The Pseudo-Smarandache Function and the Classical Functions of Number Theory

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Abstract: The Pseudo-Smarandache function has a simple definition: Given any integer $n > 0$, the value of the Pseudo-Smarandache function is the smallest integer m such that n evenly divides the sum $1 + 2 + 3 + \ldots + m$. In this paper, several problems concerning this function will be presented and solved. Most will involve the standard number theory functions such as Euler's phi function and the sum of divisors function.[1]

The Pseudo-Smarandache function has the definition

Given any integer $n \geq 1$, the value of the Pseudo-Smarandache function Z(n) is the smallest integer m such that n evenly divides

$$\sum_{k=1}^{m} k$$

Note that this summation is equivalent to the expression

$$\frac{m(m+1)}{2}$$

The purpose of this paper will be to present some theorems concerning the interactions of this function with the classical theorems of number theory.

Basic Theorems

Theorem 1: If p is an odd prime, then $Z(p) = p - 1$.

Proof: Clearly, p divides

$$\frac{(p-1)p}{2}$$

and there is no smaller number that satisfies the definition.

Theorem 2: $Z(2^k) = 2^{k+1} - 1$.

Proof: Since only one of m and m+1 is even, it follows that $Z(2^k)$ is the smallest ratio

$$\frac{m(m+1)}{2}$$

where the even number contains k + 1 instances of 2. This number is clearly 2^{k+1} and the value of m is smallest when $m + 1 = 2^{k+1}$.

Definition: Given any integer $n \geq 2$, the Euler phi function $\phi(n)$ is the number of integers k, $1 \leq k < n$, such that k and n are relatively prime.

Our first theorem concerning the combination of ϕ and Z is trivial.

Theorem : There are an infinite number of integers n such that $\phi(n) = Z(n)$.

Proof: It is well-known that if p is an odd prime $\phi(p) = p - 1$.

So we modify the statement to make it harder.

Modified theorem: There are an infinite number of composite integers n such that $\phi(n) = Z(n)$.

Proof: Let n = 2p, where p is an odd prime of the form p = 4k + 1. It is well-known that this is an infinite set.

Consider the fraction

$$\frac{(p-1)p}{2}.$$

Replacing p by the chosen form

$$\frac{(4k+1-1)(4k+1)}{2} = \frac{4k(4k+1)}{2} = 2k(4k+1)$$

Clearly,

$$2(4k+1) \mid 2k(4k+1)$$

and p = 4k + 1 is the smallest such number. Therefore,
$Z(2p) = p - 1$. It is well-known that $\phi(2p) = p - 1$ for p an odd prime.

Unsolved Question: Is there another infinite set of composite numbers such that $Z(n) = \phi(n)$?

Another equation involving these two functions has an infinite family of solutions.

Theorem: There are an infinite number of solutions to the expression

$$Z(n) + \phi(n) = n.$$

Proof: Let $n = 2^{2j} + 2^{2j+1}$, where $j \geq 1$. Factoring it, we have

$n = 2^{2j} * 3$. Using the well-known formula for the computation of the phi function

$\phi(n) = (2 - 1)2^{2j-1}(3 - 1)3^0 = 2^{2j}$

It is easy to verify that if k is odd,

$$3 \mid 2^k + 1.$$

From this, it follows that

$$2^{2j} * 3 \mid \frac{2^{2j+1}(2^{2j+1} + 1)}{2}$$

and it is easy to see that 2^{2j+1} is the smallest such m. Therefore,

$$Z(2^{2j} * 3) = 2^{2j+1}$$

and

$$Z(n) + \phi(n) = n.$$

Unsolved Question: Is there another infinite family of solutions to the equation

$$Z(n) + \phi(n) = n?$$

Another classic number theory function is the sigma or sum of divisors function.

Given any integer $n \geq 1$, $\sigma(n)$ is the sum of all the divisors of n.

Theorem: There are an infinite number of solutions to the equation

$$\sigma(n) = Z(n).$$

Proof: It has already been proven that $Z(2^k) = 2^{k+1} - 1$. It is well-known that $\sigma(p^k) = p^{k-1} - 1$.

A computer search up through n $= 10{,}000$ yielded no solutions not of this type.

Unsolved Question: Is there another infinite family of solutions to the equation

$$\sigma(n) = Z(n)?$$

The final classic function of number theory is the number of integral divisors function.

Definition: For n ≥ 1, the divisors function d(n) is the number if integers m, where $1 \leq m \leq n$, such that m evenly divides n.

Question: How many solutions are there to the equation

$$Z(n) = d(n)?$$

A computer search up through n $= 10{,}000$ yielded only the solutions n = 1, 3 and 10.

Question: How many solutions are there to the equation

$$Z(n) + d(n) = n?$$

A computer search up through n = 10,000 yielded only the solution n = 56, as d(56) = 8 and Z(56) = 48.

It is unknown if there are any additional solutions to this problem.

There are many other problems involving the classic functions that can be defined. One such example is

Question: How many solutions are there to the equation

$$Z(n) + \phi(n) = d(n)?$$

A computer search up through 10,000 failed to find a single solution.

The author continues to work on this set of problems and hopes to present additional solutions in the future.

1. This paper was presented at the Spring, 1998 meeting of the Iowa section of The Mathematical Association of America.