now let B be a right normal band and $B = \cup_{\alpha\in Y}B_\alpha$ is a semilattice composition of the right zero band B_α. For $a \in S$, if $a^\# \in E_\alpha$ we denote $a^\triangle = \alpha$. Take $M = \{(a, s) \in S \times B | x \in B_{a^\triangle}$. Define a multiplication "$\circ$ " on M as follows:

$$(a, x) \circ (b, y) = (ab, y\varphi_{b^\triangle,(ab)^\triangle}), i.e. = (ab, zy),$$

when $z \in B_(ab)^\triangle$. Notice that $ab = abb^\#$, we have $(ab)^\# = (ab)^\# b^\#$. It follows that $(ab)^\triangle = (ab)^\triangle b^\triangle$. This means that $(ab)^\triangle \leq b^\triangle$("$\triangle$ " is the natural order). Accordingly, $y\varphi_{b^\triangle,(ab)^\triangle} \in B_{(ab)^\triangle}$. So M is well defined and with respect to "$\circ$ ", M is closed.

Lemma 3.5. M is a semigroup.

Proof. Because with respect to "$\circ$ ", M is closed. We only need to show that "$\circ$ " satisfies the associative law. Let $(a, x), (b, y), (c, z) \in M$. Then by the above statement, we can show that $(abc)^\triangle \leq (bc)^\triangle \leq c^\triangle$. Thus

$$\begin{aligned}((a, x) \circ (b, y)) \circ (c, z) &= (ab, y\varphi_{b^\triangle,(ab)^\triangle}) \circ (c, z) = (abc, z\varphi_{c^\triangle,(abc)^\triangle}) \\ &= (a, x) \circ (bc, z\varphi_{c^\triangle,(bc)^\triangle}) = (a, x) \circ ((b, y) \circ (c, z)).\end{aligned}$$

So " $\circ$" is associative. Hence M is indeed a semigroup.

Definition 3.6. We call $(M, \circ)$ above the weak–spined product of S and B , and denote it by $WS(S, B)$.

Lemma 3.7. If S satisfies the identity $abc = acb$, the $WS(S, B)$ satisfies the identity $abcd = acbd$.

Proof. Let $(a.i), (b, j), (c, k), (d, l) \in WS(S, B)$. Then

$$(a.i) \circ (b, j) \circ (c, k) \circ (d, l) = (abcd, l\varphi_{d^\triangle,(abcd)^\triangle}) = (abcd, l\varphi_{d^\triangle,(acbd)^\triangle}) = (a.i) \circ (c, k) \circ (b, j) \circ (d, l).$$

Hence $WS(S, B)$ satisfies the identity.

Theorem 3.8. A Smarandache #-rpp semigroup is a PI-Smarandache #-rpp semigroup if and only if it is isomorphic to the weak spined product of a Smarandache #-rpp semigroup satisfying the identity $abc = acb$ and a right normal band.

Proof. By Lemma 3.7, it suffices to prove the “only if ”part. Suppose that S is a PI-Smarandache #-rpp semigroup with normal band $E(S)$. Then by Lemma 3.2 and Lemma 3.3, S/ε is a Smarandache #-rpp semigroup satisfying the identity $abc = aeb$. Let Y be the structure decomposition of $E(S)$. By Lemma 2.2, we have $E(S)/\mathcal{L} = \cup_{\alpha\in Y} E_\alpha/\mathcal{L}$ that is a right normal band. Notice that ε is idempotent pure and $\varepsilon \cap (E(S) \times E(S)) = \mathcal{R}$, we can easily know that $E(S/\varepsilon) = E(S)/\mathcal{R} = \cup_{\alpha\in Y} E_\alpha/\mathcal{R}$. Thus we can consider the weak spined product $WS(S/\varepsilon, E(S)/\mathcal{L})$.

Define

$$\theta : S \to WS(S/\varepsilon, E(S)/\mathcal{L}), a \mapsto (ae, \overline{a^\#}),$$

where $\overline{a^\#}$ is the $\mathcal{L}$ - class of containing $a^\#$. In order to prove the theorem, we need only to show that θ is an isomorphism. By Lemma 3.1, θ is well defined and injective. Take any element $(x, \overline{e}) \in WS(S/\varepsilon, E(S)/\mathcal{L})(e \in E(S))$, since $x \in S/\varepsilon$, there exists $s \in S$ such that $x = s\varepsilon$. By the definition of $WS(S/\varepsilon, E(S)/\mathcal{L}), s^\# \mathcal{D}^E a$. This means that θ is onto. For all $s, t \in S$, since $st = (st)t^\#$, by the definition of $\mathcal{L}^\#$, we have $(st)^\# = (st)^\# t^\#$. Furthermore, by Lemma 3.1, we can get $\overline{(st)^\#} \in B((st)\varepsilon)^\triangle$. Then $\theta(st) = ((st)\varepsilon, \overline{(st)^\#}) = ((st)\varepsilon, \overline{(st)^\# t^\#}) = (s\varepsilon, \overline{s^\#})(t\varepsilon, \overline{t^\#}) = \theta(s)\theta(t)$. To sum up,$\theta$ is an isomorphism of S onto $WS(S/\varepsilon, E(S)/\mathcal{L})$. The proof is completed.

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