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# Some Remarks on Fuzzy N-Normed Spaces

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**Abstract**: It is shown that every fuzzy $n$-normed space naturally induces a locally convex topology, and that every finite dimensional fuzzy $n$-normed space is complete.

**Key Words**: Fuzzy $n$-normed spaces, $n$-seminorm, Smarandache space.

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## §1. Introduction

A Smarandache space is such a space that a straight line passing through a point $p$ may turn an angle $\theta_p \geq 0$. If $\theta_p > 0$ , then $p$ is called a non-Euclidean. Otherwise, we call it an Euclidean point. In this paper, normed spaces are considered to be Euclidean, i.e., every point is Euclidean. In [7], S. Gähler introduced $n$-norms on a linear space. A detailed theory of $n$-normed linear space can be found in [8], [10], [12]-[13]. In [8], H. Gunawan and M. Mashadi gave a simple way to derive an $(n-1)$- norm from the $n$-norm in such a way that the convergence and completeness in the $n$-norm is related to those in the derived $(n-1)$-norm. A detailed theory of fuzzy normed linear space can be found in [1], [3]-[6], [9], [11] and [15]. In [14], A. Narayanan and S. Vijayabalaji have extend $n$-normed linear space to fuzzy $n$-normed linear space. In section 2, we quote some basic definitions, and we show that a fuzzy $n$-norm is closely related to an ascending system of $n$-seminorms. In Section 3, we introduce a locally convex topology in a fuzzy $n$-normed space, and in Section 4 we consider finite dimensional fuzzy $n$-normed linear spaces.

## §2. Fuzzy $n$-norms and ascending families of $n$-seminorms

Let $n$ be a positive integer, and let $X$ be a real vector space of dimension at least $n$. We recall the definitions of an $n$-seminorm and a fuzzy $n$-norm [14].

**Definition** 2.1 *A function $(x_1, x_2, \ldots, x_n) \mapsto \|x_1, \ldots, x_n\|$ from $X^n$ to $[0, \infty)$ is called an $n$-seminorm on $X$ if it has the following four properties:*

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*(S1)* $\|x_1, x_2, \ldots, x_n\| = 0$ *if* $x_1, x_2, \ldots, x_n$ *are linearly dependent;*

*(S2)* $\|x_1, x_2, \ldots, x_n\|$ *is invariant under any permutation of* $x_1, x_2, \ldots, x_n$*;*

*(S3)* $\|x_1, \ldots, x_{n-1}, cx_n\| = |c|\|x_1, , \ldots, x_{n-1}, x_n\|$ *for any real* $c$*;*

*(S4)* $\|x_1, \ldots, x_{n-1}, y+z\| \leqslant \|x_1, \ldots, x_{n-1}, y\| + \|x_1, \ldots, x_{n-1}, z\|$.

*An n-seminorm is called a n-norm if* $\|x_1, x_2, \ldots, x_n\| > 0$ *whenever* $x_1, x_2, \ldots, x_n$ *are linearly independent.*

**Definition** 2.2 *A fuzzy subset* $N$ *of* $X^n \times \mathbb{R}$ *is called a fuzzy n-norm on* $X$ *if and only if:*

*(F1) For all* $t \leqslant 0$, $N(x_1, x_2, \ldots, x_n, t) = 0$*;*

*(F2) For all* $t > 0$, $N(x_1, x_2, \ldots, x_n, t) = 1$ *if and only if* $x_1, x_2, \ldots, x_n$ *are linearly dependent;*

*(F3)* $N(x_1, x_2, \ldots, x_n, t)$ *is invariant under any permutation of* $x_1, x_2, \ldots, x_n$*;*

*(F4) For all* $t > 0$ *and* $c \in \mathbb{R}$, $c \neq 0$,

$$N(x_1, x_2, \ldots, cx_n, t) = N(x_1, x_2, \ldots, x_n, \frac{t}{|c|});$$

*(F5) For all* $s, t \in \mathbb{R}$,

$$N(x_1, \ldots, x_{n-1}, y+z, s+t) \geq \min\left\{N(x_1, \ldots, x_{n-1}, y, s), N(x_1, \ldots, x_{n-1} z, t)\right\}.$$

*(F6)* $N(x_1, x_2, \ldots, x_n, t)$ *is a non-decreasing function of* $t \in \mathbb{R}$ *and*

$$\lim_{t \to \infty} N(x_1, x_2, \ldots, x_n, t) = 1.$$

The following two theorems clarify the relationship between definitions 2.1 and 2.2.

**Theorem** 2.1 *Let* $N$ *be a fuzzy n-norm on* $X$*. As in [14] define for* $x_1, x_2, \ldots, x_n \in X$ *and* $\alpha \in (0, 1)$

$$\|x_1, x_2, \ldots, x_n\|_\alpha := \inf\left\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\right\}. \tag{2.1}$$

*Then the following statements hold.*

*(A1) For every* $\alpha \in (0, 1)$, $\|\bullet, \bullet, \ldots, \bullet\|_\alpha$ *is an n-seminorm on* $X$*;*

*(A2) If* $0 < \alpha < \beta < 1$ *and* $x_1, \ldots, x_n \in X$ *then*

$$\|x_1, x_2, \ldots, x_n\|_\alpha \leqslant \|x_1, x_2, \ldots, x_n\|_\beta;$$

*(A3) If* $x_1, x_2, \ldots, x_n \in X$ *are linearly independent then*

$$\lim_{\alpha \to 1-} \|x_1, x_2, \ldots, x_n\|_\alpha = \infty.$$

*Proof* (A1) and (A2) are shown in Theorem 3.4 in [14]. Let $x_1, x_2, \dots, x_n \in X$ be linearly independent, and $t > 0$ be given. We set $\beta := N(x_1, x_2, \dots, x_n, t)$. It follows from (F2) that $\beta \in [0, 1)$. Then (F6) shows that, for $\alpha \in (\beta, 1)$,

$$\|x_1, x_2, \dots, x_n\|_\alpha \geqslant t.$$

This proves (A3). □

We now prove a converse of Theorem 2.2.

**Theorem** 2.2 *Suppose we are given a family* $\|\bullet, \bullet, \dots, \bullet\|_\alpha$, $\alpha \in (0, 1)$, *of* $n$*-seminorms on* $X$ *with properties (A2) and (A3). We define*

$$N(x_1, x_2, \dots, x_n, t) := \inf\{\alpha \in (0, 1) : \|x_1, x_2, \dots, x_n\|_\alpha \geqslant t\}. \tag{2.2}$$

*where the infimum of the empty set is understood as* 1*. Then* $N$ *is a fuzzy* $n$*-norm on* $X$*.*

*Proof* (F1) holds because the values of an $n$-seminorm are nonnegative.

(F2): Let $t > 0$. If $x_1, \dots, x_n$ are linearly dependent then $N(x_1, \dots, x_n, t) = 1$ follows from property (S1) of an $n$-seminorm. If $x_1, \dots, x_n$ are linearly independent then $N(x_1, \dots, x_n, t) < 1$ follows from (A3).

(F3) is a consequence of property (S2) of an $n$-seminorm.

(F4) is a consequence of property (S3) of an $n$-seminorm.

(F5): Let $\alpha \in (0, 1)$ satisfy

$$\alpha < \min\{N(x_1, \dots, x_{n-1}, y, s), N(x_1, \dots, x_{n-1}, z, s)\}. \tag{2.3}$$

It follows that $\|x_1, \dots, x_{n-1}, y\|_\alpha < s$ and $\|x_1, \dots, x_{n-1}, z\|_\alpha < t$. Then (S4) gives

$$\|x_1, \dots, x_{n-1}, y + z\|_\alpha < s + t.$$

Using (A2) we find $N(x_1, \dots, x_{n-1}, y + z, s + t) \geqslant \alpha$ and, since $\alpha$ is arbitrary in (2.3), (F5) follows.

(F6): Definition 2.2 shows that $N$ is non-decreasing in $t$. Moreover, $\lim_{t\to\infty} N(x_1, \dots, x_n, t)$ $= 1$ because seminorms have finite values. □

It is easy to see that Theorems 2.1 and 2.2 establish a one-to-one correspondence between fuzzy $n$-norms with the additional property that the function $t \mapsto N(x_1, \dots, x_n, t)$ is left-continuous for all $x_1, x_2, \dots, x_n$ and families of $n$-seminorms with properties (A2), (A3) and the additional property that $\alpha \mapsto \|x_1, \dots, x_n\|_\alpha$ is left-continuous for all $x_1, x_2, \dots, x_n$.

**Example** 2.3(Example 3.3 in [14]). Let $\|\bullet, \bullet, \dots, \bullet\|$ be a $n$-norm on $X$. Then define $N(x_1, x_2, \dots, x_n, t) = 0$ if $t \leqslant 0$ and, for $t > 0$,

$$N(x_1, x_2, \dots, x_n, t) = \frac{t}{t + \|x_1, x_2, \dots, x_n\|}.$$

Then the seminorms (2.1) are given by

$$\|x_1, x_2, \ldots, x_n\|_\alpha = \frac{\alpha}{1-\alpha}\|x_1, x_2, \ldots, x_n\|.$$

## §3. Locally convex topology generated by a fuzzy $n$-norm

In this section $(X, N)$ is a fuzzy $n$-normed space, that is, $X$ is real vector space and $N$ is fuzzy $n$-norm on $X$. We form the family of $n$-seminorms $\|\bullet, \bullet, \ldots, \bullet\|_\alpha$, $\alpha \in (0,1)$, according to Theorem 2.1. This family generates a family $\mathcal{F}$ of seminorms

$$\|x_1, \ldots, x_{n-1}, \bullet\|_\alpha, \quad \text{where } x_1, \ldots, x_{n-1} \in X \text{ and } \alpha \in (0,1).$$

The family $\mathcal{F}$ generates a locally convex topology on $X$; see [2, Def.(37.9)], that is, a basis of neighborhoods at the origin is given by

$$\{x \in X : p_i(x) \leqslant \epsilon_i \text{ for } i = 1, 2, \ldots, n\},$$

where $p_i \in \mathcal{F}$ and $\epsilon_i > 0$ for $i = 1, 2 \ldots, n$. We call this the locally convex topology generated by the fuzzy $n$-norm $N$.

**Theorem** 3.1 *The locally convex topology generated by a fuzzy $n$-norm is Hausdorff.*

*Proof* Given $x \in X$, $x \neq 0$, choose $x_1, \ldots, x_{n-1} \in X$ such that $x_1, \ldots, x_{n-1}, x$ are linearly independent. By Theorem 2.1(A3) we find $\alpha \in (0,1)$ such that $\|x_1, \ldots, x_{n-1}, x\|_\alpha > 0$. The desired statement follows; see [2,Theorem (37.21)]. □

Some topological notions can be expressed directly in terms of the fuzzy-norm $N$. For instance, we have the following result on convergence of sequences. We remark that the definition of convergence of sequences in a fuzzy $n$-normed space as given in [16, Definition 2.2] is meaningless.

**Theorem** 3.2 *Let $\{x_k\}$ be a sequence in $X$ and $x \in X$. Then $\{x_k\}$ converges to $x$ in the locally convex topology generated by $N$ if and only if*

$$\lim_{k\to\infty} N(a_1, \ldots, a_{n-1}, x_k - x, t) = 1 \tag{3.1}$$

*for all $a_1, \ldots, a_{n-1} \in X$ and all $t > 0$.*

*Proof* Suppose that $\{x_k\}$ converges to $x$ in $(X, N)$. Then, for every $\alpha \in (0,1)$ and all $a_1, a_2, \ldots, a_{n-1} \in X$, there is $K$ such that, for all $k \geqslant K$, $\|a_1, a_2, \ldots, a_{n-1}, x_k - x\|_\alpha < \epsilon$. The latter implies

$$N(a_1, a_2, \ldots, a_{n-1}, x_k - x, \epsilon) \geqslant \alpha.$$

Since $\alpha \in (0,1)$ and $\epsilon > 0$ are arbitrary we see that (3.1) holds. The converse is shown in a similar way. □

In a similar way we obtain the following theorem.

**Theorem** 3.3 *Let $\{x_k\}$ be a sequence in $X$. Then $\{x_k\}$ is a Cauchy sequence in the locally convex topology generated by $N$ if and only if*

$$\lim_{k,m\to\infty} N(a_1,\ldots,a_{n-1},x_k-x_m,t)=1 \tag{3.2}$$

*for all $a_1,\ldots,a_{n-1}\in X$ and all $t>0$.*

It should be noted that the locally convex topology generated by a fuzzy $n$-norm is not metrizable, in general. Therefore, in many cases it will be necessary to consider nets $\{x_i\}$ in place of sequences. Of course, Theorems 3.2 and 3.3 generalize in an obvious way to nets.

## §4. Fuzzy $n$-norms on finite dimensional spaces

In this section $(X,N)$ is a fuzzy $n$-normed space and $X$ has finite dimension at least $n$. Since the locally convex topology generated by $N$ is Hausdorff by Theorem 3.1. Tihonov's theorem [2, Theorem (23.1)] implies that this locally convex topology is the only one on $X$. Therefore, all fuzzy $n$-norms on $X$ are equivalent in the sense that they generate the same locally convex topology.

In the rest of this section we will give a direct proof of this fact (without using Tihonov's theorem). We will set $X=\mathbb{R}^d$ with $d\geqslant n$.

**Lemma** 4.1 *Every $n$-seminorm on $X=\mathbb{R}^d$ is continuous as a function on $X^n$ with the euclidian topology.*

*Proof* For every $j=1,2,\ldots,n$, let $\{x_{j,k}\}_{k=1}^{\infty}$ be a sequence in $X$ converging to $x_j\in X$. Therefore, $\lim\limits_{k\to\infty}\|x_{j,k}-x_j\|=0$, where $\|x\|$ denotes the euclidian norm of $x$. From property (S4) of an $n$-seminorm we get

$$|\|x_{1,k},x_{2,k},\ldots,x_{n,k}\|-\|x_1,x_{2,k},\ldots,x_{n,k}\||\le\|x_{1,k}-x_1,x_{2,k},\ldots,x_{n,k}\|.$$

Expressing every vector in the standard basis of $\mathbb{R}^d$ we see that there is a constant $M$ such that

$$\|y_1,y_2,\ldots,\ y_n\|\le M\,\|y_1\|\ldots\|y_n\|\ \text{ for all } y_j\in X.$$

Therefore,

$$\lim_{k\to\infty}\ \|x_{1,k}-x_1,\ x_{2,k},\ldots,\ x_{n,k}\|=0$$

and so

$$\lim_{k\to\infty}\ |\|x_{1,k},\ x_{2,k},\ldots,\ x_{n,k}\|-\|x_1,\ x_{2,k},\ldots,\ x_{n,k}\||=0.$$

We continue this procedure until we reach

$$\lim_{k\to\infty}\|x_{1,k},\ x_{2,k},\ldots,\ x_{n,k}\|=\|x_1,\ x_2,\ldots,\ x_n\|\,.$$

□

**Lemma** 4.2 *Let $(\mathbb{R}^d, N)$ be a fuzzy n-normed space. Then $\|x_1, x_2, \ldots, x_n\|_\alpha$ is an n-norm if $\alpha \in (0,1)$ is sufficiently close to 1.*

*Proof* We consider the compact set

$$S = \left\{(x_1,\ x_2, \ldots,\ x_n) \in \mathbb{R}^{dn} : x_1,\ x_2, \ldots,\ x_n \text{ is an orthonormal system in } \mathbb{R}^d\right\}.$$

For each $\alpha \in (0,1)$ consider the set

$$S_\alpha = \{(x_1, x_2, \ldots, x_n) \in S : \|x_1, x_2, \ldots, x_n\|_\alpha > 0\}.$$

By Lemma 4.1, $S_\alpha$ is an open subset of $S$. We now show that

$$S = \bigcup_{\alpha \in (0,1)} S_\alpha. \tag{4.1}$$

If $(x_1,\ x_2, \ldots,\ x_n) \in S$ then $(x_1,\ x_2, \ldots,\ x_n)$ is linearly independent and therefore there is $\beta$ such that $N(x_1, x_2, \ldots, x_n, 1) < \beta < 1$. This implies that $\|x_1,\ x_2, \ldots,\ x_n\|_\beta \geq 1$ so (4.1) is proved. By compactness of $S$, we find $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_m$ such that

$$S = \bigcup_{i=1}^{m} S_{\alpha_i}.$$

Let $\alpha = \max\{\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_m\}$. Then $\|x_1,\ x_2, \ldots,\ x_n\|_\alpha > 0$ for every $(x_1,\ x_2, \ldots,\ x_n) \in S$.

Let $x_1,\ x_2, \ldots,\ x_n \in X$ be linearly independent. Construct an orthonormal system $e_1,\ e_2, \ldots,\ e_n$ from $x_1,\ x_2, \ldots,\ x_n$ by the Gram-Schmidt method. Then there is $c > 0$ such that

$$\|x_1,\ x_2, \ldots,\ x_n\|_\alpha = c\,\|e_1,\ e_2, \ldots,\ e_n\|_\alpha > 0.$$

This proves the lemma. □

**Theorem** 4.1 *Let $N$ be a fuzzy n-norm on $\mathbb{R}^d$, and let $\{x_k\}$ be a sequence in $\mathbb{R}^d$ and $x \in \mathbb{R}^d$.*

*(a) $\{x_k\}$ converges to $x$ with respect to $N$ if and only if $\{x_k\}$ converges to $x$ in the euclidian topology.*

*(b) $\{x_k\}$ is a Cauchy sequence with respect to $N$ if and only if $\{x_k\}$ is a Cauchy sequence in the euclidian metric.*

*Proof* (a) Suppose $\{x_k\}$ converges to $x$ with respect to euclidian topology. Let $a_1, a_2, \ldots, a_{n-1} \in X$. By Lemma 4.1, for every $\alpha \in (0,1)$,

$$\lim_{k \to \infty} \|a_1,\ a_2, \ldots,\ a_{n-1},\ x_k - x\|_\alpha = 0.$$

By definition of convergence in $(\mathbb{R}^d, N)$, we get that $\{x_k\}$ converges to $x$ in $(\mathbb{R}^d, N)$. Conversely, suppose that $\{x_k\}$ converges to $x$ in $(\mathbb{R}^d, N)$. By Lemma 4.2, there is $\alpha \in (0,1)$ such that $\|y_1, y_2, \ldots, y_n\|_\alpha$ is an $n$-norm. By definition, $\{x_k\}$ converges to $x$ in the $n$-normed space $(\mathbb{R}^d, \|\cdot\|_\alpha)$. It is known from [8, Proposition 3.1] that this implies that $\{x_k\}$ converges to $x$ with respect to euclidian topology.

(b) is proved in a similar way. □

**Theorem** 4.2 *A finite dimensional fuzzy $n$-normed space $(X, N)$ is complete.*

*Proof* This follows directly from Theorem 3.4. □

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