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On Smarandachely Harmonic Graphs

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Abstract: A graph G is said to be Smarandachely harmonic graph with property P if its vertices can be labeled $1, 2, \cdots, n$ such that the function $f_P : A \to Q$ defined by

$$f_p(H) = \frac{\prod\limits_{v \in V(H)} f(v)}{\sum\limits_{v \in V(H)} f(v)}, \quad H \in A$$

is injective. Particularly, if A is the collection of all paths of length 1 in G (That is, $A = E(G)$), then a Smarandachely harmonic graph is called Strongly harmonic graph. In this paper we show that all cycles, wheels, trees and grids are strongly harmonic graphs. Also we give an upper bound and a lower bound for $\mu(n)$, the maximum number of edges in a strongly harmonic graph of order n.

Key Words: Graph labeling, Smarandachely harmonic graph, strongly harmonic graph.

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§1. Introduction

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. After it was introduced in late 1960's thousands of research articles on graph labelings and their applications have been published.

Recently in 2001, L. W. Beineke and S. M. Hegde [7] introduced the concept of strongly multiplicative graph. A graph with n vertices is said to be Strongly multiplicative if the vertices of G can be labeled with distinct integers $1, 2, \cdots, n$ such that the values on the edge obtained as the product of the labels of their end vertices are all distinct. They have proved that certain classes of graphs are strongly multiplicative. They have also obtained an upper bound for $\lambda(n)$, the maximum number of edges for a given strongly multiplicative graph of order n. In [3], C. Adiga, H.N. Ramaswamy and D. D. Somashekara gave a sharper upper bound for $\lambda(n)$. Further C. Adiga, H. N. Ramaswamy and D. D. Somashekara [1] gave a lower bound for $\lambda(n)$ and proved that the complete bipartite graph $K_{r,r}$ is strongly multiplicative if and only if $r \leq 4$. In 2003, C. Adiga, H. N. Ramaswamy and D. D. Somashekara [2] gave a formula for $\lambda(n)$ and

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also showed that every wheel is strongly multiplicative. Seoud and Zid [9] and Germina and Ajitha [8] have made further contributions to this concept of strongly multiplicative graphs.

In 2000, C. Adiga, and D. D. Somashekara [4] have introduced the concept of Strongly $\star$ - graph and showed that certain classes of graphs are strongly $\star$ - graphs. Also they have obtained a formula, upper and lower bounds for the maximum number of edges in a strongly $\star$ - graph of order n. Baskar Babujee and Vishnupriya [6] have also proved that certain class of graphs are strongly $\star$ - graphs.

A graph with n vertices is said to be Strongly quotient graph if its vertices can be labeled $1, 2, \cdots, n$ so that the values on the edges obtained as the quotient of the labels of their end vertices are all distinct. In [5], C. Adiga, M. Smitha and R. Kaeshgas Zafarani showed that certain class of graphs are strongly quotient graphs. They have also obtained a formula, upper and two different lower bounds for the maximum number of edges in a strongly quotient graph of order n.

In this sequel, we shall introduce the concept of Strongly Harmonic graphs.

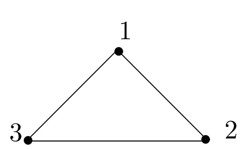
Definition 1.1 *A labeling of a graph G of order n is an injective mapping $f : V(G) \to \{1, 2, \ldots, n\}$.*

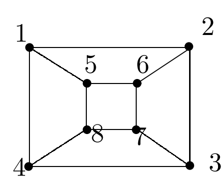
Definition 1.2 *Let G be a graph of order n and A be the set of all paths in G. Then G is said to be Smarandachely harmonic graph with property P if its vertices can be labeled $1, 2, \cdots, n$ such that the function $f_P : A \to Q$ defined by*

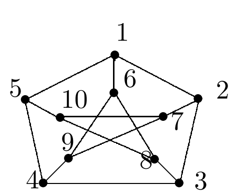
$$f_p(H) = \frac{\prod\limits_{v \in V(H)} f(v)}{\sum\limits_{v \in V(H)} f(v)}, \quad H \in A$$

is injective. In particular if A is the collection of all paths of length 1 in G (That is, $A = E(G)$), then a Smarandachely harmonic graph is called Strongly harmonic graph.

For example, the following graphs are strongly harmonic graphs.







In Section 2, we show that certain class of graphs are strongly harmonic. In Section 3, we give upper and lower bounds for $\mu(n)$, the maximum number of edges in a strongly harmonic graph of order n.

§2. Some Classes of Strongly Harmonic Graphs

Theorem 2.1 *The complete graph K_n is strongly harmonic graph if and only if $n \leq 11$.*

Proof For $n \leq 11$ it is easy to see that K_n are strongly harmonic graphs. When $n = 12$, we have $\frac{4 \cdot 3}{4+3} = \frac{12}{7} = \frac{24}{14} = \frac{12 \cdot 2}{12+2}$. Therefore K_{12} is not strongly harmonic graph and hence any complete graph K_n, for $n \geq 12$ is not strongly harmonic. □

Theorem 2.2 *For all $n \geq 3$, the cycle C_n is strongly harmonic graph.*

Proof Let $C_n = [v_1, v_2, \ldots, v_n, v_1]$ be a cycle of order n. Then consider the following labelling of the graph $v_1 = 1, v_2 = 2, \ldots, v_n = n$. Then the value of the edge $v_k v_{k+1}$ is $\frac{k(k+1)}{2k+1}$, for $1 \leq k < n$. The value of the edge $v_n v_1$ is $\frac{n}{n+1}$. Since $\frac{2}{3} < \frac{n}{n+1} < \frac{6}{5} < \cdots < \frac{n(n-1)}{2n-1}$ for all $n \geq 3$, it follows that the values of the edges are all distinct, proving that every cycle C_n, $n \geqslant 3$, is strongly harmonic. □

Theorem 2.3 *Every wheel is strongly harmonic.*

Proof Consider the wheel W_{n+1}, whose rim is the cycle $v_1, v_2, \ldots, v_n, v_1$ and whose hub is the vertex w.

Case (i) $n + 1$ is odd.

Let p be a prime such that $\frac{n}{2} < p < n$. Such a prime p exists by Bertrand's hypothesis. Consider the following labeling of graphs:

$$v_1 = 1, v_2 = 2, \cdots, v_{p-1} = p-1, v_p = p+1, \cdots, v_n = n+1, w = p.$$

The value of the edge $v_k v_{k+1}$ is $\frac{k(k+1)}{2k+1}$ for $1 \leq k < p-1$ and the value of the edge $v_k v_{k+1}$ is $\frac{(k+1)(k+2)}{2k+3}$ for $p \leq k < n$. The value of the edge $v_{p-1} v_p$ is $\frac{(p-1)(p+1)}{2p}$ and the value of the edge $v_n v_1$ is $\frac{n+1}{n+2}$. Since

$$\frac{2}{3} < \frac{n+1}{n+2} < \frac{6}{5} < \cdots < \frac{(p-2)(p-1)}{2p-3} < \frac{(p-1)(p+1)}{2p} < \frac{(p+1)(p+2)}{2p+3}$$
$$< \cdots < \frac{n(n+1)}{2n+1},$$

the value of the rim edges are all distinct.

The value of the spoke edges are $\frac{p}{p+1}, \frac{2p}{p+2}, \cdots, \frac{(n+1)p}{n+1+p}$. Since $\frac{p}{p+1} < \frac{2p}{p+2} < \cdots < \frac{(p-1)p}{2p-1} < \frac{(p+1)p}{2p+1} < \cdots < \frac{(n+1)p}{n+1+p}$, the value of the spoke edges are all distinct. The numerator in the values of spoke edges are all divisible by p and the numerator in the values of the rim edges are not divisible by p. Hence the value of the edges of the wheel are all distinct. Hence when $n+1$ is odd, the wheel is strongly harmonic.

Case (ii) $n+1$ is even.

Let p be a prime such that $\frac{n+1}{2} < p < n+1$. Proof follows in the same lines as in case(i).

Hence by the choice of p edges of the wheel are all distinct. Therefore wheel is strongly harmonic. □

Theorem 2.4 *Every tree is strongly harmonic graph.*

Proof Label the vertices of the tree using breadth - first search method. To show that the labeling is strongly harmonic it suffices to consider the following two cases.

Case (i) Let $e_1 = (a, b)$ and $e_2 = (a, c)$ be the edges with a common vertex as shown in the Fig.2.1.

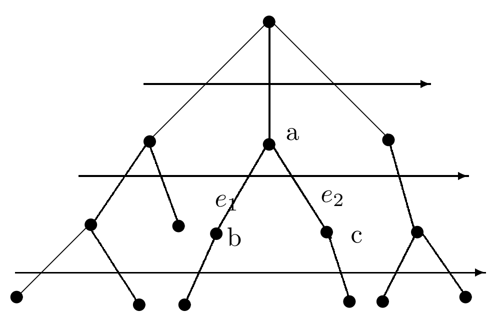


Fig.2.1

From the breadth - first search method of labelling it follows that $a < b < c$. This implies that $\frac{ab}{a+b} < \frac{ac}{a+c}$. Hence the values of the edges with common vertex form a strictly increasing sequence of rational numbers.

Case (ii) Let $e_1 = (a, c)$ and $e_2 = (b, d)$, where the edges e_1 and e_2 fall in the same level as shown in the Fig.2.2 or in two consecutive level as shown in Fig.2.3.

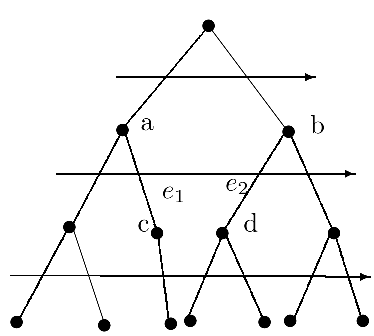


Fig.2.2

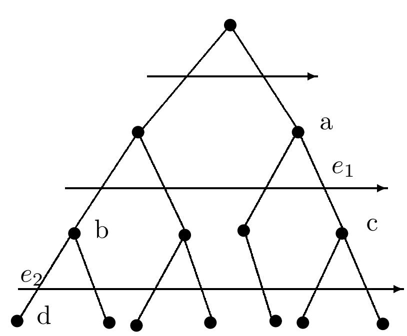


Fig.2.3

From the breadth - first search method of labeling it follows that $a < b < c < d$. This implies that $\frac{ac}{a+c} < \frac{bd}{b+d}$. Hence as indicated by the arrows, the values of the edges form a strictly increasing sequence of rational numbers.

Thus the values of the edges are all distinct. So each tree is strongly harmonic graph. □

Theorem 2.5 *Every grid is strongly harmonic graph.*

Proof Label the vertices of the grid using breadth - first search method. To show that the labeling is strongly harmonic it suffices to consider the following three cases. The first three cases are similar to the two cases considered in the proof of the Theorem 2.4. The last case is when $e_1 = (a, c)$ and $e_2 = (b, c)$ as shown in the Fig.2.4.

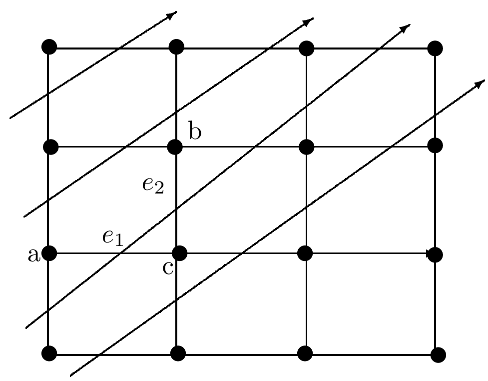


Fig.2.4

In this case from the breadth - first search method of labeling it follows that $a < b < c$ which implies that $\frac{ac}{a+c} < \frac{bc}{b+c}$.

Therefore, as indicated by the arrows, the values of the edges form a strictly increasing sequence of rational numbers. Thus, the values of the edges are all distinct proving that every grid is strongly harmonic. □

§3. Upper and Lower Bounds for $\mu(n)$

In this section we give an upper and a lower bound for $\mu(n)$.

Theorem 3.1 *If $\mu(n)$ denotes the number of edges in a strongly harmonic graph of order n, then*

$$\mu(n) \leq \frac{n(n-1)}{2} - \sum_{k=1}^{2} \left[\frac{\sqrt{4nk + k^2} + k}{4k} \right]$$

$$- \sum_{k=1}^{[\frac{n+12}{48}]} \left[\frac{\sqrt{(4k-1)(4n+4k-1)} + (4k-1)}{16k-4} \right]$$

$$-\sum_{k=1}^{[\frac{n+24}{96}]}\left[\frac{\sqrt{(8k-2)(4n+8k-2)}+(8k-2)}{32k-8}\right]$$

$$+2+\left[\frac{n+12}{48}\right]+\left[\frac{n+24}{96}\right], \quad (1)$$

where $[x]$ *denotes the greatest integer less than or equal to* x*.*

Proof Given n, the total number of edges in a complete graph of order n is $\frac{n(n-1)}{2}$.

For $7k \le t \le n$, and $t \equiv -k (\text{mod } 4k)$ where $1 \le k \le 2$ the values of the edges e_1 with end vertices $\left(\frac{t+k}{4}, \frac{t^2-k^2}{4k}\right)$ and e_2 with end vertices $\left(\frac{t+k}{2}, \frac{t-k}{2}\right)$ are equal provided $\frac{t^2-k^2}{4k} \le n$ or $t \le \sqrt{4nk+k^2}$. Since $t = 4km - k$, for some positive integer m, we have

$$7k \le 4km - k \le \sqrt{4nk+k^2}.$$

This double inequality yields

$$2 \le m \le \left(\frac{\sqrt{4nk+k^2}+k}{4k}\right).$$

Therefore, the number of such pairs of edges with equal values is

$$\left[\frac{\sqrt{4nk+k^2}+k}{4k}\right]-1. \quad (2)$$

Next for $28k-7 \le t \le n$, and $t \equiv -(4k-1)(\text{mod } (16k-4))$ and $48k-12 \le n$, the values of the edges e_1 with end vertices $\left(\frac{t+(4k-1)}{4}, \frac{t^2-(4k-1)^2}{16k-4}\right)$ and e_2 with end vertices $\left(\frac{t+(4k-1)}{2}, \frac{t-(4k-1)}{2}\right)$ are equal provided $\frac{t^2-(4k-1)^2}{16k-4} \le n$ or $t \le \sqrt{(4k-1)(4n+4k-1)}$. Since $t = (16k-4)m - (4k-1)$, for some positive integer m, we have

$$28k-7 \le (16k-4)m-(4k-1) \le \sqrt{(4k-1)(4n+4k-1)}.$$

This double inequality yields

$$2 \le m \le \left(\frac{\sqrt{(4k-1)(4n+4k-1)}+(4k-1)}{16k-4}\right).$$

Therefore, the number of such pairs of edges with equal values is

$$\left[\frac{\sqrt{(4k-1)(4n+4k-1)}+(4k-1)}{16k-4}\right]-1. \quad (3)$$

For $56k-14 \le t \le n$, and $t \equiv -(8k-2)(\text{mod } (32k-8))$ and $96k-24 \le n$, the values of the edges e_1 with end vertices $\left(\frac{t+(8k-2)}{4}, \frac{t^2-(8k-2)^2}{32k-8}\right)$ and e_2 with end vertices

$\left(\dfrac{t+(8k-2)}{2}, \dfrac{t-(8k-2)}{2}\right)$ are equal and proceeding as above we find that the number of such pairs of edges with equal values is

$$\left[\frac{\sqrt{(4k-1)(2n+4k-1)}+(4k-1)}{16k-4}\right]-1. \tag{4}$$

From equations (2), (3) and (4), we get

$$\mu(n) \leq \frac{n(n-1)}{2} - \sum_{k=1}^{2}\left(\left[\frac{\sqrt{4nk+k^2}+k}{4k}\right]-1\right)$$
$$-\sum_{k=1}^{[\frac{n+12}{48}]}\left(\left[\frac{\sqrt{(4k-1)(4n+4k-1)}+(4k-1)}{16k-4}\right]-1\right)$$
$$-\sum_{k=1}^{[\frac{n+24}{96}]}\left(\left[\frac{\sqrt{(4k-1)(2n+4k-1)}+(4k-1)}{16k-4}\right]-1\right)$$

which yields (1). □

Theorem 3.2

$$\mu(n) \geq n + \sum_{k=2}^{n-2} f(k), \ \ n \geq 4, \tag{5}$$

where $f(k) = min\left\{n - \left[\dfrac{nk(k-1)}{k(k-1)+n}\right], n-k\right\}$.

Proof Let $A = \left\{\dfrac{rs}{r+s}; 1 \leq r < s \leq n\right\}$. Then clearly $\mu(n) = |A|$. Consider the array of rational numbers:

$$\begin{array}{cccccc}
\frac{1\cdot 2}{1+2} & \frac{1\cdot 3}{1+3} & \frac{1\cdot 4}{1+4} & \cdots & \frac{1\cdot(n-1)}{1+(n-1)} & \frac{1\cdot n}{1+n} \\
 & \frac{2\cdot 3}{2+3} & \frac{2\cdot 4}{2+4} & \cdots & \frac{2\cdot(n-1)}{2+(n-1)} & \frac{2\cdot n}{2+n} \\
 & & \frac{3\cdot 4}{3+4} & \cdots & \frac{3\cdot(n-1)}{3+(n-1)} & \frac{3\cdot n}{3+n} \\
 & & & & & \cdots \\
 & & & & & \cdots \\
 & & & & \frac{(n-2)\cdot(n-1)}{(n-2)+(n-1)} & \frac{(n-2)\cdot n}{(n-2)+n} \\
 & & & & & \frac{(n-1)\cdot n}{(n-1)+n}
\end{array}$$

Now, let A_1 denote the set of all elements of the first row. Let $A_k,\ 2 \le k \le n-2$ denote the set of all elements of the k-th row which are greater than $\dfrac{(k-1)\cdot n}{(k-1)+n}$ and hence greater than every element of the $(k-1)$-th row. Let $A_{n-1} = \dfrac{(n-1)\cdot n}{(n-1)+n}$. Clearly $A_i \cap A_j = \emptyset$ for $i \ne j$ and $A_i \subset A$, for all $i = 1, 2, \ldots, n-1$. Hence

$$\mu(n) = |A| \ge \sum_{i=1}^{n-1} |A_i|. \tag{6}$$

Now one can easily see that

$$\left.\begin{aligned} |A_1| &= n-1, \\ |A_k| &= n - max\left\{\left[\frac{nk(k-1)}{k(k-1)+n}\right], k\right\}, \quad 2 \le k \le n-2 \\ &= min\left\{n - \left[\frac{nk(k-1)}{k(k-1)+n}\right], n-k\right\} = f(k), \\ \text{and} \quad |A_k| &= 1. \end{aligned}\right\} \tag{7}$$

Using (7) in (6) we obtain (5). □

The following table gives the values of $\mu(n)$ and upper and lower bounds for $\mu(n)$ found using Theorems 3.1 and 3.2, respectively.

n	$\mu(n)$	Upper bound	Lower bound
4	6	6	6
5	10	10	10
6	15	15	15
7	21	21	21
8	28	28	28
9	36	36	34
10	45	45	41
11	55	55	48
12	64	65	55
13	76	77	63
14	89	90	71
15	102	104	80

n	$\mu(n)$	Upper bound	Lower bound
16	117	119	90
17	133	135	97
18	150	152	107
19	168	170	117
20	183	191	127

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