A series expansion for a real function

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Abstract

We show that a class $\mathcal{C}^\infty(\mathbb{R})$ function can be written as an $n-$summation of terms involving its derivative. For many functions, under certain conditions, this summation can become a particular series expansion.

Theorem

Let be $f(x)$ a class $\mathcal{C}^\infty(\mathbb{R})$ function with $\mathcal{D}$ domain. Given $a \in \mathcal{D}$ the function satisfies the following identity for all $n \in \mathbb{N}^+$

$$f(x) = f(a) - \sum_{k=1}^{n} \frac{(-1)^k}{k!} \left(x^k \frac{d^k f(x)}{dx^k} - a^k \frac{d^k f(x)}{dx^k}\Big|_{x=a} \right) + \mathcal{R}_n(x) \,, \qquad (1)$$

with

$$\mathcal{R}_n(x) = \frac{(-1)^n}{n!} \int_a^x x^n \frac{d^{n+1} f(x)}{dx^{n+1}} \, dx \,.$$

In particular, if

$$\lim_{n \to +\infty} \mathcal{R}_n(x) = 0$$

and $a = 0$ then

$$f(x) = f(0) - \sum_{k=1}^{\infty} \frac{(-x)^k}{k!} \frac{d^k f(x)}{dx^k} \,. \qquad (2)$$

Proof

We define the $\hat{P}$ operator that, applied to a continuous function, it gives its primitive with zero constant. We define also the operator $\hat{D}$ as

$$\hat{D} = \frac{d}{dx}\,.$$

Now consider the identity

$$J := \int_a^x \frac{df(x)}{dx}\,dx = f(x) - f(a)\,, \tag{3}$$

we can integrate it for parties, considering the product

$$1 \cdot \frac{df(x)}{dx}\,,$$

obtaining

$$J = \left[\hat{P}(1)\hat{D}f(x)\right]_a^x - \int_a^x \hat{P}(1)\hat{D}^2 f(x)\,dx\,.$$

Iterating for many times we can write

$$J = -\sum_{k=1}^{n}(-1)^k\left[\hat{P}^k(1)\hat{D}^k f(x)\right]_a^x + (-1)^n\int_a^x \hat{P}^n(1)\hat{D}^{n+1}f(x)\,dx\,.$$

Being

$$\hat{P}^k(1) = \frac{x^k}{k!}\,,$$

the above expression, using (3), becomes

$$f(x) = f(a) - \sum_{k=1}^{n}(-1)^k\left[\frac{x^k}{k!}\hat{D}^k f(x)\right]_a^x + (-1)^n\int_a^x \frac{x^n}{n!}\hat{D}^{n+1}f(x)\,dx$$

and this is an identity $\forall n \in \mathbb{N}^+$, equation (1). Taking the limit for $n \to +\infty$ we have

$$f(x) = f(a) - \sum_{k=1}^{\infty}(-1)^k\left[\frac{x^k}{k!}\hat{D}^k f(x)\right]_a^x + \lim_{n\to+\infty}\mathcal{R}_n(x)\,,$$

where

$$\mathcal{R}_n(x) := \frac{(-1)^n}{n!}\int_a^x x^n\hat{D}^{n+1}f(x)\,dx\,.$$

In many cases we have

$$\lim_{n\to+\infty} \mathcal{R}_n(x) = 0 \tag{4}$$

and, under this condition, we can write the series expansion

$$f(x) = f(a) - \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \left(x^k \frac{d^k f(x)}{dx^k} - a^k \frac{d^k f(x)}{dx^k}\Big|_{x=a} \right) .$$

Putting $a = 0$, naturally if it is possible seen the domain $\mathcal{D}$, we obtain the series

$$f(x) = f(0) - \sum_{k=1}^{\infty} \frac{(-x)^k}{k!} \frac{d^k f(x)}{dx^k} ,$$

that is equation (2). In general for the condition (4) we can write

$$\lim_{n\to+\infty} |\mathcal{R}_n(x)| \leq \lim_{n\to+\infty} \frac{1}{n!} \left| \int_a^x \left| x^n \hat{D}^{n+1} f(x) \right| dx \right| . \tag{5}$$

Suppose that $x \in [-b, b]$, with $b \in \mathbb{R}^+$, so if

$$\lim_{n\to+\infty} \frac{1}{\sqrt{n}} \left(\frac{b \cdot e}{n} \right)^n \left| \frac{d^{n+1} f(x)}{dx^{n+1}} \right| = 0$$

uniformly, we have

$$\lim_{n\to+\infty} \mathcal{R}_n(x) = 0 .$$

In fact from (5) we can write, using Stirling formula [1] and the uniform limit condition above,

$$\begin{aligned} \lim_{n\to+\infty} |\mathcal{R}_n(x)| &\leq \left| \lim_{n\to+\infty} \int_a^x \frac{e^n}{n^n \sqrt{2\pi n}} \left| x^n \hat{D}^{n+1} f(x) \right| dx \right| \\ &\leq \left| \int_a^x \lim_{n\to+\infty} \left(\frac{b^n e^n}{n^n \sqrt{2\pi n}} \left| \hat{D}^{n+1} f(x) \right| \right) dx \right| \\ &= 0 . \end{aligned}$$

This could be a usefull method to verify (4) for many functions, in particular we see that all functions that have limited derivative of all orders, like $\sin(x)$, satisfy this condition, so for them it is possible to write (2).

Example

For example we can derive the well known series expansion

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!} . \tag{6}$$

Let be

$$f(x) = e^x \,,$$

and $a = 0$, for all $x \in [-b, b]$, with $b \in \mathbb{R}^+$ we can verify that

$$\lim_{n\to+\infty} \mathcal{R}_n(x) = \lim_{n\to+\infty} \frac{(-1)^n}{n!} \int_0^x x^n e^x \, dx = 0 \,,$$

in fact

$$|\mathcal{R}_n(x)| \leq \frac{|x|}{n!} \max_{x\in[-b,b]} |x^n e^x| \leq \frac{b^{n+1} e^b}{n!}$$

and, taking the limit for $n \to \infty$,

$$0 \leq \lim_{n\to+\infty} |\mathcal{R}_n(x)| \leq \lim_{n\to+\infty} \frac{b^{n+1} e^b}{n!} = 0 \,,$$

independently by x. So we apply the equation (2), hence

$$e^x = 1 - e^x \sum_{k=1}^{\infty} \frac{(-x)^k}{k!} \,.$$

from wich

$$\sum_{k=1}^{\infty} \frac{(-x)^k}{k!} = \frac{1 - e^x}{e^x} = e^{-x} - 1 \,,$$

that is (6) with the change $x \to -x$.

References

[1] D. Freedman P. Diaconis. An elementary proof of stirling's formula. *Amer. Math. Monthly*, 93:123–125, 1986.