The formula of $\pi(N)$

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Abstract

The formula of prime-counting function $\pi(N = 6n + 3)$ is described below.

$$\pi(N = 6n + 3) = 2n + 2 - \frac{2}{3}\sum_{k=1}^{n}\left\{\frac{\pi\beta(6k-1)}{\pi\beta(6k-1)-1} + \frac{\pi\beta(6k+1)}{\pi\beta(6k+1)-1}\right\}$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left\{\frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right) + sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}\right\}$$

where, $\beta(6k-1) = \tau(6k-1) - 2, \beta(6k+1) = \tau(6k+1) - 2, \ldots$

1. Introduction

We study $6n \pm 1$ type number because all of the prime number is $6n \pm 1$ type with the exception of 2,3 and $4n \pm 1$ type has the multiple of 3. We know that the numerical expression of $\tau(N), \sigma(N)$ is not exist when the prime factorization of N is unknown, but if the prime factorization of N is known then the numerical expression of $\tau(N), \sigma(N)$ is exist. For solving this problem, we obtain $\tau(6n \pm 1), \sigma(6n \pm 1)$ by using the characteristics of $6n \pm 1$ type when we don't know the prime factorization of N. And, we define $\rho(N)$ if N is a prime number then 0 else 1, and we study $\pi(N)$ by using $\rho(N)$. And by using the contents of the above, we study the numerical expression about that the prime number $6x \pm 1$ appears for the first time after the prime number $6p \pm 1$.

2. The formula of $\pi(N)$

Definition 1. Unless otherwise stated, all of the numbers that are used in the contents of the following is a natural number.

Definition 2. $[\ ]$ is Gauss expression, that is, floor function. For example, $[1.3] = 1$

Definition 3. $\overline{\mathbb{R}}$ is set of real number except integer.

Definition 4. For arbitrary d

$$\beta(N) = \left\{\begin{matrix} 0, & if\ N\ is\ a\ prime\ number \\ d, & if\ N\ is\ 1\ or\ a\ composite\ number \end{matrix}\right\},\ \rho(N) = \left\{\begin{matrix} 0, & if\ N\ is\ a\ prime\ number \\ 1, & if\ N\ is\ 1\ or\ a\ composite\ number \end{matrix}\right\}$$

Definition 5. "→,➔" is an expression to simplify the distinction between the formula when we expand the numberical expression. For example, when we expand $a + 1 = 0$ to obtain $a = -1$, we express $a + 1 = 0$ ➔ $a = -1$.

Theorem 1. Characteristics of $N = 6n \pm 1$ type composite number

For a composite number of $N = 6n \pm 1$ type

If $P|N,\ T|N,\ N = 6n \pm 1 = PT(1 < P < N, 1 < T < N)$ then $P \equiv \pm 1 (\text{mod } 6),\ T \equiv \pm 1 (\text{mod } 6)$

When $N = 6n + 1$, if $P = 6p + 1, T = 6t + 1$ then $N = P + 6tP,\ n = p + tP$

if $P = 6p - 1, T = 6t - 1$ then $N = -P + 6tP,\ n = -p + tP$

When $N = 6n - 1$, if $P = 6p + 1, T = 6t - 1$ then $N = -P + 6tP,\ n = -p + tP$

if $P = 6p - 1, T = 6t + 1$ then $N = P + 6tP,\ n = p + tP$

Further, the above formula always holds no matter if P is a prime number or a composite number.

Proof 1. Because N is a composite number, let us define $N = PT$ and $P = 6p + r, T = 6t + s$

$N = PT = (6p + r)(6t + s) = 6(6pt + ps + tr) + rs$ and $N \equiv \pm 1 (\text{mod } 6)$, so,

$N \equiv 6(6pt + ps + tr) + rs \equiv rs \equiv \pm 1 (\text{mod } 6)$. Because r, s is the one of $0, \pm 1, \pm 2, \pm 3$,

$r \equiv \pm 1 (\text{mod } 6), s \equiv \pm 1 (\text{mod } 6)$. That is, $P \equiv \pm 1 (\text{mod } 6),\ T \equiv \pm 1 (\text{mod } 6)$

In the case of $N = 6n + 1$, because $P \equiv 1 (\text{mod } 6), T \equiv 1 (\text{mod } 6)$ or $P \equiv -1 (\text{mod } 6), T \equiv -1 (\text{mod } 6)$, let us define $P \equiv 1 (\text{mod } 6), T \equiv 1 (\text{mod } 6)$, that is, $P = 6p + 1, T = 6t + 1$

$N = 6n + 1 = PT = (6p + 1)(6t + 1) = 36pt + 6p + 6t + 1 = 6p + 1 + 6t(6p + 1)$

$= P + 6tP = 6(6pt + p + t) + 1 = 6\big(p + t(6p + 1)\big) + 1 = 6(p + tP) + 1$

Therefore, $N = 6n + 1 = P + 6tP,\ n = p + tP$

Let us define $P \equiv -1 (\text{mod } 6), T \equiv -1 (\text{mod } 6)$, that is, $P = 6p - 1, T = 6t - 1$

$N = 6n + 1 = PT = (6p - 1)(6t - 1) = 36pt - 6p - 6t + 1 = -(6p - 1) + 6t(6p - 1)$

$= -P + 6tP = 6(6pt - p - t) + 1 = 6\big(-p + t(6p - 1)\big) + 1 = 6(-p + tP) + 1$

Therefore, $N = 6n + 1 = -P + 6tP,\ n = -p + tP$

In the case of $N = 6n - 1$, because $P \equiv 1 (\text{mod } 6), T \equiv -1 (\text{mod } 6)$ or $P \equiv -1 (\text{mod } 6), T \equiv 1 (\text{mod } 6)$, let us define $P \equiv 1 (\text{mod } 6), T \equiv -1 (\text{mod } 6)$, that is, $P = 6p + 1, T = 6t - 1$

$N = 6n - 1 = PT = (6p + 1)(6t - 1) = 36pt - 6p + 6t - 1 = -(6p + 1) + 6t(6p + 1)$

$= -P + 6tP = 6(6pt - p + t) - 1 = 6\big(-p + t(6p + 1)\big) - 1 = 6(-p + tP) - 1$

Therefore, $N = 6n - 1 = -P + 6tP,\ n = -p + tP$

Let us define $P \equiv -1 (\text{mod } 6), T \equiv 1 (\text{mod } 6)$, that is, $P = 6p - 1, T = 6t + 1$

$N = 6n - 1 = PT = (6p - 1)(6t + 1) = 36pt + 6p - 6t - 1 = 6p - 1 + 6t(6p - 1)$

$= P + 6tP = 6(6pt + p - t) - 1 = 6\big(p + t(6p - 1)\big) - 1 = 6(p + tP) - 1$

Therefore, $N = 6n - 1 = P + 6tP,\ n = p + tP$

Further, because $P|N,\ P$ is self-evidently a prime number or a composite number according to unique factorization theorem. Therefore, the above formula always holds no matter if P is a prime number or a composite number. ■

Theorem 2. $\beta(N = 6n \pm 1), \tau(N = 6n \pm 1), \sigma(N = 6n \pm 1)$

p(P) / n(N)	N=6n+1=PT=(6p+1)(6t+1) type				N=6n+1=PT=(6p-1)(6t-1) type			
	1(P=7)	**2(P=13)**	**3(P=19)**	**…**	**1(P=5)**	**2(P=11)**	**3(P=17)**	**…**
1(N=7)								
2(N=13)								
3(N=19)								
4(N=25)					N=25(t=1)			
…								
8(N=49)	N=49(t=1)							
9(N=55)					N=55(t=2)	N=55(t=1)		
10(N=61)								
…								
14(N=85)					N=85(t=3)		N=85(t=1)	
15(N=91)	N=91(t=2)	N=91(t=1)						
…								

(Table 2.1. $N = 6n + 1$ type)

We indicate $N = 6n + 1$ of $N = 6n + 1$ type number on the record, P of $N = PT$ on the column in Table 2.1. Because $N = PT = (6p + 1)(6t + 1)$ or $N = PT = (6p - 1)(6t - 1)$ in the case of $N = 6n + 1$ according to theorem 1, we indicate $N = PT = (6p + 1)(6t + 1)$ on the left side of column, and we indicate $N = PT = (6p - 1)(6t - 1)$ on the right side of column.
As in the example of $p = 1(P = 7)$, if we indicate the multiple of each p column of $N = PT = (6p + 1)(6t + 1)$ type, then $N = 49(t = 1), N = 91(t = 2)$ is displayed,
t in $(t = 1), (t = 2)$ is t of $6t + 1$ and t means the $t'th$ multiple of p column. That is, $N = 49(n = 8, t = 1)$ is the first$(t = 1)$ multiple of 7, $N = 91(n = 15, t = 2)$ is the second $(t = 2)$ multiple of 7. If we write the multiple of each column on the cell as like this, as in the example of $N = 49$, the record is a composite because some cells is filled, as in the example of $N = 17$, the record is a prime because all cells is not filled.
And, as in the example of 91, the first multiple of 2 column is same with the second multiple of 1 column, that is, the $t'th$ multiple of p column is duplicated with the $p'th$ multiple of t column. The blue cell means such duplication.

Definition 6. In Table 2.1,
Let us define $l(N)$ as the sum of the number written to all the cells in N or less.
Let us define $r(N)$ as the sum of the number except duplication of the numbers written to all the cells in N or less.

Theorem 2.1. $\boldsymbol{N = 6n + 1}$ **type**

$$l(N = 6n + 1) = \sum_{p=1}^{\left[\frac{n-1}{7}\right]} \left[\frac{n-p}{6p+1}\right] + \sum_{p=1}^{\left[\frac{n+1}{5}\right]} \left[\frac{n+p}{6p-1}\right]$$

$$r(N = 6n + 1) = \sum_{p=1}^{\left[\frac{-1+\sqrt{6n+1}}{6}\right]} \left(\left[\frac{n-p}{6p+1}\right] - (p-1)\right) + \sum_{p=1}^{\left[\frac{1+\sqrt{6n+1}}{6}\right]} \left(\left[\frac{n+p}{6p-1}\right] - (p-1)\right)$$

$l(6n+1) - l(6(n-1)+1)$ is the number of nontrivial divisor of $6n+1$.

$\tau(N = 6n + 1) = 2 + l(6n+1) - l(6(n-1)+1)$

$$\sigma(N = 6n + 1) = 1 + (6n+1) + \sum_{p=1}^{\left[\frac{n-1}{7}\right]} \left(\left[\frac{n-p}{6p+1}\right](6p+1)\right) + \sum_{p=1}^{\left[\frac{n+1}{5}\right]} \left(\left[\frac{n+p}{6p-1}\right](6p-1)\right)$$
$$- \left\{ \sum_{p=1}^{\left[\frac{(n-1)-1}{7}\right]} \left(\left[\frac{(n-1)-p}{6p+1}\right](6p+1)\right) + \sum_{p=1}^{\left[\frac{(n-1)+1}{5}\right]} \left(\left[\frac{(n-1)+p}{6p-1}\right](6p-1)\right) \right\}$$

- $\beta(N = 6n + 1)$ could be used by the one of formula below.

$\beta(N = 6n + 1) = l(6n+1) - l(6(n-1)+1)$

$$= \sum_{p=1}^{\left[\frac{n-1}{7}\right]} \left[\frac{n-p}{6p+1}\right] + \sum_{p=1}^{\left[\frac{n+1}{5}\right]} \left[\frac{n+p}{6p-1}\right]$$
$$- \left\{ \sum_{p=1}^{\left[\frac{(n-1)-1}{7}\right]} \left[\frac{(n-1)-p}{6p+1}\right] + \sum_{p=1}^{\left[\frac{(n-1)+1}{5}\right]} \left[\frac{(n-1)+p}{6p-1}\right] \right\}$$

$\beta(N = 6n + 1) = r(6n+1) - r(6(n-1)+1)$

$$= \sum_{p=1}^{\left[\frac{-1+\sqrt{6n+1}}{6}\right]} \left(\left[\frac{n-p}{6p+1}\right] - (p-1)\right) + \sum_{p=1}^{\left[\frac{1+\sqrt{6n+1}}{6}\right]} \left(\left[\frac{n+p}{6p-1}\right] - (p-1)\right)$$
$$- \left\{ \sum_{p=1}^{\left[\frac{-1+\sqrt{6(n-1)+1}}{6}\right]} \left(\left[\frac{(n-1)-p}{6p+1}\right] - (p-1)\right) + \sum_{p=1}^{\left[\frac{1+\sqrt{6(n-1)+1}}{6}\right]} \left(\left[\frac{(n-1)+p}{6p-1}\right] - (p-1)\right) \right\}$$

$$\beta(N = 6n + 1) = \tau(N) - 2 = \sum_{p=1}^{N} \left(\left[\frac{N}{p}\right] - \left[\frac{N-1}{p}\right]\right) - 2 = \sum_{p=1}^{6n+1} \left(\left[\frac{6n+1}{p}\right] - \left[\frac{6n}{p}\right]\right) - 2$$

$\beta(N = 6n + 1) = \sigma(6n+1) - (1 + (6n+1))$

Theorem 2.2. $\boldsymbol{N = 6n - 1}$ **type**

$$l(N = 6n-1) = \sum_{p=1}^{\left[\frac{n-1}{5}\right]} \left[\frac{n+p}{6p+1}\right] + \sum_{p=1}^{\left[\frac{n+1}{7}\right]} \left[\frac{n-p}{6p-1}\right]$$

$$r(N = 6n-1) = \sum_{p=1}^{\left[\frac{\sqrt{6n}}{6}\right]} \left(\left[\frac{n+p}{6p+1}\right] - (p-1)\right) + \sum_{p=1}^{\left[\frac{\sqrt{6n}}{6}\right]} \left(\left[\frac{n-p}{6p-1}\right] - (p-1)\right)$$

$l(6n-1) - l(6(n-1)-1)$ is the number of nontrivial divisor of $6n-1$.

$\tau(N = 6n-1) = 2 + l(6n-1) - l(6(n-1)-1)$

$$\sigma(N = 6n-1) = 1 + (6n-1) + \sum_{p=1}^{\left[\frac{n-1}{5}\right]} \left(\left[\frac{n+p}{6p+1}\right](6p+1)\right) + \sum_{p=1}^{\left[\frac{n+1}{7}\right]} \left(\left[\frac{n-p}{6p-1}\right](6p-1)\right)$$

$$- \left\{ \sum_{p=1}^{\left[\frac{(n-1)-1}{5}\right]} \left(\left[\frac{(n-1)+p}{6p+1}\right](6p+1)\right) + \sum_{p=1}^{\left[\frac{(n-1)+1}{7}\right]} \left(\left[\frac{(n-1)-p}{6p-1}\right](6p-1)\right) \right\}$$

- $\beta(N = 6n-1)$ could be used by the one of formula below.

$$\beta(N = 6n-1) = l(6n-1) - l(6(n-1)-1)$$

$$= \sum_{p=1}^{\left[\frac{n-1}{5}\right]} \left[\frac{n+p}{6p+1}\right] + \sum_{p=1}^{\left[\frac{n+1}{7}\right]} \left[\frac{n-p}{6p-1}\right]$$

$$- \left\{ \sum_{p=1}^{\left[\frac{(n-1)-1}{5}\right]} \left[\frac{(n-1)+p}{6p+1}\right] + \sum_{p=1}^{\left[\frac{(n-1)+1}{7}\right]} \left[\frac{(n-1)-p}{6p-1}\right] \right\}$$

$$\beta(N = 6n-1) = r(6n-1) - r(6(n-1)-1)$$

$$= \sum_{p=1}^{\left[\frac{\sqrt{6n}}{6}\right]} \left(\left[\frac{n+p}{6p+1}\right] - (p-1)\right) + \sum_{p=1}^{\left[\frac{\sqrt{6n}}{6}\right]} \left(\left[\frac{n-p}{6p-1}\right] - (p-1)\right)$$

$$- \left\{ \sum_{p=1}^{\left[\frac{\sqrt{6(n-1)}}{6}\right]} \left(\left[\frac{(n-1)+p}{6p+1}\right] - (p-1)\right) + \sum_{p=1}^{\left[\frac{\sqrt{6(n-1)}}{6}\right]} \left(\left[\frac{(n-1)-p}{6p-1}\right] - (p-1)\right) \right\}$$

$$\beta(N = 6n-1) = \tau(N) - 2 = \sum_{p=1}^{N} \left(\left[\frac{N}{p}\right] - \left[\frac{N-1}{p}\right]\right) - 2 = \sum_{p=1}^{6n-1} \left(\left[\frac{6n-1}{p}\right] - \left[\frac{6n-2}{p}\right]\right) - 2$$

$$\beta(N = 6n-1) = \sigma(6n-1) - (1 + (6n-1))$$

Proof 2.1. $N = 6n+1$ type

In the case of $N = 6n+1 = PT = (6p+1)(6t+1)$

Let us define $A_{p,t} = 6a_{p,t}+1$ as an arbitray $t'th$ multiple of p column in table 2.1.

According to theorem 1

$A_{p,t-1} = 6(p+(t-1)P)+1, A_{p,t} = 6(p+tP)+1, A_{p,t+1} = 6(p+(t+1)P)+1$ and

$a_{p,t+1} - a_{p,t} = (p+(t+1)P) - (p+tP) = P, a_{p,t} - a_{p,t-1} = (p+tP) - (p+(t-1)P) = P$

So, $\{a_{p,t}\}$ is arithmetic progression, let us define d as the common difference, $d = P$,

$a_{p,t} = p + tP = a_{p,1} + (t-1)d = 7p + 1 + (t-1)P$

Because $a_{p,t} \le n$, so, $p + tP \le n \to t \le \dfrac{n-p}{P}$, so, length of $a_{p,t}$ is $\left[\dfrac{n-p}{P}\right] = \left[\dfrac{n-p}{6p+1}\right]$

Because $a_{p,1} \le n$, so, $a_{p,1}(= 7p+1) \le n \to p \le \dfrac{n-1}{7}$, so, number of p column is $\left[\dfrac{n-1}{7}\right]$

Therefore, let us define $l_+(N)$ as $l(N)$ of $N = (6p+1)(6t+1)$, $l_+(N = 6n+1) = \displaystyle\sum_{p=1}^{\left[\frac{n-1}{7}\right]} \left[\frac{n-p}{6p+1}\right]$

And, a general term of the $m'th$ multiple of t column is $a_{t,m} = t + m(6t+1)$.

If $m = p$, then $a_{t,p} = t + p(6t+1) = t + 6tp + p = p + t(6p+1) = p + tP = a_{p,t}$.

That is, $t'th$ term of p as the $t'th$ multiple of p column is duplicated with $p'th$ term of t as the $p'th$ multiple of t column. To exclude such duplication, in the case of $t < p$, we exclude only the $t'th$ duplicated multiple of $a_{p,t}$ in $1 \le t < p$, because $a_{t,m}$ has already same thing.

Let us define $\{c_{p,t}\}$ as arithmetic progression except duplication of $a_{p,t}$, the common difference is $d = P$, but the initial term should be $t = p$. So

$c_{p,1} = a_{p,p} = p + pP = p + p(6p+1) = 6p^2 + 2p$

The length of $c_{p,t}$ is the length of $a_{p,t} - (p-1)$, so, $\left[\dfrac{n-p}{P}\right] - (p-1) = \left[\dfrac{n-p}{6p+1}\right] - (p-1)$

Because $c_{p,1} \le n$, so, $c_{p,1}(= 6p^2+2p) \le n \to 1 \le p \le \dfrac{-1+\sqrt{6n+1}}{6}$

Therefore, Let us define $r_+(N)$ as $r(N)$ of $N = (6p+1)(6t+1)$

$$r_+(N = 6n+1) = \sum_{p=1}^{\left[\frac{-1+\sqrt{6n+1}}{6}\right]} \left(\left[\frac{n-p}{6p+1}\right] - (p-1)\right).$$

For reference, $c_{p,1}$ is perfect squre because $6(6p^2+2p)+1 = 36p^2+12p+1 = (6p+1)^2$.

In the case of $N = 6n + 1 = PT = (6p - 1)(6t - 1)$

Let us define $B_{p,t} = 6b_{p,t} + 1$ as an arbitray $t'th$ multiple of p column in table 2.1.

According to theorem 1

$B_{p,t-1} = 6(-p + (t-1)P) + 1, B_{p,t} = 6(-p + tP) + 1, B_{p,t+1} = 6(-p + (t+1)P) + 1$ and $b_{p,t+1} - b_{p,t} = (-p + (t+1)P) - (-p + tP) = P, b_{p,t} - b_{p,t-1} = (-p + tP) - (-p + (t - 1)P=P.$

So, $\{b_{p,t}\}$ is arithmetic progression, let us define d as the common difference, $d = P$, a general term is $b_{p,t} = -p + tP = b_{p,1} + (t-1)d = 5p - 1 + (t-1)P$

Because $b_{p,t} \le n, \text{so}, -p + tP \le n \to t \le \dfrac{n+p}{P}, \text{so, the length of } b_{p,t}$ is $\left[\dfrac{n+p}{P}\right] = \left[\dfrac{n+p}{6p-1}\right]$

Because $b_{p,1} \le n, \text{so}, b_{p,1} = 5p - 1 \le n \to p \le \dfrac{n+1}{5}, \text{so, the number of } p \text{ column is } \left[\dfrac{n+1}{5}\right]$

Therefore, let us define $l_-(N)$ as $l(N)$ of $N = (6p-1)(6t-1), l_-(N = 6n+1) = \displaystyle\sum_{p=1}^{\left[\frac{n+1}{5}\right]} \left[\frac{n+p}{6p-1}\right]$

And, a general term of the $m'th$ multiple of t column is $b_{t,m} = -t + m(6t - 1)$.

If $m = p$, then $b_{t,p} = -t + p(6t-1) = -t + 6tp - p = -p + t(6p-1) = -p + tP = b_{p,t}$

According to the same principle as the above $N = (6p+1)(6t+1)$, let us difine $\{d_{p,t}\}$ as arithmetic progression except duplication of $b_{p,t}$, the common difference is $d = P$, but the initial term should be $t = p$. So, $d_{p,1} = b_{p,p} = -p + pP = -p + p(6p-1) = 6p^2 - 2p$

The length of $d_{p,t}$ is the length of $b_{p,t} - (p-1), \text{so}, \left[\dfrac{n+p}{P}\right] - (p-1) = \left[\dfrac{n+p}{6p-1}\right] - (p-1)$

Because, $d_{p,1} \le n, \text{so}, d_{p,1} = 6p^2 - 2p \le n \to 1 \le p \le \dfrac{1+\sqrt{6n+1}}{6}$

Therefore, let us define $r_-(N)$ as $r(N)$ of $N = (6p-1)(6t-1)$

$$r_-(N = 6n+1) = \sum_{p=1}^{\left[\frac{1+\sqrt{6n+1}}{6}\right]} \left(\left[\frac{n+p}{6p-1}\right] - (p-1)\right)$$

For reference, $d_{p,1}$ is perfect squre because $6(6p^2 - 2p) + 1 = 36p^2 - 12p + 1 = (6p-1)^2$.

By summarizing the above contents, $l(N) = l_+(N) + l_-(N)$ and $r(N) = r_+(N) + r_-(N)$, so

$$l(N = 6n+1) = \sum_{p=1}^{\left[\frac{n-1}{7}\right]} \left[\frac{n-p}{6p+1}\right] + \sum_{p=1}^{\left[\frac{n+1}{5}\right]} \left[\frac{n+p}{6p-1}\right]$$

$$r(N = 6n+1) = \sum_{p=1}^{\left[\frac{-1+\sqrt{6n+1}}{6}\right]} \left(\left[\frac{n-p}{6p+1}\right] - (p-1)\right) + \sum_{p=1}^{\left[\frac{1+\sqrt{6n+1}}{6}\right]} \left(\left[\frac{n+p}{6p-1}\right] - (p-1)\right)$$

Let us define $U = \left\lfloor\frac{n-p}{6p+1}\right\rfloor - \left\lceil\frac{(n-1)-p}{6p+1}\right\rceil, V = \left\lfloor\frac{n+p}{6p-1}\right\rfloor - \left\lceil\frac{(n-1)+p}{6p-1}\right\rceil, R = \sum_{P>1}^{P<N} U + \sum_{P>1}^{P<N} V$

If $P|6n+1$ then $P \nmid 6(n-1)+1$, so

The length of $a_{p,t}$ is $\left\lfloor\frac{n-p}{6p+1}\right\rfloor = \frac{n-p}{6p+1} = t$, and, $\frac{(n-1)-p}{6p+1} < t \rightarrow \left\lceil\frac{(n-1)-p}{6p+1}\right\rceil = t-1$,

Therefore, $U = \left\lfloor\frac{n-p}{6p+1}\right\rfloor - \left\lceil\frac{(n-1)-p}{6p+1}\right\rceil = t-(t-1) = 1$,

The length of $b_{p,t}$ is $\left\lfloor\frac{n+p}{6p-1}\right\rfloor = \frac{n+p}{6p-1} = t$, and, $\frac{(n-1)+p}{6p-1} < t \rightarrow \left\lceil\frac{(n-1)+p}{6p-1}\right\rceil = t-1$

Therefore, $V = \left\lfloor\frac{n+p}{6p-1}\right\rfloor - \left\lceil\frac{(n-1)+p}{6p-1}\right\rceil = t-(t-1) = 1$

If $P \nmid 6n+1$,

let us define $m < n < o, P|6m+1, P|6o+1, \frac{6m+1}{P} = 6t \pm 1, \frac{6o+1}{P} = 6(t+1) \pm 1$

$t < \frac{n-p}{6p+1} < t+1 \rightarrow \left\lfloor\frac{n-p}{6p+1}\right\rfloor = t$, and, $t \leq \frac{(n-1)-p}{6p+1} < t+1 \rightarrow \left\lceil\frac{(n-1)-p}{6p+1}\right\rceil = t$, so,

$U = \left\lfloor\frac{n-p}{6p+1}\right\rfloor - \left\lceil\frac{(n-1)-p}{6p+1}\right\rceil = t-t = 0$

$t < \frac{n+p}{6p-1} < t+1 \rightarrow \left\lfloor\frac{n+p}{6p-1}\right\rfloor = t$, and, $t \leq \frac{(n-1)+p}{6p-1} < t+1 \rightarrow \left\lceil\frac{(n-1)+p}{6p-1}\right\rceil = t$, so,

$V = \left\lfloor\frac{n+p}{6p-1}\right\rfloor - \left\lceil\frac{(n-1)+p}{6p-1}\right\rceil = t-t = 0$

By summarizing the above contents, if $P|6n+1$ then $U = V = 1$,

if $P \nmid 6n+1$, then $U = V = 0$, so,

$$R = \sum_{P>1}^{P<N} U + \sum_{P>1}^{P<N} V = \sum_{6p+1|N}^{1<P<N} U + \sum_{6p+1\nmid N}^{1<P<N} U + \sum_{6p-1|N}^{1<P<N} V + \sum_{6p-1\nmid N}^{1<P<N} V$$

$$= \sum_{6p+1|N}^{1<P<N} U + \sum_{6p-1|N}^{1<P<N} V + \sum_{6p+1\nmid N}^{1<P<N} U + \sum_{6p-1\nmid N}^{1<P<N} V = \sum_{6p+1|N}^{1<P<N} 1 + \sum_{6p-1|N}^{1<P<N} 1 + \sum_{6p+1\nmid N}^{1<P<N} 0 + \sum_{6p-1\nmid N}^{1<P<N} 0$$

$$= \sum_{P|N}^{1<P<N} 1 + \sum_{P\nmid N}^{1<P<N} 0 = \sum_{P|N}^{1<P<N} 1$$

R is the number of nontrivial divisor of N because the number of nontrivial divisor of N is $\sum_{P|N}^{1<P<N} 1$,

And, if $W=\left[\frac{n-p}{6p+1}\right]+\left[\frac{n+p}{6p-1}\right]$, $X=\left[\frac{(n-1)-p}{6p+1}\right]+\left[\frac{(n-1)+p}{6p-1}\right]$, then $W-X=U+V$

$$l(6n+1)-l(6(n-1)+1)=\sum_{P>1}^{P<N}W-\sum_{P>1}^{P<N}X=\sum_{P>1}^{P<N}(W-X)=\sum_{P>1}^{P<N}(U+V)=\sum_{P>1}^{P<N}U+\sum_{P>1}^{P<N}V$$

,so, $l(6n+1)-l(6(n-1)+1)=R$.

Therefore, $l(6n+1)-l(6(n-1)+1)$ is the number of non-trivial divisor of N.

Because $\tau(N=6n+1)$ is sum of the number of trivial divisor and the number of non-trivial divisor, and the number of trivial divisor of N is 2,

$\tau(N=6n+1)=2+l(6n+1)-l(6(n-1)+1)$

Because $\sigma(N=6n+1)$ is sum of trivial divisor and non-trivial divisor, and the trivial divisor of N are $1, N$,

$$\sigma(N=6n+1)=1+N+\sum_{P|N}^{1<P<N}P=1+N+\sum_{P|N}^{1<P<N}(1\times P)+\sum_{P\nmid N}^{1<P<N}(0\times P)$$

$$=1+N+\sum_{(P=6p+1)|N}^{1<P<N}(1\times P)+\sum_{(P=6p-1)|N}^{1<P<N}(1\times P)+\sum_{(P=6p+1)\nmid N}^{1<P<N}(0\times P)+\sum_{(P=6p-1)\nmid N}^{1<P<N}(0\times P)$$

$$=1+N+\sum_{(P=6p+1)|N}^{1<P<N}(U\times P)+\sum_{(P=6p-1)|N}^{1<P<N}(V\times P)+\sum_{(P=6p+1)\nmid N}^{1<P<N}(U\times P)+\sum_{(P=6p-1)\nmid N}^{1<P<N}(V\times P)$$

$$=1+N+\sum_{P>1}^{P<N}(U\times P)+\sum_{P>1}^{P<N}(V\times P)=1+N+\sum_{P>1}^{P<N}(W\times P)-\sum_{P>1}^{P<N}(X\times P)\text{, so,}$$

$$\sigma(N=6n+1)=1+(6n+1)+\sum_{p=1}^{\left[\frac{n-1}{7}\right]}\left(\left[\frac{n-p}{6p+1}\right](6p+1)\right)+\sum_{p=1}^{\left[\frac{n+1}{5}\right]}\left(\left[\frac{n+p}{6p-1}\right](6p-1)\right)$$

$$-\left\{\sum_{p=1}^{\left[\frac{(n-1)-1}{7}\right]}\left(\left[\frac{(n-1)-p}{6p+1}\right](6p+1)\right)+\sum_{p=1}^{\left[\frac{(n-1)+1}{5}\right]}\left(\left[\frac{(n-1)+p}{6p-1}\right](6p-1)\right)\right\}$$

In addition, because $l(6n+1)-l(6(n-1)+1)$ is the number of non-trivial divisor of N

if N is a composite number, then, $l(6n+1)-l(6(n-1)+1)>0$,

if N is a prime number,then, $l(6n+1)-l(6(n-1)+1)=0$.

Therefore,

$$\beta(N=6n+1)=l(6n+1)-l(6(n-1)+1)$$

$$=\sum_{p=1}^{\left[\frac{n-1}{7}\right]}\left[\frac{n-p}{6p+1}\right]+\sum_{p=1}^{\left[\frac{n+1}{5}\right]}\left[\frac{n+p}{6p-1}\right]$$

$$-\left\{\sum_{p=1}^{\left[\frac{(n-1)-1}{7}\right]}\left[\frac{(n-1)-p}{6p+1}\right]+\sum_{p=1}^{\left[\frac{(n-1)+1}{5}\right]}\left[\frac{(n-1)+p}{6p-1}\right]\right\}$$

For the length of $c_{p,t}, d_{p,t}$

if $P|6n+1$, for the length of $c_{p,t}$, $\left[\frac{n-p}{6p+1}\right]-(p-1)-\left(\left[\frac{(n-1)-p}{6p+1}\right]-(p-1)\right)=1,$

for the length of $d_{p,t}$, $\left[\frac{n+p}{6p-1}\right]-(p-1)-\left(\left[\frac{(n-1)+p}{6p-1}\right]-(p-1)\right)=1.$

if $P\nmid 6n+1$, for the length of $c_{p,t}$, $\left[\frac{n-p}{6p+1}\right]-(p-1)-\left(\left[\frac{(n-1)-p}{6p+1}\right]-(p-1)\right)=0,$

for the length of $d_{p,t}$, $\left[\frac{n+p}{6p-1}\right]-(p-1)-\left(\left[\frac{(n-1)+p}{6p-1}\right]-(p-1)\right)=0.$

So, if N is a composite number then $r(6n+1)-r(6(n-1)+1)>0$,

If N is a prime number then $r(6n+1)-r(6(n-1)+1)=0$.

Therefore,

$$\beta(N=6n+1)=r(6n+1)-r(6(n-1)+1)$$

$$=\sum_{p=1}^{\left[\frac{-1+\sqrt{6n+1}}{6}\right]}\left(\left[\frac{n-p}{6p+1}\right]-(p-1)\right)+\sum_{p=1}^{\left[\frac{1+\sqrt{6n+1}}{6}\right]}\left(\left[\frac{n+p}{6p-1}\right]-(p-1)\right)$$

$$-\left\{\sum_{p=1}^{\left[\frac{-1+\sqrt{6(n-1)+1}}{6}\right]}\left(\left[\frac{(n-1)-p}{6p+1}\right]-(p-1)\right)+\sum_{p=1}^{\left[\frac{1+\sqrt{6(n-1)+1}}{6}\right]}\left(\left[\frac{(n-1)+p}{6p-1}\right]-(p-1)\right)\right\}$$

And, if N is a composite number then $\tau(6n+1)-2>0$, If N is a prime number then $\tau(6n+1)-2=0$,so, $\beta(N=6n+1)=\tau(6n+1)-2$.

And, $\tau(N=6n+1)=\sum_{p=1}^{N}\left(\left[\frac{N}{p}\right]-\left[\frac{N-1}{p}\right]\right)$ is also satistied, so,

$$\beta(N=6n+1)=\ \tau(N=6n+1)-2=\sum_{p=1}^{N}\left(\left[\frac{N}{p}\right]-\left[\frac{N-1}{p}\right]\right)-2$$

$$=\sum_{p=1}^{6n+1}\left(\left[\frac{6n+1}{p}\right]-\left[\frac{6n+1-1}{p}\right]\right)-2=\sum_{p=1}^{6n+1}\left(\left[\frac{6n+1}{p}\right]-\left[\frac{6n}{p}\right]\right)-2.$$

if N is a composite number then $\sigma(6n+1)-(1+6n+1)>0$, If N is a prime number then $\sigma(6n+1)-(1+6n+1)=0$,so, $\beta(N=6n+1)=\sigma(6n+1)-(1+6n+1)$.

■

Proof 2.2. $N = 6n - 1$ type

p(P) / n(N)	N=6n-1=PT=(6p+1)(6t-1) type				N=6n-1=PT=(6p-1)(6t+1) type			
	1(P=7)	**2(P=13)**	**3(P=19)**	**…**	**1(P=5)**	**2(P=11)**	**3(P=17)**	**…**
1(N=5)								
2(N=11)								
…								
6(N=35)	35(t=1)				35(t=1)			
…								
11(N=65)		65(t=1)			65(t=2)			
12(N=71)								
13(N=77)	77(t=2)					77(t=1)		
…								
16(N=95)			95(t=1)		95(t=3)			
…								

(Table 2.2. $N = 6n - 1$ type)

We indicate $N = 6n - 1$ of $N = 6n - 1$ type number on the record, P of $N = PT$ on the column in Table 2.2. Because $N = PT = (6p + 1)(6t - 1)$ or $N = PT = (6p - 1)(6t + 1)$ in the case of $N = 6n - 1$ according to theorem 1, we indicate $N = PT = (6p + 1)(6t - 1)$ on the left side of column, and we indicate $N = PT = (6p - 1)(6t + 1)$ on the right side of column.
The remaining contents is the same as table 2.1

However, in the case of duplication unlike $N = 6n + 1$, as in the example of $N = 65$, the first multiple of 2 column of $(6p + 1)(6t - 1)$ type is same with the second multiple of 1 column of $(6p - 1)(6t + 1)$ type.
That is, the $t'th$ multiple of p column of $(6p + 1)(6t - 1)$ type is duplicated with the $p'th$ multiple of t column of $(6p - 1)(6t + 1)$ type. The blue cell means such duplication.

In the case of $N = 6n - 1 = PT = (6p + 1)(6t - 1)$

Let us define $A_{p,t} = 6a_{p,t} - 1$ as an arbitray $t'th$ multiple of p column in table 2.2.

According to theorem 1,

$A_{p,t-1} = 6(-p + (t - 1)P) - 1, A_{p,t} = 6(-p + tP) - 1, A_{p,t+1} = 6(-p + (t + 1)P) - 1$ and $a_{p,t+1} - a_{p,t} = (-p + (t + 1)P) - (-p + tP) = P, a_{p,t} - a_{p,t-1} = (-p + tP) - (-p + (t - 1)P = P$. So, $\{ap,t\}$ is arithmetic progression, let us define d as the common difference, $d=P$,

a general term is $a_{p,t} = -p + tP = a_{p,1} + (t - 1)d = 5p + 1 + (t - 1)P$

Because $a_{p,t} \leq n$, so, $-p + tP \leq n \rightarrow t \leq \frac{n + p}{P}$, so, the length of $a_{p,t}$ is $\left[\frac{n + p}{P}\right] = \left[\frac{n + p}{6p + 1}\right]$

Because $a_{p,1} \leq n$, so, $a_{p,1} = 5p + 1 \leq n \rightarrow p \leq \frac{n - 1}{5}$, so, the number of p column is $\left[\frac{n - 1}{5}\right]$

Therefore, let us define $l_+(N)$ as $l(N)$ of $N = (6p + 1)(6t - 1)$, $l_+(N = 6n - 1) = \sum_{p=1}^{\left[\frac{n-1}{5}\right]} \left[\frac{n + p}{6p + 1}\right]$

In the case of $N = 6n - 1 = PT = (6p - 1)(6t + 1)$

Let us define $B_{p,t} = 6b_{p,t} - 1$ as an arbitray $t'th$ multiple of p column in table 2.2.

According to theorem 1,

$B_{p,t-1} = 6(p + (t - 1)P) - 1, B_{p,t} = 6(p + tP) - 1, B_{p,t+1} = 6(p + (t + 1)P) - 1$

$b_{p,t+1} - b_{p,t} = (p + (t + 1)P) - (p + tP) = P, b_{p,t} - b_{p,t-1} = (p + tP) - (p + (t - 1)P) = P$

So, $\{b_{p,t}\}$is arithmetic progression, let us define d as the common difference, $d = P$,

a general term is $b_{p,t} = p + tP = b_{p,1} + (t - 1)d = 7p - 1 + (t - 1)P$.

Because $b_{p,t} \leq n$, so, $p + tP \leq n \rightarrow t \leq \frac{n - p}{P}$, so, the length of $b_{p,t}$ is $\left[\frac{n - p}{P}\right] = \left[\frac{n - p}{6p - 1}\right]$

Because $b_{p,1} \leq n$, so, $b_{p,1} = 7p - 1 \leq n \rightarrow p \leq \frac{n + 1}{7}$, so, the number of p column is $\left[\frac{n + 1}{7}\right]$

Therefore, let us define $l_-(N)$ as $l(N)$ of $N = (6p - 1)(6t + 1)$, $l_-(N = 6n - 1) = \sum_{p=1}^{\left[\frac{n+1}{7}\right]} \left[\frac{n - p}{6p - 1}\right]$

By summarizing the above contents, because $l(N) = l_+(N) + l_-(N)$,

$$l(N = 6n - 1) = \sum_{p=1}^{\left[\frac{n-1}{5}\right]} \left[\frac{n + p}{6p + 1}\right] + \sum_{p=1}^{\left[\frac{n+1}{7}\right]} \left[\frac{n - p}{6p - 1}\right]$$

In the case of $N = 6n-1 = PT = (6p+1)(6t-1)$, $a_{p,t} = -p + tP$
In the case of $N = 6n-1 = PT = (6p-1)(6t+1)$, $b_{p,t} = p + tP$
Because a general term of $m'th$ multiple of t column in $b_{p,t}$ is $b_{t,m} = t + m(6t-1)$,
if $m = p$ then $b_{t,p} = t + p(6t-1) = t + 6tp - p = -p + t(6p+1) = a_{p,t}$.
We exclude the t'th duplication multiple in $1 \le t < p$ in each $a_{p,t}$ and $b_{p,t}$ to avoid the such duplication.

Let us define $\{c_{p,t}\}$ as arithmetic progression except duplication of $a_{p,t}$, the common difference is $d = 6p+1$, but the initial term should be $t = p$.
So, $c_{p,1} = a_{p,p} = -p + pP = -p + p(6p+1) = 6p^2$

The length of $c_{p,t}$ is the length of $a_{p,t} - (p-1)$, so, $\left[\frac{n+p}{P}\right] - (p-1) = \left[\frac{n+p}{6p+1}\right] - (p-1)$

Because, $c_{p,1} \le n$, so, $c_{p,1} = 6p^2 \le n \to 1 \le p \le \frac{\sqrt{6n}}{6}$

Therefore, let us define $r_+(N)$ as $r(N)$ of $N = (6p+1)(6t-1)$

$$r_+(N = 6n-1) = \sum_{p=1}^{\left[\frac{\sqrt{6n}}{6}\right]} \left(\left[\frac{n+p}{6p+1}\right] - (p-1)\right)$$

For reference, $c_{p,1}$ is not perfect squre because $6(6p^2) - 1 = 36p^2 - 1$

If we define $\{d_{p,t}\}$ as arithmetic progression except duplication of $b_{p,t}$, the common difference is $d = 6p-1$, but the initial term should be $t = p$.
So, $d_{p,1} = b_{p,p} = p + pP = p + p(6p-1) = 6p^2$

The length of $d_{p,t}$ is the length of $b_{p,t} - (p-1)$, so, $\left[\frac{n-p}{P}\right] - (p-1) = \left[\frac{n-p}{6p-1}\right] - (p-1)$

Because $d_{p,1} \le n$, so, $d_{p,1} = 6p^2 \le n \to 1 \le p \le \frac{\sqrt{6n}}{6}$

Therefore, let us define $r_-(N)$ as $r(N)$ of $N = (6p-1)(6t+1)$

$$r_-(N = 6n-1) = \sum_{p=1}^{\left[\frac{\sqrt{6n}}{6}\right]} \left(\left[\frac{n-p}{6p-1}\right] - (p-1)\right)$$

For reference, $d_{p,1}$ is not also perfect squre because $6(6p^2) - 1 = 36p^2 - 1$

By summarizing the above contents, because $r(N) = r_+(N) + r_-(N)$

$$r(N = 6n-1) = \sum_{p=1}^{\left[\frac{\sqrt{6n}}{6}\right]} \left(\left[\frac{n+p}{6p+1}\right] - (p-1)\right) + \sum_{p=1}^{\left[\frac{\sqrt{6n}}{6}\right]} \left(\left[\frac{n-p}{6p-1}\right] - (p-1)\right)$$

In addition, for the same reason as $N = 6n+1$ (detail proof is omitted)

$\tau(N = 6n-1) = 2 + l(6n-1) - l(6(n-1)-1)$

$$\sigma(N = 6n-1) = 1 + (6n-1) + \sum_{p=1}^{\left[\frac{n-1}{5}\right]} \left(\left[\frac{n+p}{6p+1}\right](6p+1)\right) + \sum_{p=1}^{\left[\frac{n+1}{7}\right]} \left(\left[\frac{n-p}{6p-1}\right](6p-1)\right)$$

$$-\left\{\sum_{p=1}^{\left[\frac{(n-1)-1}{5}\right]} \left(\left[\frac{(n-1)+p}{6p+1}\right](6p+1)\right) + \sum_{p=1}^{\left[\frac{(n-1)+1}{7}\right]} \left(\left[\frac{(n-1)-p}{6p-1}\right](6p-1)\right)\right\}$$

If N is a composite number, $l(6n-1) - l(6(n-1)-1) > 0, r(6n-1) - r(6(n-1)-1) > 0$

If N is a prime number, $l(6n-1) - l(6(n-1)-1) > 0, r(6n-1) - r(6(n-1)-1) = 0.$

Therefore,

$$\beta(N = 6n-1) = l(6n-1) - l(6(n-1)-1)$$

$$= \sum_{p=1}^{\left[\frac{n-1}{5}\right]} \left[\frac{n+p}{6p+1}\right] + \sum_{p=1}^{\left[\frac{n+1}{7}\right]} \left[\frac{n-p}{6p-1}\right]$$

$$-\left\{\sum_{p=1}^{\left[\frac{(n-1)-1}{5}\right]} \left[\frac{(n-1)+p}{6p+1}\right] + \sum_{p=1}^{\left[\frac{(n-1)+1}{7}\right]} \left[\frac{(n-1)-p}{6p-1}\right]\right\}$$

$$\beta(N = 6n-1) = r(6n-1) - r(6(n-1)-1)$$

$$= \sum_{p=1}^{\left[\frac{\sqrt{6n}}{6}\right]} \left(\left[\frac{n+p}{6p+1}\right] - (p-1)\right) + \sum_{p=1}^{\left[\frac{\sqrt{6n}}{6}\right]} \left(\left[\frac{n-p}{6p-1}\right] - (p-1)\right)$$

$$-\left\{\sum_{p=1}^{\left[\frac{\sqrt{6(n-1)}}{6}\right]} \left(\left[\frac{(n-1)+p}{6p+1}\right] - (p-1)\right) + \sum_{p=1}^{\left[\frac{\sqrt{6(n-1)}}{6}\right]} \left(\left[\frac{(n-1)-p}{6p-1}\right] - (p-1)\right)\right\}$$

$$\beta(N = 6n-1) = \tau(N = 6n-1) - 2 = \sum_{p=1}^{N} \left(\left[\frac{N}{p}\right] - \left[\frac{N-1}{p}\right]\right) - 2$$

$$= \sum_{p=1}^{6n-1} \left(\left[\frac{6n-1}{p}\right] - \left[\frac{6n-1-1}{p}\right]\right) - 2 = \sum_{p=1}^{6n-1} \left(\left[\frac{6n-1}{p}\right] - \left[\frac{6n-2}{p}\right]\right) - 2$$

$$\beta(N = 6n-1) = \sigma(6n-1) - (1 + (6n-1))$$

■

Theorem 3. $\boldsymbol{\rho(N)}$

$$\rho(N)=\left[\frac{\beta(N)}{\beta(N)-w}\right],0<w<\frac{1}{2},w\in\overline{\mathbb{R}},w=\frac{1}{e},\frac{1}{\pi},\frac{1}{N}(N>2),\ldots$$

If N is not a prime number then

$$\rho(N)=\frac{\beta(N)}{\beta(N)-w}-\frac{1}{2}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(\frac{2k\pi\beta(N)}{\beta(N)-w}\right)}{k}=\frac{\beta(N)}{\beta(N)-w}-\frac{1}{2}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{e^{2j\frac{k\pi\beta(N)}{\beta(N)-w}}-e^{-2j\frac{k\pi\beta(N)}{\beta(N)-w}}}{2jk}$$

if N is a prime number then

$$\rho(N)=\left\{\frac{\beta(N)}{\beta(N)-w}-\frac{1}{2}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(2k\pi\frac{\beta(N)}{\beta(N)-w}\right)}{k}\right\}+\frac{1}{2}$$

$$=\left\{\frac{\beta(N)}{\beta(N)-w}-\frac{1}{2}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{e^{2j\frac{k\pi\beta(N)}{\beta(N)-w}}-e^{-2j\frac{k\pi\beta(N)}{\beta(N)-w}}}{2jk}\right\}+\frac{1}{2}$$

Especially, if $=\frac{1}{\pi}$,

if N is not a prime number then

$$\rho(N)=\frac{\pi\beta(N)}{\pi\beta(N)-1}-\frac{1}{2}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(\frac{2k\pi^2\beta(N)}{\pi\beta(N)-1}\right)}{k}$$

$$=\frac{\pi\beta(N)}{\pi\beta(N)-1}-\frac{1}{2}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{e^{2j\frac{k\pi^2\beta(N)}{\pi\beta(N)-1}}-e^{-2j\frac{k\pi^2\beta(N)}{\pi\beta(N)-1}}}{2jk}$$

if N is a prime number then

$$\rho(N)=\left\{\frac{\pi\beta(N)}{\pi\beta(N)-1}-\frac{1}{2}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(\frac{2k\pi^2\beta(N)}{\pi\beta(N)-1}\right)}{k}\right\}+\frac{1}{2}$$

$$=\left\{\frac{\pi\beta(N)}{\pi\beta(N)-1}-\frac{1}{2}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{e^{2j\frac{k\pi^2\beta(N)}{\pi\beta(N)-1}}-e^{-2j\frac{k\pi^2\beta(N)}{\pi\beta(N)-1}}}{2jk}\right\}+\frac{1}{2}$$

Proof 3. In the case of $\beta(N) = 0$, if $w \neq 0$ then $\rho(N) = 0$, because $\rho(N) = \left\lfloor \frac{\beta(N)}{\beta(N)-w} \right\rfloor = \left\lfloor \frac{0}{0-w} \right\rfloor$

In the case of $\beta(N) > 0$,

if we want to make $\rho(N) = \left\lfloor \frac{\beta(N)}{\beta(N)-w} \right\rfloor = 1$, then $\beta(N) - w > 0$ when $1 \leq \frac{\beta(N)}{\beta(N)-w} < 2$,

so, $\beta(N) - w \leq \beta(N) < 2(\beta(N) - w)$ and because the left side of the inequality $\beta(N) - w \leq \beta(N)$ is $-w \leq 0 \rightarrow 0 \leq w$, but if the case of the above $\beta(N) = 0$ is satisfied, then $w \neq 0$, so, $0 < w$

The right side of the inequality $\beta(N) < 2(\beta(N) - w) \rightarrow \beta(N) < 2\beta(N) - 2w \rightarrow 2w < \beta(N) \rightarrow w < \frac{\beta(N)}{2}$. Therefore, by summarizing the above contents, $0 < w < \frac{\beta(N)}{2}$,

If $\rho(N)$ is always held regardless of the value of $\beta(N)$, then $\beta(N) = 1$ as the minimum value of $\beta(N)$. So, $0 < w < \frac{\beta(N)}{2} \rightarrow 0 < w < \frac{1}{2}$. Therefore, $0 < w < \frac{1}{2}$, $w \in \overline{\mathbb{R}}$.

And, $0 < \frac{1}{e} < \frac{1}{2}, \frac{1}{e} \in \overline{\mathbb{R}}, 0 < \frac{1}{\pi} < \frac{1}{2}, \frac{1}{\pi} \in \overline{\mathbb{R}}$, If $N > 2$ then $0 < \frac{1}{N} < \frac{1}{2}, \frac{1}{N} \in \overline{\mathbb{R}}$.

Therefore, $w = \frac{1}{e}, \frac{1}{\pi}, \frac{1}{N} (N > 2), \ldots$

When N is not a prime number, $\beta(N) > 0, 0 < w < \frac{1}{2}, w \in \overline{\mathbb{R}}$, so, $1 < \frac{\beta(N)}{\beta(N)-w} < 2 \rightarrow \frac{\beta(N)}{\beta(N)-w} \in \overline{\mathbb{R}}$

For an arbitrary $x \in \overline{\mathbb{R}}, \lfloor x \rfloor = x - \frac{1}{2} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{sin(2k\pi x)}{k}$

[3] ,So,

$$\rho(N) = \left\lfloor \frac{\beta(N)}{\beta(N) - w} \right\rfloor = \frac{\beta(N)}{\beta(N) - w} - \frac{1}{2} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{sin\left(2k\pi \frac{\beta(N)}{q(N) - w}\right)}{k}.$$

In addition, $\sin(2a) = 2\sin(a)\, cos(a), cos(a) = \frac{e^{ja}+e^{-ja}}{2}, sin(a) = \frac{e^{ja}-e^{-ja}}{2j}$ [4],[5] ,so,

$$\sin(2a) = 2\sin(a)\, cos(a) = 2\frac{e^{ja} - e^{-ja}}{2j}\frac{e^{ja} + e^{-ja}}{2} = \frac{e^{j2a} - e^{-j2a}}{2j}.$$

$$\text{Therefore, } \rho(N) = \frac{\beta(N)}{\beta(N) - w} - \frac{1}{2} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{e^{2j\frac{k\pi\beta(N)}{\beta(N)-w}} - e^{-2j\frac{k\pi\beta(N)}{\beta(N)-w}}}{2jk}$$

When N is a prime number, because $\beta(N) = 0$

$$\frac{\beta(N)}{\beta(N)-w} - \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(2k\pi\frac{\beta(N)}{\beta(N)-w}\right)}{k} = \frac{0}{0-w} - \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(2k\pi\frac{0}{0-w}\right)}{k} = -\frac{1}{2},$$

$$\frac{\beta(N)}{\beta(N)-w} - \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{e^{2j\frac{k\pi\beta(N)}{\beta(N)-w}} - e^{-2j\frac{k\pi\beta(N)}{\beta(N)-w}}}{2jk} = \frac{0}{0-w} - \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{e^{2jk\pi\frac{0}{0-w}} - e^{-2jk\pi\frac{0}{0-w}}}{2jk} = -\frac{1}{2}$$

And, because $\rho(N) = 0$

$$\rho(N) = 0 = -\frac{1}{2} + \frac{1}{2} = \left\{\frac{\beta(N)}{\beta(N)-w} - \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(2k\pi\frac{\beta(N)}{\beta(N)-w}\right)}{k}\right\} + \frac{1}{2}$$

$$= \left\{\frac{\beta(N)}{\beta(N)-w} - \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{e^{2j\frac{k\pi\beta(N)}{\beta(N)-w}} - e^{-2j\frac{k\pi\beta(N)}{\beta(N)-w}}}{2jk}\right\} + \frac{1}{2}$$

Especially, if $=\frac{1}{\pi}$,

when N is not a prime number, then

$$\rho(N) = \frac{\beta(N)}{\beta(N)-\frac{1}{\pi}} - \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(\frac{2k\pi\beta(N)}{\beta(N)-\frac{1}{\pi}}\right)}{k} = \frac{\pi\beta(N)}{\pi\beta(N)-1} - \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(\frac{2k\pi^2\beta(N)}{\pi\beta(N)-1}\right)}{k}$$

$$= \frac{\beta(N)}{\beta(N)-\frac{1}{\pi}} - \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{e^{2j\frac{k\pi\beta(N)}{\beta(N)--\frac{1}{\pi}}} - e^{-2j\frac{k\pi\beta(N)}{\beta(N)-\frac{1}{\pi}}}}{2jk}$$

$$= \frac{\pi\beta(N)}{\pi\beta(N)-1} - \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{e^{2j\frac{k\pi^2\beta(N)}{\pi\beta(N)-1}} - e^{-2j\frac{k\pi^2\beta(N)}{\pi\beta(N)-1}}}{2jk}$$

when N is a prime number, then

$$\rho(N) = \left\{\frac{\pi\beta(N)}{\pi\beta(N)-1} - \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(\frac{2k\pi^2\beta(N)}{\pi\beta(N)-1}\right)}{k}\right\} + \frac{1}{2} = -\frac{1}{2} + \frac{1}{2} = 0$$

$$= \left\{\frac{\pi\beta(N)}{\pi\beta(N)-1} - \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{e^{2j\frac{k\pi^2\beta(N)}{\pi\beta(N)-1}} - e^{-2j\frac{k\pi^2\beta(N)}{\pi\beta(N)-1}}}{2jk}\right\} + \frac{1}{2} = -\frac{1}{2} + \frac{1}{2} = 0$$

■

Theorem 4. $\boldsymbol{\pi(N)}$

For $0 < w < \frac{1}{2}, w \in \overline{\mathbb{R}}, w = \frac{1}{e}, \frac{1}{\pi}, \frac{1}{N} (N > 2), \ldots$

$$\pi(6n+3) = 2n+2-\left\{\sum_{k=1}^{n} \rho(6k-1) + \sum_{k=1}^{n} \rho(6k+1)\right\} = \pi(6n+1) = \pi(6n+2) = \pi(6n+4)$$

$$= 2n+2-\frac{2}{3}\sum_{k=1}^{n}\left\{\frac{\beta(6k-1)}{\beta(6k-1)-w}+\frac{\beta(6k+1)}{\beta(6k+1)-w}\right\}$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left\{\frac{sin\left(\frac{2m\pi\beta(6k-1)}{\beta(6k-1)-w}\right)+sin\left(\frac{2m\pi\beta(6k+1)}{\beta(6k+1)-w}\right)}{m}\right\}$$

$$= 2n+2-\frac{2}{3}\sum_{k=1}^{n}\left\{\frac{\pi\beta(6k-1)}{\pi\beta(6k-1)-1}+\frac{\pi\beta(6k+1)}{\pi\beta(6k+1)-1}\right\}$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left\{\frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right)+sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}\right\}$$

$$= 2+\frac{2n}{3}-\frac{2}{3}\sum_{k=1}^{n}\left(\frac{1}{\pi\beta(6k-1)-1}+\frac{1}{\pi\beta(6k+1)-1}\right)$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right)+sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}\right)$$

$$= 2+\frac{4n}{3}-\frac{1}{3}\sum_{k=1}^{n}\left(\frac{\pi\beta(6k-1)+1}{\pi\beta(6k-1)-1}+\frac{\pi\beta(6k+1)+1}{\pi\beta(6k+1)-1}\right)$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right)+sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}\right)$$

Proof 4. If N is a prime number, then $1-\rho(N)=1$. If N is 1 or a composite number then

$1-\rho(N)=0.\mathrm{So},\pi(N)=\sum_{k=1}^{N}\{1-\rho(k)\}$

If $N=6n+3$ then

$\pi(N)=\pi(6n+3)$

$$=\sum_{k=1}^{6n+3}\{1-\rho(k)\}=\sum_{k=1}^{6n+3}1-\sum_{k=1}^{6n+3}\rho(k)=6n+3-\sum_{k=1}^{3}\rho(k)-\sum_{k=4}^{6n+3}\rho(k)$$

$$=6n+3-\{\rho(1)+\rho(2)+\rho(3)\}$$

$$-\sum_{k=1}^{n}\{\rho(6k-2)+\rho(6k-1)+\rho(6k+0)+\rho(6k+1)+\rho(6k+2)+\rho(6k+3)\}$$

$\rho(1)=1$ and 2,3 is prime so $\rho(2)=0,\rho(3)=0$ and

$6k-2,6k+0,6k+2,6k+3$ is composite because the multiple of 2 or 3,so,

$\rho(6k-2)=1,\rho(6k+0)=1,\rho(6k+2)=1,\rho(6k+3)=1$. Therefore,

$\pi(N)=\pi(6n+3)$

$$=6n+3-\{1+0+0\}-\left\{\sum_{k=1}^{n}1+\sum_{k=1}^{n}\rho(6k-1)+\sum_{k=1}^{n}1+\sum_{k=1}^{n}\rho(6k+1)+\sum_{k=1}^{n}1+\sum_{k=1}^{n}1\right\}$$

$$=6n+3-\{1\}-\left\{4n+\sum_{k=1}^{n}\rho(6k-1)+\sum_{k=1}^{n}\rho(6k+1)\right\}$$

$$=2n+2-\left\{\sum_{k=1}^{n}\rho(6k-1)+\sum_{k=1}^{n}\rho(6k+1)\right\} \text{------------- } (4.1)$$

$\mathrm{Therefore},\pi(6n+3)=2n+2-\left\{\sum_{k=1}^{n}\rho(6k-1)+\sum_{k=1}^{n}\rho(6k+1)\right\}$

And, $\pi(6n+1)=\pi(6n+3)-\{1-\rho(6n+2)\}-\{1-\rho(6n+3)\}=\pi(6n+3)$

$\pi(6n+2)=\pi(6n+3)-\{1-\rho(6n+3)\}=\pi(6n+3)$

$\pi(6n+4)=\pi(6n+3)+\{1-\rho(6n+4)\}=\pi(6n+3)$

Now,let us define $\mathbb{P}_-$ as a set of prime of $6k-1$ type, $\mathbb{P}_+$ as a set of prime of $6k+1$ type, $\mathbb{C}_-$ as a set of composite of $6k-1$ type, $\mathbb{C}_+$ as a set of prime of $6k+1$ type, and let us define

$$A = \frac{\beta(6k-1)}{\beta(6k-1)-w} - \frac{1}{2} + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi\beta(6k-1)}{\beta(6k-1)-w}\right)}{m},$$

$$B = \frac{\beta(6k+1)}{\beta(6k+1)-w} - \frac{1}{2} + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi\beta(6k+1)}{\beta(6k+1)-w}\right)}{m}$$

According to theorem 3,

if $6k-1 \in \mathbb{C}_-$ then $\rho(6k-1) = A$, if $6k-1 \in \mathbb{P}_-$ then $\rho(6k-1) = A + \frac{1}{2}$,

if $6k+1 \in \mathbb{C}_+$ then $\rho(6k+1) = B$, if $6k+1 \in \mathbb{P}_+$ then $\rho(6k+1) = B + \frac{1}{2}$, and,

let us express $\sum_{\mathbb{Z}}^{n} u(k)$ with the sum of $u(k)$, only if $u(k) \in \mathbb{Z}$ in $1 \le k \le n$ for a certain $u(k)$, $\mathbb{Z}$

because $\mathbb{C}_- \cap \mathbb{P}_- = \emptyset$, $\mathbb{C}_+ \cap \mathbb{P}_+ = \emptyset$, so,

$$\sum_{k=1}^{n}\rho(6k-1) = \sum_{\mathbb{C}_-}^{n}\rho(6k-1) + \sum_{\mathbb{P}_-}^{n}\rho(6k-1),$$

$$\sum_{k=1}^{n}\rho(6k+1) = \sum_{\mathbb{C}_+}^{n}\rho(6k+1) + \sum_{\mathbb{P}_+}^{n}\rho(6k+1)$$

So, if we apply the above contents to (4.1) then

$$\pi(N) = 2n+2-\left\{\sum_{\mathbb{C}_-}^{n}\rho(6k-1) + \sum_{\mathbb{P}_-}^{n}\rho(6k-1) + \sum_{\mathbb{C}_+}^{n}\rho(6k+1) + \sum_{\mathbb{P}_+}^{n}\rho(6k+1)\right\}$$

$$= 2n+2-\left\{\sum_{\mathbb{C}_-}^{n}A + \sum_{\mathbb{P}_-}^{n}\left(A+\frac{1}{2}\right) + \sum_{\mathbb{C}_+}^{n}B + \sum_{\mathbb{P}_+}^{n}\left(B+\frac{1}{2}\right)\right\}$$

$$= 2n+2-\left\{\sum_{\mathbb{C}_-}^{n}A + \sum_{\mathbb{P}_-}^{n}A + \sum_{\mathbb{P}_-}^{n}\frac{1}{2} + \sum_{\mathbb{C}_+}^{n}B + \sum_{\mathbb{P}_+}^{n}B + \sum_{\mathbb{P}_+}^{n}\frac{1}{2}\right\}$$

$$= 2n+2-\left\{\sum_{\mathbb{C}_-}^{n}A + \sum_{\mathbb{P}_-}^{n}A + \sum_{\mathbb{C}_+}^{n}B + \sum_{\mathbb{P}_+}^{n}B + \sum_{\mathbb{P}_-}^{n}\frac{1}{2} + \sum_{\mathbb{P}_+}^{n}\frac{1}{2}\right\} \text{------------- (4.2)}$$

$$\sum_{\mathbb{C}_-}^{n} A + \sum_{\mathbb{P}_-}^{n} A = \sum_{k=1}^{n} A \, , \sum_{\mathbb{C}_+}^{n} B + \sum_{\mathbb{P}_+}^{n} B = \sum_{k=1}^{n} B,$$

so, if we apply this to (4.2) then

$$\pi(N) = 2n + 2 - \left\{ \sum_{k=1}^{n} A + \sum_{k=1}^{n} B + \sum_{\mathbb{P}_-}^{n} \frac{1}{2} + \sum_{\mathbb{P}_+}^{n} \frac{1}{2} \right\} \text{------------- (4.3)}$$

If we define $\pi_-(N)$ as the number of $6n-1$ type prime number of N or less, $\pi_+(N)$ as the number of $6n+1$ type prime number of N or less then $\pi(N) = 2 + \pi_-(N) + \pi_+(N)$ because all prime is $6n-1$ or $6n+1$ type except 2,3 and

$$\sum_{\mathbb{P}_-}^{n} \frac{1}{2} = \frac{1}{2} \sum_{\mathbb{P}_-}^{n} 1 = \frac{\pi_-(N)}{2}, \sum_{\mathbb{P}_+}^{n} \frac{1}{2} = \frac{1}{2} \sum_{\mathbb{P}_+}^{n} 1 = \frac{\pi_+(N)}{2}$$

,so,if we apply this to (4.3) then

$$\pi(N) = 2n + 2 - \left\{ \sum_{k=1}^{n} A + \sum_{k=1}^{n} B + \frac{\pi_-(N)}{2} + \frac{\pi_+(N)}{2} \right\}$$

$$= 2n + 2 - \left\{ \sum_{k=1}^{n} A + \sum_{k=1}^{n} B + \frac{\pi(N) - 2}{2} \right\} \text{------------- (4.4)}$$

If we arrange (4.4) then

$$\pi(N) + \frac{\pi(N) - 2}{2} = 2n + 2 - \left\{ \sum_{k=1}^{n} A + \sum_{k=1}^{n} B \right\} \to \frac{3\pi(N) - 2}{2} = 2n + 2 - \left\{ \sum_{k=1}^{n} A + \sum_{k=1}^{n} B \right\} \to$$

$$\frac{3\pi(N)}{2} = 2n + 3 - \left\{ \sum_{k=1}^{n} A + \sum_{k=1}^{n} B \right\} \to \pi(N) = \frac{2}{3} \left\{ 2n + 3 - \left\{ \sum_{k=1}^{n} A + \sum_{k=1}^{n} B \right\} \right\} \to$$

$$\pi(N) = 2 + \frac{4n}{3} - \frac{2}{3} \left\{ \sum_{k=1}^{n} A + \sum_{k=1}^{n} B \right\} \text{------------- (4.5)}$$

If we substitute A,B to (4.5) then

$$\pi(N) = 2 + \frac{4n}{3} - \frac{2}{3}\left\{\sum_{k=1}^{n}\left(\frac{\beta(6k-1)}{\beta(6k-1)-w} - \frac{1}{2} + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi\beta(6k-1)}{\beta(6k-1)-w}\right)}{m}\right)\right.$$

$$\left. + \sum_{k=1}^{n}\left(\frac{\beta(6k+1)}{\beta(6k+1)-w} - \frac{1}{2} + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi\beta(6k+1)}{\beta(6k+1)-w}\right)}{m}\right)\right\}$$

$$= 2 + \frac{4n}{3} + \frac{2n}{3}$$

$$- \frac{2}{3}\left\{\sum_{k=1}^{n}\left(\frac{\beta(6k-1)}{\beta(6k-1)-w} + \frac{\beta(6k+1)}{\beta(6k+1)-w} + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi\beta(6k-1)}{\beta(6k-1)-w}\right)}{m}\right.\right.$$

$$\left.\left. + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi\beta(6k+1)}{\beta(6k+1)-w}\right)}{m}\right)\right\}$$

$$= 2n + 2 - \frac{2}{3}\sum_{k=1}^{n}\left(\frac{\beta(6k-1)}{\beta(6k-1)-w} + \frac{\beta(6k+1)}{\beta(6k+1)-w}\right)$$

$$- \frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi\beta(6k-1)}{\beta(6k-1)-w}\right) + sin\left(\frac{2m\pi\beta(6k+1)}{\beta(6k+1)-w}\right)}{m}\right) \text{------------- (4.6)}$$

If we substitute $w = \frac{1}{\pi}$ to (4.6) especially, then

$$\pi(N) = 2n + 2 - \frac{2}{3}\sum_{k=1}^{n}\left(\frac{\beta(6k-1)}{\beta(6k-1)-\frac{1}{\pi}} + \frac{\beta(6k+1)}{\beta(6k+1)-\frac{1}{\pi}}\right)$$

$$- \frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi\beta(6k-1)}{\beta(6k-1)-\frac{1}{\pi}}\right) + sin\left(\frac{2m\pi\beta(6k+1)}{\beta(6k+1)-\frac{1}{\pi}}\right)}{m}\right)$$

$$= 2n + 2 - \frac{2}{3}\sum_{k=1}^{n}\left(\frac{\pi\beta(6k-1)}{\pi\beta(6k-1)-1} + \frac{\pi\beta(6k+1)}{\pi\beta(6k+1)-1}\right)$$

$$- \frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right) + sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}\right) \text{------------- (4.7)}$$

And, if we modify (4.7) then

$$\pi(N) = 2n + 2 - \frac{4n}{3} + \frac{4n}{3} - \frac{2}{3}\sum_{k=1}^{n}\left(\frac{\pi\beta(6k-1)}{\pi\beta(6k-1)-1} + \frac{\pi\beta(6k+1)}{\pi\beta(6k+1)-1}\right)$$

$$- \frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right) + sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}\right)$$

$$= 2 + \frac{2n}{3} + \frac{2}{3}\sum_{k=1}^{n} 2 - \frac{2}{3}\sum_{k=1}^{n}\left(\frac{\pi\beta(6k-1)}{\pi\beta(6k-1)-1} + \frac{\pi\beta(6k+1)}{\pi\beta(6k+1)-1}\right)$$

$$- \frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right) + sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}\right)$$

$$= 2 + \frac{2n}{3} + \frac{2}{3}\sum_{k=1}^{n}\left(1 - \frac{\pi\beta(6k-1)}{\pi\beta(6k-1)-1} + 1 - \frac{\pi\beta(6k+1)}{\pi\beta(6k+1)-1}\right)$$

$$- \frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right) + sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}\right)$$

$$= 2 + \frac{2n}{3} + \frac{2}{3}\sum_{k=1}^{n}\left(\frac{\pi\beta(6k-1)-1-\pi\beta(6k-1)}{\pi\beta(6k-1)-1} + \frac{\pi\beta(6k+1)-1-\pi\beta(6k+1)}{\pi\beta(6k+1)-1}\right)$$

$$- \frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right) + sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}\right)$$

$$= 2 + \frac{2n}{3} - \frac{2}{3}\sum_{k=1}^{n}\left(\frac{1}{\pi\beta(6k-1)-1} + \frac{1}{\pi\beta(6k+1)-1}\right)$$

$$- \frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right) + sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}\right) \text{-------------} \ (4.8)$$

And, we modify (4.8) then

$$\pi(N) = 2 + \frac{2n}{3} + \frac{2n}{3} - \frac{2n}{3} - \frac{2}{3}\sum_{k=1}^{n}\left(\frac{1}{\pi\beta(6k-1)-1} + \frac{1}{\pi\beta(6k+1)-1}\right)$$

$$- \frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right) + sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}\right)$$

$$= 2 + \frac{4n}{3} - \frac{1}{3}\sum_{k=1}^{n} 2 - \frac{1}{3}\sum_{k=1}^{n}\left(\frac{2}{\pi\beta(6k-1)-1} + \frac{2}{\pi\beta(6k+1)-1}\right)$$

$$- \frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right) + sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}\right)$$

$$= 2 + \frac{4n}{3} - \frac{1}{3}\sum_{k=1}^{n}\left(1 + \frac{2}{\pi\beta(6k-1)-1} + 1 + \frac{2}{\pi\beta(6k+1)-1}\right)$$

$$- \frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right) + sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}\right)$$

$$= 2 + \frac{4n}{3} - \frac{1}{3}\sum_{k=1}^{n}\left(\frac{\pi\beta(6k-1)-1+2}{\pi\beta(6k-1)-1} + \frac{\pi\beta(6k+1)-1+2}{\pi\beta(6k+1)-1}\right)$$

$$- \frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right) + sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}\right)$$

$$= 2 + \frac{4n}{3} - \frac{1}{3}\sum_{k=1}^{n}\left(\frac{\pi\beta(6k-1)+1}{\pi\beta(6k-1)-1} + \frac{\pi\beta(6k+1)+1}{\pi\beta(6k+1)-1}\right)$$

$$- \frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right) + sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}\right) \blacksquare$$

Theorem 5. Next prime of $6n \pm 1$ type

If we define $P = 6p + 1$ as an arbitrary prime number of $6n + 1$ type

and if we define $X = 6x + 1$ as the first prime number of $6n + 1$ type after P,

then, the following equation is satisfied.

$$x = p + 1 + \sum_{k=p+1}^{x-1} \rho(6k+1)$$

$$= p + 1 + \frac{1}{2}\sum_{k=p+1}^{x-1} \frac{\pi\beta(6k+1)+1}{\pi\beta(6k+1)-1} + \frac{1}{\pi}\sum_{k=p+1}^{x-1}\sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}$$

$$= p + 1 + \sum_{k=p+1}^{x} \rho(6k+1)$$

$$= p + \frac{3}{2} + \frac{1}{2}\sum_{k=p+1}^{x} \frac{\pi\beta(6k+1)+1}{\pi\beta(6k+1)-1} + \frac{1}{\pi}\sum_{k=p+1}^{x}\sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}$$

If we define $P = 6p - 1$ as an arbitrary prime number of $6n - 1$ type

and if we define $X = 6x - 1$ as the first prime number of $6n - 1$ type after P,

then, the following equation is satisfied..

$$x = p + 1 + \sum_{k=p+1}^{x-1} \rho(6k-1)$$

$$= p + 1 + \frac{1}{2}\sum_{k=p+1}^{x-1} \frac{\pi\beta(6k-1)+1}{\pi\beta(6k-1)-1} + \frac{1}{\pi}\sum_{k=p+1}^{x-1}\sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right)}{m}$$

$$= p + 1 + \sum_{k=p+1}^{x} \rho(6k-1)$$

$$= p + \frac{3}{2} + \frac{1}{2}\sum_{k=p+1}^{x} \frac{\pi\beta(6k-1)+1}{\pi\beta(6k-1)-1} + \frac{1}{\pi}\sum_{k=p+1}^{x}\sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right)}{m}$$

Proof 5. In the case of $P = 6p + 1,\ X = 6x + 1,$
let us define $P = 6p + 1$ as an arbitrary prime number of $6n + 1$ type
and let us define $X = 6x + 1$ as the first prime number of $6n + 1$ type after P.
$\rho(6k + 1) = 1$ because $6k + 1$ is a composite number in $p < k < x$ and
$\rho(6x + 1) = 0$ because $6x + 1$ is a prime number. Therefore,

$$x = \sum_{k=1}^{x} 1 = \sum_{k=1}^{p} 1 + \sum_{k=p+1}^{x-1} 1 + \sum_{k=x}^{x} 1 + \sum_{k=x}^{x} 0 = \sum_{k=1}^{p} 1 + \sum_{k=x}^{x} 1 + \sum_{k=p+1}^{x-1} 1 + \sum_{k=x}^{x} 0$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \rho(6k + 1) + 0 = p + 1 + \sum_{k=p+1}^{x-1} \rho(6k + 1) + \sum_{k=x}^{x} \rho(6k + 1)$$

$$= p + 1 + \sum_{k=p+1}^{x} \rho(6k + 1)$$

And,for $p < k < x,\ \rho(6k + 1) = \left[\frac{\beta(6k+1)}{\beta(6k+1)-w}\right],\quad 1 < \frac{\beta(6k+1)}{\beta(6k+1)-w} < 2$,that is, $\frac{\beta(6k+1)}{\beta(6k+1)-w} \in \overline{\mathbb{R}}$,

so,according to theorem 3,if we arrange the above formula then

$$x = p + 1 + \sum_{k=p+1}^{x-1} \rho(6k + 1)$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \left\{ \frac{\pi\beta(6k + 1)}{\pi\beta(6k + 1) - 1} - \frac{1}{2} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(6k + 1)}{\pi\beta(6k + 1) - 1}\right)}{m} \right\}$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \left\{ \frac{2\pi\beta(6k + 1)}{2(\pi\beta(6k + 1) - 1)} - \frac{\pi\beta(6k + 1) - 1}{2(\pi\beta(6k + 1) - 1)} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(6k + 1)}{\pi\beta(6k + 1) - 1}\right)}{m} \right\}$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \left\{ \frac{1}{2}\left(\frac{\pi\beta(6k + 1) + 1}{\pi\beta(6k + 1) - 1}\right) + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(6k + 1)}{\pi\beta(6k + 1) - 1}\right)}{m} \right\}$$

$$= p + 1 + \frac{1}{2} \sum_{k=p+1}^{x-1} \frac{\pi\beta(6k + 1) + 1}{\pi\beta(6k + 1) - 1} + \frac{1}{\pi} \sum_{k=p+1}^{x-1} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(6k + 1)}{\pi\beta(6k + 1) - 1}\right)}{m}$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \rho(6k + 1) + \sum_{k=x}^{x} 0 = p + 1 + \sum_{k=p+1}^{x-1} \rho(6k + 1) + \sum_{k=x}^{x} \rho(6k + 1)$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \rho(6k+1) + \sum_{k=x}^{x} \left\{ \frac{\pi\beta(6k+1)}{\pi\beta(6k+1)-1} - \frac{1}{2} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m} + \frac{1}{2} \right\}$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \rho(6k+1) + \sum_{k=x}^{x} \left\{ \frac{\pi\beta(6k+1)}{\pi\beta(6k+1)-1} - \frac{1}{2} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m} \right\} + \sum_{k=x}^{x} \frac{1}{2}$$

$$= p + 1 + \sum_{k=p+1}^{x} \left\{ \frac{\pi\beta(6k+1)}{\pi\beta(6k+1)-1} - \frac{1}{2} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m} \right\} + \frac{1}{2}$$

$$= p + \frac{3}{2} + \frac{1}{2} \sum_{k=p+1}^{x} \frac{\pi\beta(6k+1)+1}{\pi\beta(6k+1)-1} + \frac{1}{\pi} \sum_{k=p+1}^{x} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}$$

In the case of $P = 6p - 1,\ X = 6x - 1$,
let us define $P = 6p - 1$ as an arbitrary prime number of $6n - 1$ type
and let us define $X = 6x - 1$ as the first prime number of $6n - 1$ type after P.
$\rho(6k - 1) = 1$ because $6k - 1$ is a composite number in $p < k < x$ and
$\rho(6x - 1) = 0$ because $6x - 1$ is a prime number. Therefore,

$$x = \sum_{k=1}^{x} 1 = \sum_{k=1}^{p} 1 + \sum_{k=p+1}^{x-1} 1 + \sum_{k=x}^{x} 1 + \sum_{k=x}^{x} 0 = \sum_{k=1}^{p} 1 + \sum_{k=x}^{x} 1 + \sum_{k=p+1}^{x-1} 1 + \sum_{k=x}^{x} 0$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \rho(6k-1) + 0 = p + 1 + \sum_{k=p+1}^{x-1} \rho(6k-1) + \sum_{k=x}^{x} \rho(6k-1)$$

$$= p + 1 + \sum_{k=p+1}^{x} \rho(6k-1)$$

And,for $p < k < x,\ \rho(6k-1) = \left[\frac{\beta(6k-1)}{\beta(6k-1)-w}\right]$, $\quad 1 < \frac{\beta(6k-1)}{\beta(6k-1)-w} < 2$,that is, $\frac{\beta(6k-1)}{\beta(6k-1)-w} \in \overline{\mathbb{R}}$,

so,according to theorem 3.

$$x = p + 1 + \sum_{k=p+1}^{x-1} \rho(6k-1)$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \left\{ \frac{\pi\beta(6k-1)}{\pi\beta(6k-1)-1} - \frac{1}{2} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right)}{m} \right\}$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \left\{ \frac{2\pi\beta(6k-1)}{2(\pi\beta(6k-1)-1)} - \frac{\pi\beta(6k-1)-1}{2(\pi\beta(6k-1)-1)} + \frac{1}{\pi}\sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right)}{m} \right\}$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \left\{ \frac{1}{2}\left(\frac{\pi\beta(6k-1)+1}{\pi\beta(6k-1)-1}\right) + \frac{1}{\pi}\sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right)}{m} \right\}$$

$$= p + 1 + \frac{1}{2}\sum_{k=p+1}^{x-1} \frac{\pi\beta(6k-1)+1}{\pi\beta(6k-1)-1} + \frac{1}{\pi}\sum_{k=p+1}^{x-1}\sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right)}{m}$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \rho(6k-1) + \sum_{k=x}^{x} 0 = p + 1 + \sum_{k=p+1}^{x-1} \rho(6k-1) + \sum_{k=x}^{x} \rho(6k-1)$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \rho(6k-1) + \sum_{k=x}^{x} \left\{ \frac{\pi\beta(6k-1)}{\pi\beta(6k-1)-1} - \frac{1}{2} + \frac{1}{\pi}\sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right)}{m} + \frac{1}{2} \right\}$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \rho(6k-1) + \sum_{k=x}^{x} \left\{ \frac{\pi\beta(6k-1)}{\pi\beta(6k-1)-1} - \frac{1}{2} + \frac{1}{\pi}\sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right)}{m} \right\} + \sum_{k=x}^{x} \frac{1}{2}$$

$$= p + 1 + \sum_{k=p+1}^{x} \left\{ \frac{\pi\beta(6k-1)}{\pi\beta(6k-1)-1} - \frac{1}{2} + \frac{1}{\pi}\sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right)}{m} \right\} + \frac{1}{2}$$

$$= p + \frac{3}{2} + \frac{1}{2}\sum_{k=p+1}^{x} \frac{\pi\beta(6k-1)+1}{\pi\beta(6k-1)-1} + \frac{1}{\pi}\sum_{k=p+1}^{x}\sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right)}{m}$$

■

Theorem 6.

The below formula is not finished but we write here, because we think that if we arrange this formula more then it would be useful.

If $1 \le \beta(6k-1) \le u, 1 \le \beta(6k+1) \le v, Max(u,v) = M, T(k) = \left(\frac{2\pi^2\beta(k)}{\pi\beta(k)-1}\right)$ then

$$\lim_{N\to\infty}\left(\frac{\pi}{2}\left(\frac{N+3}{3}-\frac{N}{lnN}\right)\left(\frac{\pi-3}{\pi-1}\right)\right) \le \sum_{k=1}^{\infty}\left(T(6k-1)\sum_{m=1}^{\infty}\left(\frac{sin(mT(6k-1))}{mT(6k-1)}\right)\right) + \sum_{k=1}^{\infty}\left(T(6k+1)\sum_{m=1}^{\infty}\left(\frac{sin(mT(6k+1))}{mT(6k+1)}\right)\right) \le \lim_{N\to\infty}\left(\frac{\pi}{2}\left(\frac{N+3}{3}-\frac{N}{lnN}\right)\right)$$

Proof 6.

Let us define below contents to simplify the formula of theorem 4 .

$$b_- = \frac{\pi\beta(6k-1)}{\pi\beta(6k-1)-1}, b_+ = \frac{\pi\beta(6k+1)}{\pi\beta(6k+1)-1}$$

$$s_- = \frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right)}{m}, s_+ = \frac{sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}$$

If we apply the above definition to theorem 4 then

$$\pi(N=6n+3) = 2n+2-\frac{2}{3}\sum_{k=1}^{n}(b_- + b_+) - \frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}(s_- + s_+) \text{ ------------- } (6.1)$$

Let us define $\mathbb{P}_-$ as a set of prime of $6k-1$ type, $\mathbb{P}_+$ as a set of prime of $6k+1$ type, $\mathbb{C}_-$ as a set of composite of $6k-1$ type, $\mathbb{C}_+$ as a set of prime of $6k+1$ type.

If $6k-1 \in \mathbb{P}_-$ then $\beta(6k-1) = 0$ so $b_- = 0$, if $6k+1 \in \mathbb{P}_+$ then $\beta(6k+1) = 0$ so $b_+ = 0$ and $\mathbb{C}_- \cap \mathbb{P}_- = \emptyset, \mathbb{C}_+ \cap \mathbb{P}_+ = \emptyset$. If we express (6.1) again according to the above contents then

$$\pi(N=6n+3) = 2n+2-\frac{2}{3}\left(\sum_{\mathbb{C}_-}^{n} b_- + \sum_{\mathbb{P}_-}^{n} b_- + \sum_{\mathbb{C}_+}^{n} b_+ + \sum_{\mathbb{P}_+}^{n} b_+\right) - \frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}(s_- + s_+)$$

$$= 2n+2-\frac{2}{3}\left(\sum_{\mathbb{C}_-}^{n} b_- + \sum_{\mathbb{C}_+}^{n} b_+\right) - \frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}(s_- + s_+) \text{ ------------- } (6.2)$$

If $1 \le \beta(6k-1) \le u, 1 \le \beta(6k+1) \le v$ then

$$\frac{\pi}{\pi-1} - \frac{\pi u}{\pi u-1} = \frac{\pi\pi u - \pi - \pi\pi u + \pi u}{(\pi-1)(\pi u-1)} = \frac{\pi u - \pi}{(\pi-1)(\pi u-1)} \ge 0 \to \frac{\pi}{\pi-1} \ge \frac{\pi u}{\pi u-1}$$

If $Max(u,v) = M$ then

$$\frac{\pi u}{\pi u-1}-\frac{\pi M}{\pi M-1}=\frac{\pi M-\pi u}{(\pi u-1)(\pi M-1)}\geq 0\rightarrow\frac{\pi u}{\pi u-1}\geq\frac{\pi M}{\pi M-1}\rightarrow$$

$$\frac{\pi M}{\pi M-1}\leq\frac{\pi u}{\pi u-1}\leq b_-\leq\frac{\pi}{\pi-1},\qquad\frac{\pi M}{\pi M-1}\leq\frac{\pi v}{\pi v-1}\leq b_+\leq\frac{\pi}{\pi-1}\text{-------------}\ (6.3)$$

If we define $\pi_-(N)$ as the number of $6n-1$ type prime number of N or less, $\pi_+(N)$ as the number of $6n+1$ type prime number of N or less then

$$\sum_{\mathbb{C}_-}^{n}1=n-\pi_-(N),\sum_{\mathbb{C}_+}^{n}1=n-\pi_+(N)\text{-------------}\ (6.4)$$

If we apply (6.3), (6.4) for using (6.2) then

$$\frac{2}{3}\left(\sum_{\mathbb{C}_-}^{n}\frac{\pi M}{\pi M-1}+\sum_{\mathbb{C}_+}^{n}\frac{\pi M}{\pi M-1}\right)\leq\frac{2}{3}\left(\sum_{\mathbb{C}_-}^{n}b_-+\sum_{\mathbb{C}_+}^{n}b_+\right)\leq\frac{2}{3}\left(\sum_{\mathbb{C}_-}^{n}\frac{\pi}{\pi-1}+\sum_{\mathbb{C}_+}^{n}\frac{\pi}{\pi-1}\right)\rightarrow$$

$$\frac{2}{3}\left(\frac{\pi M}{\pi M-1}\sum_{\mathbb{C}_-}^{n}1+\frac{\pi M}{\pi M-1}\sum_{\mathbb{C}_+}^{n}1\right)\leq\frac{2}{3}\left(\sum_{\mathbb{C}_-}^{n}b_-+\sum_{\mathbb{C}_+}^{n}b_+\right)\leq\frac{2}{3}\left(\frac{\pi}{\pi-1}\sum_{\mathbb{C}_-}^{n}1+\frac{\pi}{\pi-1}\sum_{\mathbb{C}_+}^{n}1\right)\rightarrow$$

$$\frac{2}{3}\left(\frac{\pi M}{\pi M-1}\right)\left(2n-\pi_-(N)-\pi_+(N)\right)\leq\frac{2}{3}\left(\sum_{\mathbb{C}_-}^{n}b_-+\sum_{\mathbb{C}_+}^{n}b_+\right)\leq\frac{2}{3}\left(\frac{\pi}{\pi-1}\right)\left(2n-\pi_-(N)-\pi_+(N)\right)$$

$\pi(N)=2+\pi_-(N)+\pi_+(N)$ because all prime is $6n-1$ or $6n+1$ type except 2,3.
If we apply this contents to the above formula and apply (6.1) then

$$\frac{2}{3}\left(\frac{\pi M}{\pi M-1}\right)\left(2n+2-\pi(N)\right)\leq\frac{2}{3}\left(\sum_{\mathbb{C}_-}^{n}b_-+\sum_{\mathbb{C}_+}^{n}b_+\right)\leq\frac{2}{3}\left(\frac{\pi}{\pi-1}\right)\left(2n+2-\pi(N)\right)\rightarrow$$

$$\frac{2}{3}\left(\frac{\pi M}{\pi M-1}\right)\left(2n+2-\pi(N)\right)\leq 2n+2-\pi(N=6n+3)-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}(s_-+s_+)$$

$$\leq\frac{2}{3}\left(\frac{\pi}{\pi-1}\right)\left(2n+2-\pi(N)\right)\rightarrow$$

$$\frac{2}{3}\left(\frac{\pi M}{\pi M-1}\right)\left(2n+2-\pi(N)\right)-\left(2n+2-\pi(N)\right)\leq-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}(s_-+s_+)$$

$$\leq\frac{2}{3}\left(\frac{\pi}{\pi-1}\right)\left(2n+2-\pi(N)\right)-\left(2n+2-\pi(N)\right)\rightarrow$$

$$\left(2n+2-\pi(N)\right)\left(\frac{2\pi M}{3(\pi M-1)}-1\right) \le -\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}(s_- + s_+) \le \left(2n+2-\pi(N)\right)\left(\frac{2\pi}{3(\pi-1)}-1\right)$$

$\rightarrow$

$$\frac{1}{3}\left(2n+2-\pi(N)\right)\left(\frac{-\pi M+3}{\pi M-1}\right) \le -\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}(s_- + s_+) \le \frac{1}{3}\left(2n+2-\pi(N)\right)\left(\frac{-\pi+3}{\pi-1}\right) \rightarrow$$

$$\frac{\pi}{2}\left(2n+2-\pi(N)\right)\left(\frac{\pi-3}{\pi-1}\right) \le \sum_{k=1}^{n}\sum_{m=1}^{\infty}(s_- + s_+) \le \frac{\pi}{2}\left(2n+2-\pi(N)\right)\left(\frac{\pi M-3}{\pi M-1}\right) \text{------------- (6.5)}$$

If $T(k) = \left(\frac{2\pi^2\beta(k)}{\pi\beta(k)-1}\right)$ then

$$s_- = \frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right)}{m} = \frac{sin\left(mT(6k-1)\right)}{mT(6k-1)}T(6k-1)$$

$$s_+ = \frac{sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m} = \frac{sin\left(mT(6k+1)\right)}{mT(6k+1)}T(6k+1)$$

So,

$$\sum_{k=1}^{n}\sum_{m=1}^{\infty}(s_- + s_+)$$

$$= \sum_{k=1}^{n}\left(T(6k-1)\sum_{m=1}^{\infty}\left(\frac{sin(mT(6k-1))}{mT(6k-1)}\right)\right)$$

$$+ \sum_{k=1}^{n}\left(T(6k+1)\sum_{m=1}^{\infty}\left(\frac{sin(mT(6k+1))}{mT(6k+1)}\right)\right)$$

If we apply the above contents to (6.5) then

$$\frac{\pi}{2}\left(2n+2-\pi(N)\right)\left(\frac{\pi-3}{\pi-1}\right)$$

$$\le \sum_{k=1}^{n}\left(T(6k-1)\sum_{m=1}^{\infty}\left(\frac{sin(mT(6k-1))}{mT(6k-1)}\right)\right)$$

$$+ \sum_{k=1}^{n}\left(T(6k+1)\sum_{m=1}^{\infty}\left(\frac{sin(mT(6k+1))}{mT(6k+1)}\right)\right) \le \frac{\pi}{2}\left(2n+2-\pi(N)\right)\left(\frac{\pi M-3}{\pi M-1}\right)$$

If we apply $\lim_{n\to\infty}$ to both sides of the above formula then

$$\lim_{n\to\infty}\left(\frac{\pi}{2}\left(2n+2-\pi(N)\right)\left(\frac{\pi-3}{\pi-1}\right)\right)$$

$$\leq \lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(T(6k-1)\sum_{m=1}^{\infty}\left(\frac{sin(mT(6k-1)}{mT(6k-1)}\right)\right)\right.$$

$$\left.+\sum_{k=1}^{n}\left(T(6k+1)\sum_{m=1}^{\infty}\left(\frac{sin(mT(6k+1)}{mT(6k+1)}\right)\right)\right)$$

$$\leq \lim_{n\to\infty}\left(\frac{\pi}{2}\left(2n+2-\pi(N)\right)\left(\frac{\pi M-3}{\pi M-1}\right)\right)\rightarrow$$

$$\lim_{n\to\infty}\left(\frac{\pi}{2}\left(2n+2-\pi(N)\right)\left(\frac{\pi-3}{\pi-1}\right)\right)$$

$$\leq \sum_{k=1}^{\infty}\left(T(6k-1)\sum_{m=1}^{\infty}\left(\frac{sin(mT(6k-1)}{mT(6k-1)}\right)\right)$$

$$+\sum_{k=1}^{\infty}\left(T(6k+1)\sum_{m=1}^{\infty}\left(\frac{sin(mT(6k+1)}{mT(6k+1)}\right)\right)$$

$$\leq \lim_{n\to\infty}\left(\frac{\pi}{2}\left(2n+2-\pi(N)\right)\left(1-\frac{2}{\pi M-1}\right)\right)$$

$N=6n+3\rightarrow 2n+2=\dfrac{N+3}{3}$, $\lim_{n\to\infty}\dfrac{2}{\pi M-1}=0$ and

if we apply prime number theory (PNT) [1] to the above formula then

$$\lim_{N\to\infty}\left(\frac{\pi}{2}\left(\frac{N+3}{3}-\frac{N}{lnN}\right)\left(\frac{\pi-3}{\pi-1}\right)\right)$$

$$\leq \sum_{k=1}^{\infty}\left(T(6k-1)\sum_{m=1}^{\infty}\left(\frac{sin(mT(6k-1)}{mT(6k-1)}\right)\right)$$

$$+\sum_{k=1}^{\infty}\left(T(6k+1)\sum_{m=1}^{\infty}\left(\frac{sin(mT(6k+1)}{mT(6k+1)}\right)\right)\leq \lim_{N\to\infty}\left(\frac{\pi}{2}\left(\frac{N+3}{3}-\frac{N}{lnN}\right)\right)$$

■

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