Scientia Magna
Vol. 2 (2006), No. 1, 45-49

Some identities involving the Smarandache ceil function

Wang Yongxing
School of Science, Xidian University, Xi'an, Shaanxi, P.R.China

Abstract For any fixed positive integer n, the Smarandache ceil function of order k is denoted by $N^* \to N$ and has the following definition:

$$S_k(n) = \min\{x \in N : n \mid x^k\} \, (\forall n \in N^*) \, .$$

In this paper, we use the elementary methods to study the arithmetical properties of $S_k(n)$, and give some identities involving the Smarandache ceil function.

Keywords Smarandache ceil function; Arithmetical properties; Identity.

§1. Introduction

For any fixed positive integer n, the Smarandache ceil function of order k is denoted by $N^* \to N$ and has the following definition:

$$S_k(n) = \min\{x \in N : n \mid x^k\} \, (\forall n \in N^*) \, .$$

For example, $S_2(1) = 1$, $S_2(2) = 2$, $S_2(3) = 3$, $S_2(4) = 2$, $S_2(5) = 5$, $S_2(6) = 6$, $S_2(7) = 7$, $S_2(8) = 4$, $S_2(9) = 3$, $\cdots\cdots$. $S_3(1) = 1$, $S_3(2) = 2$, $S_3(3) = 3$, $S_3(4) = 2$, $S_3(5) = 5$, $S_3(6) = 6$, $S_3(7) = 7$, $S_3(8) = 2$, $\cdots\cdots$.

The dual function of $S_k(n)$ is defined as

$$\overline{S_k}(n) = \max\{x \in N : x^k \mid n\} \, (\forall n \in N^*) \, .$$

For example, $\overline{S_2}(1) = 1$, $\overline{S_2}(2) = 1$, $\overline{S_2}(3) = 1$, $\overline{S_2}(4) = 2$, $\cdots$. For any primes p and q with $p \neq q$, $\overline{S_2}(p^2) = p$, $\overline{S_2}(p^{2m+1}) = p^m$ and $\overline{S_2}(p^m q^n) = \overline{S_2}(p^m)\overline{S_2}(q^n)$.

These functions were introduced by F.Smarandache who proposed many problems in [1]. There are many papers on the Smarandache ceil function and its dual. For example, Ibstedt [2] and [3] studied these functions both theoretically and computationally, and got the following conclusions:

$$(\forall a, b \in N^*)(a, b) = 1 \Rightarrow S_k(ab) = S_k(a)S_k(b),$$

$$S_k(p_1^{\alpha_1} p_2^{\alpha_2} . \cdots .p_r^{\alpha_r}) = S_k(p_1^{\alpha_1}) . \cdots .S_k(p_r^{\alpha_r}).$$

Ding Liping [4] studied the mean value properties of the Smarandache ceil function, and obtained a sharp asymptotic formula for it. That is, she proved the following conclusion:

Let real number $x \geq 2$, then for any fixed positive integer $k \geq 2$, we have the asymptotic formula

$$\sum_{n \leq x} S_k(n) = \frac{x^2}{2}\zeta(2k-1)\prod_p \left[1 - \frac{1}{p(p+1)}\left(1 + \frac{1}{p^{2k-3}}\right)\right] + O\left(x^{\frac{3}{2}+\epsilon}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, $\prod_p$ denotes the product over all prime p, and ϵ denotes any fixed positive number.

Lu Yaming [7] studied the hybrid mean value involving $\overline{S_k}(n)$ and $d(n)$, and obtained the following asymptotic formula:

$$\sum_{n \leq x} d(\overline{S_k}(n)) = \zeta(k)x + \zeta\left(\frac{1}{k}\right) + O\left(x^{\frac{1}{k+1}}\right),$$

where $\zeta(s)$ is the Riemann zeta-function and $d(n)$ is the Dirichlet divisor function.

In this paper, we use the elementary methods to study the arithmetical properties of Smarandache ceil function and its dual, and give some interesting identities involving these functions. That is, we shall prove the following:

Theorem 1. For any real number $\alpha > 1$ and integer $k \geq 2$, we have the identity:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{S_k^{\alpha}(n)} = \frac{2^{\alpha} - k - 1}{2^{\alpha} + k - 1}\prod_p \left(1 + \frac{k}{p^{\alpha} - 1}\right),$$

where $\prod_p$ denotes the product over all prime p.

Theorem 2. For any real number $\alpha > 1$ and integer $k \geq 2$, we also have the identities:

$$\sum_{n=1}^{\infty} \frac{\overline{S_k}(n)}{n^{\alpha}} = \frac{\zeta(\alpha)\zeta(k\alpha - 1)}{\zeta(k\alpha)}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\overline{S_k}(n)}{n^{\alpha}} = \frac{\zeta(\alpha)\zeta(k\alpha - 1)}{\zeta(k\alpha)}\left[\frac{(2^{\alpha} - 1)(2^{k\alpha - 1} - 1)}{2^{\alpha - 2}(2^{k\alpha} - 1)} - 1\right],$$

where $\zeta(s)$ is the Riemann zeta-function.

Taking $k = 2$, $\alpha = 2$ and 4, from our Theorems we may immediately deduce the following:

Corollary 1. Let $S_k(n)$ denotes the Smarandache ceil function, then we have the identities:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{S_2^2(n)} = \frac{1}{2}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{S_2^4(n)} = \frac{91}{102}.$$

Corollary 2. Let $\overline{S_k}(n)$ denotes the dual function of the Smarandache ceil function, then we have the identities:

$$\sum_{n=1}^{\infty} \frac{\overline{S_2}(n)}{n^2} = \frac{15}{\pi^2} \cdot \zeta(3)$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\overline{S_2}(n)}{n^2} = \frac{6}{\pi^2} \cdot \zeta(3).$$

§2. Proof of the theorems

In this section, we shall complete the proof of Theorems. First we prove Theorem 1. For any real number α with $\alpha > 1$, let

$$f(\alpha) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{S_k^{\alpha}(n)}.$$

Then from the multiplicative property of $S_k(n)$ we may get

$$\begin{aligned} f(\alpha) &= \sum_{n=1}^{\infty} \frac{1}{S_k^{\alpha}(2n-1)} - \sum_{t=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{S_k^{\alpha}(2n-1)S_k^{\alpha}(2^t)} \\ &= \left(\sum_{n=1}^{\infty} \frac{1}{S_k^{\alpha}(2n-1)}\right)\left(1 - \sum_{t=1}^{\infty} \frac{1}{S_k^{\alpha}(2^t)}\right). \end{aligned} \tag{1}$$

For any prime p, note that $S_k(p) = p$, $S_k(p^2) = p$, $\cdots$, $S_k(p^k) = p$, $S_k(p^{k+1}) = p^2$, $S_k(p^{tk+r}) = p^{t+1}$ for any integers $t \geq 0$ and $1 \leq r \leq k$. So from the Euler product formula [6] we have

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \frac{1}{S_k^{\alpha}(2n-1)}\right) &= \prod_{\substack{p \\ p\neq 2}} \left(1 + \frac{1}{S_k^{\alpha}(p)} + \frac{1}{S_k^{\alpha}(p^2)} + \cdots + \frac{1}{S_k^{\alpha}(p^k)} + \cdots\right) \\ &= \prod_{\substack{p \\ p\neq 2}} \left(1 + \frac{k}{p^{\alpha}} + \frac{k}{p^{2\alpha}} + \cdots + \frac{k}{p^{n\alpha}} + \frac{k}{p^{(n+1)\alpha}} + \cdots\right) \\ &= \prod_{\substack{p \\ p\neq 2}} \left(1 + \frac{k}{p^{\alpha}-1}\right) \\ &= \frac{2^{\alpha}-1}{2^{\alpha}+k-1} \prod_{p} \left(1 + \frac{k}{p^{\alpha}-1}\right). \end{aligned} \tag{2}$$

Similarly, we also have

$$1 - \sum_{t=1}^{\infty} \frac{1}{S_k^{\alpha}(2^t)} = 1 - \frac{k}{2^{\alpha}-1}. \tag{3}$$

Combining (1), (2) and (3) we may immediately get the identity

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{S_k^{\alpha}(n)} = \frac{2^{\alpha}-k-1}{2^{\alpha}+k-1} \prod_{p} \left(1 + \frac{k}{p^{\alpha}-1}\right).$$

This proves Theorem 1.

Now we prove Theorem 2. For any real number α with $\alpha > 1$ and integer $k \geq 2$, let

$$g(\alpha) = \sum_{n=1}^{\infty} \frac{\overline{S_k}(n)}{n^{\alpha}}.$$

Then from the multiplicative property of $\overline{S_k}(n)$ and the Euler product formula [6] we have

$$\begin{aligned}
g(\alpha) &= \prod_p \left(1 + \frac{\overline{S_k}(p)}{p^{\alpha}} + \frac{\overline{S_k}(p^2)}{p^{2\alpha}} + \frac{\overline{S_k}(p^3)}{p^{3\alpha}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{1}{p^{\alpha}} + \frac{1}{p^{2\alpha}} + \cdots + \frac{1}{p^{(k-1)\alpha}} + \frac{p}{p^{k\alpha}} + \cdots + \frac{p}{p^{(2k-1)\alpha}} + \cdots\right) \\
&= \prod_p \left(\frac{1 - \frac{1}{p^{k\alpha}}}{1 - \frac{1}{p^{\alpha}}} + \frac{p}{p^{k\alpha}} \frac{1 - \frac{1}{p^{k\alpha}}}{1 - \frac{1}{p^{\alpha}}} + \frac{p^2}{p^{2k\alpha}} \frac{1 - \frac{1}{p^{k\alpha}}}{1 - \frac{1}{p^{\alpha}}} + \cdots\right) \\
&= \prod_p \left(\frac{1 - \frac{1}{p^{k\alpha}}}{1 - \frac{1}{p^{\alpha}}}\right) \prod_p \left(1 + \frac{p}{p^{k\alpha}} + \frac{p^2}{p^{2k\alpha}} + \cdots\right) \\
&= \prod_p \left(\frac{1 - \frac{1}{p^{k\alpha}}}{1 - \frac{1}{p^{\alpha}}}\right) \prod_p \frac{1}{1 - \frac{1}{p^{k\alpha-1}}} \\
&= \frac{\zeta(\alpha)\zeta(k\alpha - 1)}{\zeta(k\alpha)}.
\end{aligned}$$

This proves the first formula of Theorem 2.

Similarly, we can also get

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\overline{S_k}(n)}{n^{\alpha}} &= \sum_{n=1}^{\infty} \frac{\overline{S_k}(2n-1)}{(2n-1)^{\alpha}} - \sum_{n=1}^{\infty}\sum_{t=1}^{\infty} \frac{\overline{S_k}(2n-1)}{(2n-1)^{\alpha}} \frac{\overline{S_k}(2^t)}{2^{t\alpha}} \\
&= \left(\sum_{n=1}^{\infty} \frac{\overline{S_k}(2n-1)}{(2n-1)^{\alpha}}\right) \left(1 - \sum_{t=1}^{\infty} \frac{\overline{S_k}(2^t)}{2^{t\alpha}}\right) \\
&= \left(1 - \sum_{t=1}^{\infty} \frac{\overline{S_k}(2^t)}{2^{t\alpha}}\right) \prod_{\substack{p \\ p \neq 2}} \left(1 + \frac{\overline{S_k}(p)}{p^{\alpha}} + \frac{\overline{S_k}(p^2)}{p^{2\alpha}} + \frac{\overline{S_k}(p^3)}{p^{3\alpha}} + \cdots\right) \\
&= \frac{\zeta(\alpha)\zeta(k\alpha-1)}{\zeta(k\alpha)} \frac{1 - \sum\limits_{t=1}^{\infty} \frac{\overline{S_k}(2^t)}{2^{t\alpha}}}{1 + \sum\limits_{t=1}^{\infty} \frac{\overline{S_k}(2^t)}{2^{t\alpha}}} \\
&= \frac{\zeta(\alpha)\zeta(k\alpha-1)}{\zeta(k\alpha)} \left[\frac{(2^{\alpha}-1)(2^{k\alpha-1}-1)}{2^{\alpha-2}(2^{k\alpha}-1)} - 1\right].
\end{aligned}$$

This completes the proof of Theorem 2.

Taking $k = 2$, then from Theorem 1 we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{S_2^{\alpha}(n)} = \frac{2^{\alpha}-3}{2^{\alpha}+1} \prod_p \left(1 + \frac{2}{p^{\alpha}-1}\right) = \frac{2^{\alpha}-3}{2^{\alpha}+1} \prod_p \frac{p^{\alpha}+1}{p^{\alpha}-1} = \frac{2^{\alpha}-3}{2^{\alpha}+1} \frac{\zeta^2(\alpha)}{\zeta(2\alpha)}, \tag{4}$$

where $\zeta(\alpha)$ is the Riemann zeta-function.

Note that $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$ and $\zeta(8) = \pi^8/9450$, from (4) we may immediately deduce Corollary 1.

Corollary 2 follows from Theorem 2 with $k = 2$.

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