On the Smarandache Irrationality Conjecture

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The Smarandache Irratioality Conjecture (see [1]) claims:

Conjecture.

Let $a(n)$ be the nth term of a Smarandache sequence. Then, the number

$$0.a(1)a(2)...a(n)...$$

is irrational.

Here is an immediate proof in the following cases:

1. $a(n) = n$;
2. $a(n) = d(n)$ =number of divisors of n;
3. $a(n) = \omega(n)$ =number of distinct prime divisors of n;
4. $a(n) = \Omega(n)$ =number of total prime divisors of n (that is, counted with repetitions);
5. $a(n) = \phi(n)$ =the Euler function of n;
6. $a(n) = \sigma(n)$ =the sum of the divisors of n;
7. $a(n) = p_n$ =the nth prime;
8. $a(n) = \pi(n)$ =the number of primes smaller than n;
9. $a(n) = S(n)$ =the Smarandache function of n;
10. $a(n) = n!$;
11. $a(n) = a^n$, where a is any fixed positive integer coprime to 10 and larger than 1;
12. $a(n)$ =any fixed non-constant polynomial in one of the above;

Here is the argument:
Assume that

$$0.a(1)a(2)...a(n)...$$

is rational. Write it under the form

$$0.a(1)a(2)...a(n)... = 0.ABBBB...,$$

where A is some block of digits and B is some other repeting block of digits. Asume that B has length t. If there exist infinitely many $a(n)$'s such that the decimal representation of $a(n)$ contains at least $2t$ consecutive zeros, then, since B has lenght t, it follows that the block of these $2t$ consecutive zeros will contain a full period B. Hence, $B = 0$ and the number has, in fact, only finitely many nonzero decimals, which is impossible because $a(n)$ is never zero.

All it is left to do is to notice that if $a(n)$ is any of the 12 sequences above, then $a(n)$ has the property that there exist arbitrarily many consecutive zero's in the decimal representation of $a(n)$. This is clear for the sequences 1, 2, 3, 4, 8 and 9 because these functions are onto, hence they have all the positive integers in their range. It is also obvious for the sequence 10 because $n!$ becomes divisible with arbitrarily large powers of 10 when n is large. For the sequence 7, fix any t and choose infinitely many primes from the progression $(10^{2t+2}k + 1)_{k \geq 0}$ whose first term is 1 and whose difference is 10^{2t+2}. This is possible by Dirichlet's theorem. Such a prime will end in ...00000001 with $2t+1$ consecutive zero's. For the sequence 5, notice that the Euler function of the primes constructed above is of the form $10^{2t+2}k$, hence it ends in $2t+2$ zeros, while for the sequence 6, notice that the divisor sum of the above primes is of the form $10^{2t+2}k+2$, hence it ends in ...000002 with $2t+1$ consecutive zeros. For the sequence 11, since a is coprime to 10, it follows that for any t there exist infinitely many n's such that $a^n \equiv 1 \pmod{10^{2t+2}}$. To see why this happens, simply choose n to be any multiple of the Euler function of 10^{2t+2}. What the above congruence says, is that a^n is of the form .....0000001 with at least $2t+1$ consecutive zero's (here is why we don't want a to be 1).

Now 12 should also be obvious. It is also clear that the argument can be extended to any base.

It certainly seems much harder to conclude if any one of those series is transcendal or not.

Reference

[1] *Smarandache Irrationality Conjectures*, at http://www.gallup.unm.edu/ smarandache/

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