# Light Cone Gauge Quantization of Strings, Dynamics of D-brane and String dualities.

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## Abstract

This review aims to show the Light cone gauge quantization of strings. It is divided up into three parts. The first consists of an introduction to bosonic and superstring theories and a brief discussion of Type II superstring theories. The second part deals with different configurations of D-branes, their charges and tachyon condensation. The third part contains the compactification of an extra dimension, the dual picture of D-branes having electric as well as magnetic field and the different dualities in string theories. In ten dimensions, there exist five consistent string theories and in eleven dimensions there is a unique M-Theory under these dualities, the different superstring theoies are the same underlying M-Theory.

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# INTRODUCTION

General relativity and quantum mechanics were the two major breakthroughs that revolutionized theoretical physics in the twentieth century. General relativity gives the idea to understand of the large-scale expansion of the Universe and gives a small correction to the predictions of Newtonian gravity for the motion of planets and the deflection of light rays, and it predicts the existence of gravitational radiation and black holes. It describes the gravitational force in terms of the curvature of spacetime which has fundamentally changed our view of space and time i.e. they are now viewed as dynamical [1].

Quantum mechanics, on the other hand, is the essential tool for understanding the sub atomic particles and microscopic physics. The evidence continues to build that it is an exact property of Nature [2]. The fundamental law of Nature is surely incomplete until general relativity and quantum mechanics are successfully reconciled and unified. String theory is a candidate which resolves this problem and a straightforward attempt to combine the General relativity and Quantum mechanics. String theory is based on the idea that particles are not point-like, but rather tiny loops (i.e. closed strings) or (open) pieces of string i.e. the 0-dimensional point particle is replaced by a 1-dimensional string [3]. This assumption leads to some features which are

# General features

Even though string theory is not yet fully formulated, and we cannot yet understand the detailed description of how the standard model of elementary particles should emerge at low energies, or how the Universe originated, there are some general features of the theory that have been well understood.

## Vibrating string

String theory predicts that all objects in our universe are composed of Vibrating strings and different vibrational modes of the strings represent different kinds of particles. Since there is just one type of string, and all particles arise from string vibrations, all particles are naturally incorporated into a single theory [3].

## Gravity

String theory attempts to reconcile the General relativity and Quantum mechanics. One of the vibrational modes of strings is the graviton particle, the quantum version of the gravity, so string theory has the remarkable property of predicting gravity [3].

## Unification of forces

There are four fundamental forces that had been recognized to exist in nature.

1. Electromagnetic force
2. Weak force
3. Strong force
4. Gravitational force

As the quantum version of electromagnetism describes the photon (a massless particle) and its interactions with charged particles, while the Yang-Mills theory describes W and Z bosons and gluons (the mediators of the weak and the strong nuclear forces) and their interactions. All of these theories make a single theory named the Standard Model of particle interactions, which is a gauge theory. The Standard Model of particle physics does not include the graviton particle and its interactions. The graviton which has spin 2 can not be described by the gauge theory. Since the standard model is believed to incomplete due to it does not incorporate the gravity forces. String theory is currently the most promising candidate to unify all the fundamental forces. This is a general feature of the string theory [3].

## Yang-Mills gauge theory

Standard model of particle physics describe the elementary particles in nature. It reconciles the special relativity and Quantum mechanics [4]. And is based on Yang-Mills theory having the gauge group

$$SU(3) \times SU(2) \times U(1)$$

However it has some shortcomings. It does not include Gravity and it has about 20 parameters that cannot be calculated and we use them as an input. While string theory predicts the gravity as well as describe all the elementary particles and has one parameter, the string length. Its value is roughly equal to the typical size of strings. Yang-Mills gauge theories arise very naturally in string theory. However, it is not yet fully understood why the gauge group

$$SU(3) \times SU(2) \times U(1)$$

Of the standard model with three generations of quarks and leptons should be singled out in nature [5, 6].

## Supersymmetry

A supersymmetry is a symmetry which relates bosons and fermions. There exists a non supersymmetric bosonic string theory which is an unrealistic theory due to the lack of fermions. In the order to get a realistic string theory which explains the beauty of nature, we need a supersymmetry. Hence supersymmetry is a general feature of string theory [3].

## Extra dimensions of space

In quantum field theory, for point particle, we let the dimensions of the spacetime to be four while superstring theories predict some additional dimensions of the spacetime. The superstring theories are only able to work in a ten dimensions or eleventh (in some cases) dimensions of spacetime. To make an ordinary four dimensional space time, there is a straight forward possibility that is, the additional six or seven dimensions can be curled up and compactified on an internal manifold having the sufficient small size, which can not be detectable at the low energies. The idea of an extra dimension was first introduce by Kaluza and Klein in 1920s. Their aim was to unify the electromagnetic force and the gravitational force. The compactification of an extra dimension can be imagined as, let us consider we have a cylinder having the radius $R$. when the cylinder is viewed from a very large distance or equivalently, when the radius of the cylinder R becomes too small then the two dimensional cylinder will look like a one dimensional line.

Generalizing this idea by letting the cylinder as a four dimensional spacetime and replacing the short circle of radius $R$ (compact space) by a six or seven dimensional manifold, hence at large distance or at the low energies, the additional dimensions (compact manifold) can not be visible. These additional dimensions or compact manifold are called Calabi-Yau manifolds [3].

## The size of the strings

Quantum field theory deals the particles as a mathematical zero dimensional point while in string theory the ordinary point particles are replaced by a one dimensional string. These one dimensional strings will have a characteristic length scale which is denoted be and can estimated by the dimensionality analysis. As string theory is a relativistic quantum theory which also includes the force of gravity, since it must involves the fundamental constants speed of light $c$ Planck's constant $\hbar$, and the gravitational constant $G$ From these, we can form a length, called the Planck length

$$\boxed{l_s = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.6 \times 10^{-33}\text{cm}},$$

The Planck mass becomes

$$\boxed{m_p = \left(\frac{\hbar c}{G}\right)^{1/2} = 2.176 \times 10^{-5}\text{g}},$$

And similarly the Planck time

$$\boxed{t_p = \left(\frac{\hbar G}{c^5}\right)^{1/2} = 5.391 \times 10^{-44}\text{s}}.$$

The Planck length scale is a natural guess for a fundamental string length scale as well as the characteristic size of compact extra spatial dimensions. At low energies string can be approximated by the point particle that explains why the quantum field theory has been successfully in the describing our world. The relativistic quantum gravity effects can be important on the above three scales of planks length, planks mass and planks time [3].

# BOSONIC STRING THEORY

The bosonic string theory is the simplest string theory that predicts and describes only a certain set of boson. As the theory does not describe any fermions, so it is an unrealistic theory but this theory is a natural place to start, because the same techniques and structures, together with some additional terms are required for analysis of more realistic theory (super string theories).
String can be regarded as 1-brane moving through a space-time which is a special case of a p-brane, p-dimensional extended object. Point particle corresponds to 0-brane. Similarly the two dimensional extended object or 2-brane are called membranes.

## 2.1 The relativistic string action

In quantum field theory, the action for point particle (0-brane) is proportional to the invariant length of the word-line or particle trajectory. Similarly when we replace the point particle by 1 dimensional string (1-brane) then the action is proportional to the proper area of the word-sheet or the area swept out by the string in D dimensional space-time.

$$S_{NG} = -T \int dA \tag{2.1}$$

The word-sheet is parameterized by the two coordinates $\xi^0 = \tau$ which is time-like and $\xi^1 = \sigma$ which is space-like. For close string the $\sigma$ will be periodic while for open string $\sigma$ will be have some finite value [3].

### 2.1.1 The Nambu-Goto string action

As one-dimensional string having[1] $X^{\mu}(\tau,\sigma)$ space-time coordinates tracing out a two dimensional word-sheet. In the order to calculate the area of word-sheet, we will need a metric. Let $\xi^{i}(i = 0,1)$ denote the word-sheet coordinates and take the metric $g_{\mu\nu}$ having the signature $(-,+,+,......,+)$ with $\mu,\nu = 0,1,....,D-1$ , describe the background geometry in which the string propagates then

$$-ds^2 = g_{\mu\nu}dX^{\mu}dX^{\nu} = g_{\mu\nu}\frac{\partial X^{\mu}}{\partial\xi^{i}}\frac{\partial X^{\nu}}{\partial\xi^{j}}d\xi^{i}d\xi^{j} \equiv g_{ij}d\xi^{i}d\xi^{j} \tag{2.2}$$

Here $\xi^0 = \tau$,$\xi^1 = \sigma$ and the minus sign is used with $ds^2$ is for having real time-like trajectory. In Minkowski flat space-time $g_{\mu\nu} = \eta_{\mu\nu}$. The Nambu-Goto action then takes the form

$$S_{NG} = -T\int\sqrt{-\det g_{ij}}d^2\xi = -T_{\circ}\int\sqrt{(\dot{X}.X')^2 - (\dot{X}^2)(X'^2)}d\tau d\sigma \tag{2.3}$$

Here $\dot{X}^{\mu} = \frac{\partial X^{\mu}}{\partial\tau}$, $X'^{\mu} = \frac{\partial X^{\mu}}{\partial\sigma}$ and by the dimensional analysis $T = \frac{T_o}{c}$ , $T_o$ is the string tension and we will use natural units in which speed of light is 1.

### 2.1.2 Equation of motions, boundary conditions and D-branes

Let us start from the Nambu-Goto string action which is in the form of lagrangian density

$$S = \int_{\tau_i}^{\tau_f} d\tau L = \int_{\tau_i}^{\tau_f} d\tau \int_{0}^{\sigma_1} d\sigma L(\dot{X}^{\mu}, X'^{\mu}) \tag{2.4}$$

Where $L$ is given as

$$L(\dot{X}^{\mu}, X'^{\mu}) = -T_{\circ}\sqrt{(\dot{X}.X') - (\dot{X})^2(X')^2} \tag{2.5}$$

The equations for the motion of string can be obtained by the action principle.

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_{0}^{\sigma_1} d\sigma \left[\frac{\partial}{\partial\tau}\left(\delta X^{\mu}P^{\tau}_{\mu}\right) + \frac{\partial}{\partial\sigma}\left(\delta X^{\mu}P^{\sigma}_{\mu}\right)\right] - \delta X^{\mu}\left(\frac{\partial P^{\tau}_{\mu}}{\partial\tau} + \frac{\partial P^{\sigma}_{\mu}}{\partial\sigma}\right) \tag{2.6}$$

[1]As we will use the $X^{\mu}$ for string coordinate while for general space-time we will called $x^{\mu}$

Where

$$P^{\tau}_{\mu} \equiv \frac{\delta L}{\delta \dot{X}^{\mu}} = -T_{\circ}\frac{(\dot{X}.X')X'_{\mu} - (X')^2\dot{X}_{\mu}}{\sqrt{(\dot{X}.X')^2 - (\dot{X})^2(X')^2}} \tag{2.7}$$

And

$$P^{\sigma}_{\mu} \equiv \frac{\delta L}{\delta X'^{\mu}} = -T_{\circ}\frac{(\dot{X}.X')\dot{X}_{\mu} - (X')^2 X'_{\mu}}{\sqrt{(\dot{X}.X')^2 - (\dot{X})^2(X')^2}} \tag{2.8}$$

For $\delta S = 0$ we get some conditions (boundary conditions) and equations of motion, which are, the equations of motion for relativistic string[2]. (closed or open) are.

$$\boxed{\frac{\partial P^{\tau}_{\mu}}{\partial \tau} + \frac{\partial P^{\sigma}_{\mu}}{\partial \sigma} = 0} \tag{2.9}$$

And the boundary conditions are [3]

A) The boundary condition for closed string is, as for the closed string the word-sheet is like a tube (cylinder type), so their boundary conditions are

$$X^{\mu}(\tau, \sigma + 2\pi) = X^{\mu}(\tau, \sigma) \tag{2.10}$$

B) While the word-sheet of open string is like a sheet, there are two types of boundary conditions for open string

i) Neumann conditions

$$\mathcal{P}^{\sigma}_{\mu}(\tau, \sigma_*) = 0 \tag{2.11}$$

ii) Dirichlet conditions

$$\frac{\partial X^{\mu}}{\partial \tau}(\tau, \sigma_*) = 0, \mu \neq 0 \tag{2.12}$$

Here $\sigma_*$ are the end points of the open string. No momentum flows off the ends of the String by implying the Neumann conditions while the endpoints are fixed in spacetime by implying the Dirichlet conditions. When the end points are fixed in space-time this means that the string is attached with some physical object, which is called D-brane. While when the Dirichlet conditions are applied to a subset of the $d$ indices (spatial

[2]For closed string $\sigma \sim [0, 2\pi]$. While for open string $\sigma \sim [0, \pi]$. The length of string is $\sigma_1$

[3]These are the momentum densities, $P^{\tau}_{\mu}, P^{\sigma}_{\mu}$. While $P^{\tau}_{\mu}$ is called momentum conjugate.

dimensions) of $X^\mu$ , i.e. to the indices $p+1, \ldots, d$, this mean that the endpoints of the string are restricted to move on a $p$-dimensional hyper-plane. This higher dimensional object is called a Dp-brane, where $p$ indicates its dimension and $D$ stands for Dirichlet. It turns out that these objects should be considered to be dynamical as well [3, 7, 8, 9, 10].

## 2.2 Constraints and wave equations

As the word-sheet have the two parameters, $(\tau, \sigma)$ by reparameterization (gauge) of $\tau$ and $\sigma$ of the word-sheet, simply using a class of gauges [4] which fix the parameterization ($\tau$ and $\sigma$) of the word-sheet [11] and give the some constraints equations which are

$$\dot{X}.X' = 0,\ \dot{X}^2 + X'^2 = 0 \tag{2.13}$$

Both these constraints equations combine to give

$$(\dot{X} \pm X')^2 = 0 \tag{2.14}$$

Using these two constraints equation to simplify the momentum densities $P^{\tau\mu}$ and $P^{\sigma\mu}$ which give[5]

$$P^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu,\ P^{\sigma\mu} = -\frac{1}{2\pi\alpha'}X'^\mu \tag{2.15}$$

Putting these momentum densities in the equation of motion which is $\partial_\tau P^{\tau\mu} + \partial_\sigma P^{\sigma\mu} = 0$, we get

$$\ddot{X}^\mu - X''^\mu = 0 \tag{2.16}$$

So by reparameterization we got the equation of motion just as a wave equation.

[4] Reparameterization of word-sheet In static gauge ($t = \tau$), our solution is not fully explicit so we use the more general gauges. A class of choices for $\tau$ is $n.X(\tau, \sigma) = \beta\alpha'(n.p)\tau$ , Where $\beta$ is equal to two for open string and one for closed string and "$n$" is chosen in such a way that $(n.p)$ is conserved. And the conservation of $(n.p)$ means $n.P^\sigma = 0$ for both closed as well as open strings And similarly the associated $\sigma$ parameterization gives $n.p = \frac{2\pi}{\beta} n.P^\tau$

[5] The string tension $T_\circ$ is equal to $T_\circ = \frac{1}{2\pi\alpha'}$, and $\alpha'$ is the Regge-slope parameter.

## 2.3 Open string Mode expansions

Let we have a space-filling D-brane for this all the string coordinates $X^\mu$ satisfy the boundary condition (Neumann) at the end points. And as the solution of a wave equation can be written in the form of superposition of waves moving to the left and right on the string.

$$X^\mu(\tau,\sigma) = \frac{1}{2}\left(f^\mu(\tau+\sigma) \ + \ g^\mu(\tau-\sigma)\right) \tag{2.17}$$

Where $f^\mu$ and $g^\mu$ are the arbitrary functions. As the Neumann boundary conditions $P^{\sigma\mu} = 0$ at the endpoints

$$\tfrac{\partial X^\mu}{\partial \sigma} = 0 \text{ , at } \sigma = 0, \pi$$

The Neumann boundary conditions at $\sigma = 0$ give

$$\frac{\partial X^\mu}{\partial \sigma}(\tau, 0) = \frac{1}{2}\left(f'^\mu(\tau) - g'^\mu(\tau)\right) = 0 \tag{2.18}$$

This equation means that the derivative of $f^\mu$ and $g^\mu$ are same then these two functions are differ by some constant i.e. $g^\mu = f^\mu + c^\mu$, so replacing this and absorbing the constant into the definition of $f^\mu$. This give

$$X^\mu(\tau,\sigma) = \frac{1}{2}\left(f^\mu(\tau+\sigma) \ + \ f^\mu(\tau-\sigma)\right) \tag{2.19}$$

Now consider boundary conditions at $\sigma = \pi$

$$\frac{\partial X^\mu}{\partial \sigma}(\tau, \pi) = \frac{1}{2}\left(f'^\mu(\tau+\pi) - f'^\mu(\tau-\pi)\right) = 0 \tag{2.20}$$

Since from this equation we see that $f'^\mu$ is a periodic with period of $2\pi$. so we can write it in term of Fourier series

$$f'^\mu(u) = f_1^\mu + \sum_{n=1}^{\infty}\left(a_n^\mu \cos nu + b_n^\mu \sin nu\right) \tag{2.21}$$

Now integrating this equation and then putting the result $f^\mu(u)$ back into equation (2.17) and simplifying, we get

$$X^\mu(\tau,\sigma) = f_0^\mu + f_1^\mu \tau + \sum_{n=1}^{\infty} (A_n^\mu \cos n\tau + B_n^\mu \sin n\tau) \cos n\sigma \tag{2.22}$$

Now replace the coefficients in above equation (2.22) by new coefficients which have some simple physical interpretation.

$$A_n^\mu \cos n\tau + B_n^\mu \sin n\tau = -i\frac{\sqrt{2\alpha'}}{\sqrt{n}} \left(a_n^{\mu *} e^{in\tau} - a_n^\mu e^{-in\tau}\right) \tag{2.23}$$

Here * denote the complex conjugate and $\sqrt{2\alpha'}$ factor is to make the $a_n^\mu$ dimensionless and the physical interpretation of these new constant and their conjugate is, they become annihilation and creation operators when considering Quantum field theory Similarly $f_1^\mu$ in equation (2.22) has also a simple physical interpretation which is as from the momentum conjugate equation

$$P^\mu = \int_0^\pi P^{\tau\mu} d\sigma = \frac{1}{2\pi\alpha'} \int_0^\pi \dot{X}^\mu d\sigma = \frac{1}{2\alpha'} f_1^\mu \tag{2.24}$$

This equation tells us that the quantity $f_1^\mu$ is proportional to the space-time momentum which is carried by the string. From above equation (2.24), putting the value of $f_1^\mu$ with the new coefficients equation (2.23) into equation (2.22), then we gets[6]

$$X^\mu(\tau,\sigma) = x_0^\mu + 2\alpha' p^\mu \tau - i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \left(a_n^{\mu *} e^{in\tau} - a_n^\mu e^{-in\tau}\right) \frac{\cos n\sigma}{\sqrt{n}} \tag{2.25}$$

In the order to make the simple expression of above equation (2.25), we replace the constant $f_o^\mu = x_o^\mu$ and defining[7]

$$\alpha_0^\mu = \sqrt{2\alpha'} p^\mu \ , \ \alpha_n^\mu = a_n^\mu \sqrt{n} \text{ and } \alpha_{-n}^\mu = a_n^{\mu *} \sqrt{n} \ , \ n \geq 1$$

And also $\alpha_{-n}^\mu = (\alpha_n^\mu)^*$, using these new coefficients (modes) and simplifying by which we get

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'} \alpha_0^\mu \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{\mathrm{n}} \alpha_n^\mu e^{-in\tau} \cos n\sigma \tag{2.26}$$

[6] If all the coefficients (oscillators) $a_n^\mu$ vanishes, then the equation represents the point particle.

[7] As $a$ is oscillators while $\alpha$ is mode.

This is the solution of wave equation and a simple expression for the open string. Similarly we can get

$$\dot{X}^\mu \pm X'^\mu = \sqrt{2\alpha'} \sum_{n \in Z} \alpha_n^\mu e^{-in(\tau \pm \sigma)} \tag{2.27}$$

## 2.4 Closed string Mode expansions

Let us consider the general solution for the wave equation, which is

$$X^\mu(\tau, \sigma) = X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma) \tag{2.28}$$

Where $X_L^\mu$ is for left-moving wave while $X_R^\mu$ is for right-moving wave of the string. The word-sheet of the closed string is a cylinder, so to describe the closed string, we need some conditions, as the closed string have no endpoints but having the periodicity condition.

$$(\tau, \sigma) \sim (\tau, \sigma + 2\pi) \tag{2.29}$$

So by this periodicity, we can write

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi) \text{ for all } \tau \text{ and } \sigma \tag{2.30}$$

Now let introduce two variables

$$u = \tau + \sigma \text{ and } v = \tau - \sigma$$

By putting these new variable in equation (2.28), which becomes

$$X^\mu(\tau, \sigma) = X_L^\mu(u) + X_R^\mu(v) \tag{2.31}$$

Now by periodicity condition $(\tau, \sigma) \sim (\tau, \sigma + 2\pi)$, the above equation becomes

$$X_L^\mu(u + 2\pi) - X_L^\mu(u) = X_R^\mu(v) - X_R^\mu(v - 2\pi) \tag{2.32}$$

From this equation we can say that both $X_L^\mu(u)$ and $X_R^\mu(v)$ are periodic with the period of $2\pi$. Therefore we can write the mode expansions

$$X_L'^{\mu}(u) = \sqrt{\frac{\alpha'}{2}} \sum_{n \in Z} \bar{\alpha}_n^{\mu} e^{-inu} \tag{2.33}$$

$$X_R'^{\mu}(v) = \sqrt{\frac{\alpha'}{2}} \sum_{n \in Z} \alpha_n^{\mu} e^{-inv} \tag{2.34}$$

By integrating these equations, we get

$$X_L^{\mu}(u) = \frac{1}{2} x_0^{L\mu} + \sqrt{\frac{\alpha'}{2}} \bar{\alpha}_0^{\mu} u + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \bar{\alpha}_n^{\mu} e^{-inu} \tag{2.35}$$

$$X_R^{\mu}(v) = \frac{1}{2} x_0^{R\mu} + \sqrt{\frac{\alpha'}{2}} \bar{\alpha}_0^{\mu} v + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \bar{\alpha}_n^{\mu} e^{-inv} \tag{2.36}$$

Now putting these two equations (2.35) and (2.36) into equation (2.32) and solving by which we get

$$\bar{\alpha}_0^{\mu} = \alpha_0^{\mu} v \tag{2.37}$$

Now by using this equality and equations (2.35) and (2.36), we get the equation (2.31) as

$$X^{\mu}(\tau, \sigma) = \frac{1}{2}(x_0^{L\mu} + x_0^{R\mu}) + \sqrt{2\alpha'} \alpha_n^{\mu} + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} e^{-in\tau} (\alpha_n^{\mu} e^{in\sigma} + \bar{\alpha}_n^{\mu} e^{-in\sigma}) \tag{2.38}$$

As from the canonical momentum conjugate equation, we can write

$$P^{\mu} = \int_0^{2\pi} P^{\tau\mu} d\sigma = \frac{1}{2\pi\alpha'} \int_0^{2\pi} \dot{X}^{\mu} d\sigma = \sqrt{\frac{2}{\alpha'}} \alpha_0^{\mu} \tag{2.39}$$

$$\alpha_0^{\mu} = \sqrt{\frac{\alpha'}{2}} P^{\mu} \tag{2.40}$$

This equation (2.40) differs by a factor of two from the corresponding open string result and also tells the same idea i.e. the quantity $\alpha_0^{\mu}$ is proportional to the space-time momentum which is carried by the closed string.

And let $x_0^{L\mu}$ and $x_0^{R\mu}$ be equal i.e. having one conjugate coordinate zero mode, without any loss of generality,

$$x_0^{L\mu} = x_0^{R\mu} \equiv x_0^{\mu}$$

Finally using the above equation then equation (2.38) becomes

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}e^{-in\tau}(\alpha_n^\mu e^{in\sigma} + \bar{\alpha}_n^\mu e^{-in\sigma}) \tag{2.41}$$

Here we required these two relations for the reality of the above equation

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^*,\ \bar{\alpha}_{-n}^\mu = (\bar{\alpha}_n^\mu)^*$$

As when $\alpha_n^\mu = \bar{\alpha}_n^\mu$ then the above equation (for closed string) is equal to equation (2.26) which is for open string. Since the closed string can be viewed as a two copies of the open strings. This is a complete mode expansion for a closed string [3, 7].

## 2.5 Light cone solution and Transverse Virasoro Modes

The light cone solution is, to represents the motion of string by using the light cone coordinates and this impose a set of conditions which is called light cone gauge. The light cone gauge is one of the choices among the general gauges. As we know that the more general gauges

$$n.X(\tau,\sigma) = \beta\alpha'(n.p)\ \tau\ ,\ n.p = \frac{2\pi}{\beta}n.P^\tau \tag{2.42}$$

Where $\beta$ is equal to two for open string and one for closed string. In the order to select the light cone gauge, we need to impose the above conditions with a vector $n^\mu$ such that $n.X = X^+$ So

$$n^\mu = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0....,0\right) \tag{2.43}$$

$$n.X = \frac{X^0 + X^1}{\sqrt{2}} = X^+\ ,\ n.P = \frac{P^0 + P^1}{\sqrt{2}} = P^+ \tag{2.44}$$

Now using this pair of equations (2.44) into equation (2.42), we get

$$X^+(\tau,\sigma) = \beta\alpha' P^+\tau\ ,\ P^+ = \frac{2\pi}{\beta}P^{+\tau} \tag{2.45}$$

This choice of gauge is called the light cone gauge. Let the transverse coordinates be denoted as

$$X^l = \left(X^2, X^3, ...X^d\right)$$

Here $(l = 2, 3, ..., d)$. Now using the constraints equation (2.14) and expanding it into light cone coordinates.

$$(\dot{X} \pm X')^2 = 0$$

$$= (\dot{X} \pm X').(\dot{X} \pm X') = 0$$

$$= -2(\dot{X}^+ \pm X'^+).\,(\dot{X}^- \pm X'^-) + (\dot{X}^l \pm X'^l)^2 = 0 \tag{2.46}$$

Here we use the dot products of light cone coordinates.

Now using the equation (2.45) and calculating the equation (2.46), we get[8]

$$\dot{X}^- \pm X'^- = \frac{1}{\beta\alpha'}\frac{1}{2P^+}\,(\dot{X}^l \pm X'^l)^2 \tag{2.47}$$

So we have developed the relation between the light cone coordinates and the transverse coordinates. And the mode expansions for open string ($\beta = 2$) in the term of light cone coordinates are

$$X^l(\tau,\sigma) = x_0^l + \sqrt{2\alpha'}\alpha_0^l\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^l e^{-in\tau}\cos n\sigma \tag{2.48}$$

And as our gauge conditions give

$$X^+(\tau,\sigma) = 2\alpha' P^+\tau = \sqrt{2\alpha'}\alpha_0^+\tau \tag{2.49}$$

With position zero mode and oscillators for the $X^+$ are

$$x_0^+ = 0,\ \alpha_n^+ = \alpha_{-n}^+ = 0,\ n = 1, 2, ..., \infty$$

Similarly

[8]We assume that $P^+ \neq 0$, the vanishing of $P^+$ means that a massless particle moving in the negative $x^1$ direction. While $P^+$ certainly satisfied $P^+ \geq 0$.

$$X^-(\tau,\sigma) = x_0^- + \sqrt{2\alpha'}\alpha_0^-\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-in\tau}\cos n\sigma \tag{2.50}$$

These are the complete set of mode expansions in light cone coordinate [12]. Now using equation (2.27) with $\mu = -$ and $\mu = l$, then we get

$$\dot{X}^- \pm X'^- = \sqrt{2\alpha'}\sum_{n\in Z}\alpha_n^- e^{-in(\tau\pm\sigma)} \tag{2.51}$$

$$\dot{X}^l \pm X'^l = \sqrt{2\alpha'}\sum_{n\in Z}\alpha_n^l e^{-in(\tau\pm\sigma)} \tag{2.52}$$

Now putting these two equations (2.51) and (2.52) into equation (2.47), by simplifying we get

$$\alpha_n^- = \frac{1}{2P^+}\frac{1}{\sqrt{2\alpha'}}\sum_{p\in Z}\alpha_{n-p}^l\alpha_p^l \tag{2.53}$$

From above equation (2.53)[9] , we have the explicit expression for the minus oscillators $\alpha_n^-$ in the term of transverse oscillators $\alpha_n^l$. This represents the full solutions [3, 11]. From the above equation (2.53), the right side has given a special type name, which is the Transverse Virasoro mode $L_n^\perp$,

$$\sqrt{2\alpha'}\alpha_n^- = \frac{1}{P^+}L_n^\perp \;,\; L_n^\perp \equiv \frac{1}{2}\sum_{p\in Z}\alpha_{n-p}^l\alpha_p^l \tag{2.54}$$

Now by using the equation (2.53) and the equation which we defined earlier that is $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$ , then we get

$$\frac{1}{\alpha'}L_0^\perp = 2P^+P^- \tag{2.55}$$

Now using the value of $\alpha_n^-$ from equation (2.54) then equation (2.51) and (2.47) are written

$$\dot{X}^- \pm X'^- = \frac{1}{P^+}\sum_{n\in Z}L_n^\perp e^{-in(\tau\pm\sigma)} = \frac{1}{4\alpha' P^+}(\dot{X}^l \pm X'^l)^2 \tag{2.56}$$

[9]A critical string only have transverse excitations, just like for a massless particle only has transverse polarization states.

Similarly for closed string ($\beta = 1$) since the equation (2.47) becomes

$$\dot{X}^- \pm X'^- = \frac{1}{\alpha'}\frac{1}{2P^+}\,(\dot{X}^l \pm X'^l)^2 \tag{2.57}$$

And as like equation (2.56) we can also write for closed strings

$$(\dot{X}^l + X'^l)^2 = 4\alpha' \sum_{n\in Z}\left(\frac{1}{2}\sum_{p\in Z}\bar{\alpha}^l_p\bar{\alpha}^l_{n-p}\right)e^{-in(\tau+\sigma)} \equiv 4\alpha' \sum_{n\in Z}\bar{L}^\perp_n e^{-in(\tau+\sigma)} \tag{2.58}$$

$$(\dot{X}^l - X'^l)^2 = 4\alpha' \sum_{n\in Z}\left(\frac{1}{2}\sum_{p\in Z}\alpha^l_p\alpha^l_{n-p}\right)e^{-in(\tau-\sigma)} \equiv 4\alpha' \sum_{n\in Z}L^\perp_n e^{-in(\tau-\sigma)} \tag{2.59}$$

So we define the Verasoro mode for closed string

$$\bar{L}^\perp_n = \frac{1}{2}\sum_{p\in Z}\bar{\alpha}^l_p\bar{\alpha}^l_{n-p}\ ,\ L^\perp_n = \frac{1}{2}\sum_{p\in Z}\alpha^l_p\alpha^l_{n-p} \tag{2.60}$$

Now using equations (2.58), (2.59) and equations (2.57), (2.41) we can easily finds

$$\sqrt{2\alpha'}\alpha^-_n = \frac{2}{P^+}L^\perp_n\ ,\ \sqrt{2\alpha'}\bar{\alpha}^-_n = \frac{2}{P^+}\bar{L}^\perp_n \tag{2.61}$$

From this equation (2.61) we cal also write

$$\bar{\alpha}^-_n = \frac{1}{2P^+}\frac{1}{\sqrt{2\alpha'}}\sum_{p\in Z}\bar{\alpha}^l_{n-p}\bar{\alpha}^l_p$$

And for $n = 0$ then $\bar{\alpha}^-_0 = \bar{\alpha}^-_0$ for equation (2.61), then, we have

$$\bar{L}^\perp_0 = L^\perp_0 \tag{2.62}$$

This equality is called level-matching and it means that one in the terms of right-moving oscillators and one in terms of left-moving oscillators are equal as level matching.

In the quantum theory level matching implies that the states of right-moving excitations of a string are equal to the states of its left-moving excitations [12]. We will explain this later.

## 2.6 Quantization and Commutations relations

In the order to quantize our classical results we need to replace the Poisson brackets by commutation relations and the fields by operators and then will solve the Virasoro constraint equations, and will describe our theory in a Fock space that describes the physical degrees of freedom [13]. Before we have written the classical equations of motion in the light cone gauge. Now we will use (to quantization) the results of our light cone analysis of the classical relativistic strings. Our Hamiltonian for open strings, as we have $X^+ = 2\alpha' p^+ \tau$ so from this we can write $\partial_\tau = 2\alpha' p^+ \partial_{X^+}$ and in addition, as $p^-$ generates $X^+$ translations so from this we can write Hamiltonian as

$$H = 2\alpha' p^+ p^-$$

For quantization and commutation relations, we will start from a set of operators.

$$(X^l(\tau, \sigma)\ ,\ x_0^-\ ,\ P^{\tau l}(\tau, \sigma)\ ,\ P^+(\tau)) \tag{2.63}$$

From this, we have the full set of basic operators of string theory because the above list has the collection of all zero modes plus the infinite set of the annihilation and creation operators [3]. First we will deal the open string and assume the space-filling D-brane. Let postulate some commutation relations from the above set of operators

$$\left[X^l(\tau, \sigma), P^{\tau j}(\tau, \sigma')\right] = i\eta^{lj}\delta(\sigma - \sigma') \tag{2.64}$$

And

$$\left[x_0^-, P^+\right] = -i \tag{2.65}$$

And all the other commutation relations are equal to zero i.e.

$$\left[X^l(\tau, \sigma), X^j(\tau, \sigma')\right] = 0, \left[P^{\tau l}(\tau, \sigma), P^{\tau j}(\tau, \sigma')\right] = 0$$

$$\left[x_0^-, X^l(\tau, \sigma)\right] = 0, \left[x_0^-, P^{\tau l}(\tau, \sigma)\right] = 0$$

$$\left[P^+, X^l(\tau, \sigma)\right] = 0\ ,\ \left[P^+, P^{\tau l}(\tau, \sigma)\right] = 0 \tag{2.66}$$

From the commutation relation in equation (2.64) we can also write as

$$\left[X^l(\tau,\sigma), \dot{X}^j(\tau,\sigma')\right] = i2\pi\alpha'\eta^{lj}\delta(\sigma-\sigma') \tag{2.67}$$

Similarly the other commutation relations from the above relations are

$$\left[X'^l(\tau,\sigma), \dot{X}^j(\tau,\sigma')\right] = i2\pi\alpha'\eta^{lj}\frac{d}{d\sigma}\delta(\sigma-\sigma') \tag{2.68}$$

$$\left[\dot{X}^l(\tau,\sigma), \dot{X}^j(\tau,\sigma')\right] = 0\ ,\ \left[X'^l(\tau,\sigma), X'^j(\tau,\sigma')\right] = 0 \tag{2.69}$$

Now by solving these equations we get

$$\left[(\dot{X}^l \pm X'^l)(\tau,\sigma), (\dot{X}^j \pm X'^j)(\tau,\sigma')\right] = \pm i4\pi\alpha'\eta^{lj}\frac{d}{d\sigma}\delta(\sigma-\sigma') \tag{2.70}$$

This commutation relation in equation (2.70) is useful to write down the commutation relations for the oscillators. The classical modes $\alpha_n^l$ will become the quantum operators with a nontrivial commutation relation. For commutation relation of oscillators, let recall the equation (2.27), we can write

$$(\dot{X}^l + X'^l)(\tau,\sigma) = \sqrt{2\alpha'}\sum_{n\in Z}\alpha_n^l e^{-in(\tau+\sigma)}\ ,\ \sigma\in[0,\pi] \tag{2.71}$$

$$(\dot{X}^l - X'^l)(\tau,-\sigma) = \sqrt{2\alpha'}\sum_{n\in Z}\alpha_n^l e^{-in(\tau+\sigma)}\ ,\ \sigma\in[-\pi,0] \tag{2.72}$$

Now let define an operator $A^l(\tau,\sigma)$

$$A^l(\tau,\sigma) \equiv \sqrt{2\alpha'}\sum_{n\in Z}\alpha_n^l e^{-in(\tau+\sigma)},\ A^l(\tau,\sigma+2\pi) = A^l(\tau,\sigma) \tag{2.73}$$

The periodicity is due to the definition of $A^l(\tau,\sigma)$ and from this definition we can write

$$A^l(\tau,\sigma) = \begin{cases} (\dot{X}^l + X'^l)(\tau,\sigma),\ \ \sigma\in[0,\pi] \\ (\dot{X}^l - X'^l)(\tau,-\sigma),\ \ \sigma\in[-\pi,0]. \end{cases} \tag{2.74}$$

So from this set of equation (2.74), there are four possibilities of commutation relations, we can summarized as

$$\left[A^l(\tau,\sigma), A^j(\tau,\sigma')\right] = 4\pi\alpha'\eta^{lj}\frac{d}{d\sigma}\delta(\sigma-\sigma')\ ,\ \sigma,\sigma'\in[-\pi,\pi] \tag{2.75}$$

Now solving this equation (2.75) by using equation (2.73), we find

$$\left[\alpha_m^l, \alpha_n^j\right] = m\ \eta^{lj}\delta_{m+n,0} \tag{2.76}$$

This shows the commutation relation between the $\alpha$ modes and also $\alpha_0^l$ commutes with others oscillators and these are equivalent to an infinite set of annihilation and creation operators. To see this let start from our defining oscillators i.e.

$$\alpha_n^\mu = a_n^\mu\sqrt{n}\ ,\ \alpha_{-n}^\mu = a_n^{\mu*}\sqrt{n}\ ,\ n\geq 1 \tag{2.77}$$

As both $(a,\alpha)$ are classical variables and now they become operators. As those classical variables which are complex conjugate of each other will now become the Hermitian operators in quantum theory i.e. the operators $x_0^l$ and $p^l$ are Hermitian as

$$(x_0^l)^\dagger = x_0^l\ ,\ (p^l)^\dagger = p^l \tag{2.78}$$

Similarly the above oscillators and modes are now operators and we will take these like

$$\alpha_n^l = a_n^l\sqrt{n} \text{ and } \alpha_{-n}^l = a_n^{l\dagger}\sqrt{n},\ n\geq 1$$

From this we have

$$(\alpha_n^l)^\dagger = \alpha_{-n}^l\ ,\ n\in Z \tag{2.79}$$

From these conditions we can replace equation (2.76) as

$$\left[\alpha_m^l, \alpha_{-n}^j\right] = m\ \eta^{lj}\delta_{m,n}\ ,\ \left[a_m^l, a_n^{j\dagger}\right] = \delta_{m,n}\eta^{lj} \tag{2.80}$$

And similarly

$$\left[\alpha_m^l, \alpha_n^j\right] = 0, \left[\alpha_m^{l\dagger}, \alpha_n^{j\dagger}\right] = 0$$

These commutation relations shows $(a_m^{l\dagger}, a_m^l)$ satisfy the commutation relations of canonical creation and annihilation operators of the quantum simple harmonic oscillator. So from this, we have $\alpha_n^l$ are annihilation operators while $\alpha_{-n}^l$ are creation operators for $n \geq 1$. Using these Hermiticity conditions (2.78) and (2.79) we can write

$$(X^l(\tau,\sigma))^\dagger = X^l(\tau,\sigma)$$

Now to find the commutation relation between $\alpha_0^l$ and $x_n^l$, for this as $\alpha_0^l = \sqrt{2\alpha'}P^l$ and using equation (2.64) and by simplification we get

$$\left[x_0^l, P^j\right] = i\ \eta^{lj} \tag{2.81}$$

By quantization, replacing the classical variables to operators, our mode expansion then become

$$X^l(\tau,\sigma) = x_0^l + 2\alpha' p^l \tau + i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left(a_n^\mu e^{-in\tau} - a_n^{l\dagger} e^{in\tau}\right)\frac{\cos n\sigma}{\sqrt{n}} \tag{2.82}$$

Here replace the $\alpha$ modes by corresponding oscillators. And this is in the term of annihilation and creation operators, expansion of the coordinate operator. Similarly now quantize the closed string, the Poisson brackets will be replaced by commutation relations and the classical variables by quantum operators. For commutation relations, as the operators content of closed strings can be treated as the two commuting copies the open string operators i.e. left moving and right moving. Similarly we can use the almost same techniques to derive the commutation relations for closed strings. The commutations relations for closed strings are

$$\left[\bar{\alpha}_m^l, \bar{\alpha}_n^j\right] = m\ \eta^{lj}\delta_{m+n,0},\ \left[\bar{\alpha}_m^l, \bar{\alpha}_n^j\right] = m\ \eta^{lj}\delta_{m+n,0}$$

$$\left[\bar{\alpha}_m^l, \alpha_n^j\right] = 0,\ \left[\bar{\alpha}_m^l, \bar{\alpha}_n^{j\dagger}\right] = \delta_{m,n}\eta^{lj}$$

$$\left[x_0^l, p^l\right] = i\eta^{lj} \rightarrow \left[x_0^l, \alpha_0^j\right] = i\eta^{lj}\sqrt{\frac{\alpha'}{2}},\ \left[x_0^l, \bar{\alpha}_0^j\right] = i\eta^{lj}\sqrt{\frac{\alpha'}{2}} \tag{2.83}$$

Our Hamiltonian for closed strings in light cone coordinates is, as we know from our light cone gauge, as $p^-$ generates $X^+$ translation and also we have $X^+ = \alpha' p^+ \tau$ so from this we can write $\partial_\tau = \alpha' p^+ \partial_{X^+}$ , so our Hamiltonian will be

$$H = \alpha' p^+ p^-$$

There is only the factor of two in the open strings and closed strings Hamiltonian due to the value of $\beta$, which is two for open string while one for closed string [3, 14].

## 2.7 Transverse Verasoro operators

As before we have write down the classical solution of motion of both open and closed strings in light cone coordinates and then we solved for $X^-$ in the term of transverse coordinates by using the constraints equation (2.47) i.e. $\alpha_n^-$ the in the term of $\alpha_n^l$ modes as shown (for open string) in equation (2.53) so

$$\sqrt{2\alpha'}\alpha_n^- = \frac{1}{P^+} L_n^\perp,\ L_n^\perp \equiv \frac{1}{2}\sum_{p\in Z} \alpha_{n-p}^l \alpha_p^l \tag{2.84}$$

Where $l$ (repeated index) is summed over transverse light cone direction. As before $L_n^\perp$ were transverse Virasoro modes but now it is called transverse Virasoro operators due to the modes become operators. As the two $\alpha$ operators $(in L_n^\perp)$ fail to commute for $n = 0$, so $L_0^\perp$ is the only operator which needs a Normal Ordering. From the definition of $L_0^\perp$ as

$$L_0^\perp = \frac{1}{2}\sum_{p\in Z} \alpha_{-p}^l \alpha_p^l \tag{2.85}$$

$$= \frac{1}{2}\alpha_0^l\alpha_0^l + \frac{1}{2}\sum_{p=1}^{\infty} \alpha_{-p}^l \alpha_p^l + \frac{1}{2}\sum_{p=1}^{\infty} \alpha_p^l \alpha_{-p}^l$$

Since the last term is not Normal Ordered, taking the last term and making it as a normal order by using the commutation relation, such that

$$\frac{1}{2}\sum_{p\,=\,1}^{\infty} \alpha_p^l \alpha_{-p}^l = \frac{1}{2}\sum_{p\,=\,1}^{\infty} \alpha_{-p}^l \alpha_p^l + \frac{1}{2}(D-2)\sum_{p\,=\,1}^{\infty} p$$

So $L_0^\perp$ then become

$$L_0^\perp = \frac{1}{2}\alpha_0^l\alpha_0^l + \sum_{p=1}^{\infty}\alpha_{-p}^l\alpha_p^l + \frac{1}{2}(D-2)\sum_{p=1}^{\infty} p \tag{2.86}$$

And also from the definition of $L_0^\perp$ (with normal ordering constant a ) in the term of $p^-$ is as

$$2\alpha' P^- \equiv \frac{1}{p^+}\left(L_n^\perp + a\right) \tag{2.87}$$

From equation (2.86) and (2.87), we can say that

$$a = \frac{1}{2}(D-2)\sum_{p\,=\,1}^{\infty} p \tag{2.88}$$

By Zeta function we find the value of $\sum\limits_{p\,=\,1}^{\infty} p$ and so

$$a = -\frac{1}{24}(D-2) \tag{2.89}$$

The Normal order $L_0^\perp$ is then become

$$L_0^\perp = \alpha' p^l p^l + \sum_{p\,=\,1}^{\infty}\alpha_{-p}^l\alpha_p^l - \frac{1}{24}(D-2) \tag{2.90}$$

This is all about the open string transverse Virasoro operators. Similarly for closed strings, the transverse Virasoro operators are

$$\begin{aligned} \sqrt{2\alpha'}\alpha_n^- &= \frac{1}{P^+}L_n^\perp,\; L_n^\perp \equiv \frac{1}{2}\sum_{p\in Z}\alpha_{n-p}^l\alpha_p^l \\ \sqrt{2\alpha'}\bar{\alpha}_n^- &= \frac{1}{P^+}\bar{L}_n^\perp,\; \bar{L}_n^\perp \equiv \frac{1}{2}\sum_{p\in Z}\bar{\alpha}_{n-p}^l\bar{\alpha}_p^l \end{aligned} \tag{2.91}$$

With $(L_0^\perp)^\dagger = L_0^\perp$, $(L_n^\perp)^\dagger = L_{-n}^\perp$ and $(\bar{L}_n^\perp)^\dagger = \bar{L}_{-n}^\perp$ due to $(\alpha_n^l)^\dagger = \alpha_{-n}^l$ and $(\bar{\alpha}_n^l)^\dagger = \bar{\alpha}_{-n}^l$. Similarly we have $(\alpha_n^-)^\dagger = \alpha_{-n}^-$. As we shown that $\alpha_m^l$ and $\alpha_n^l$ commutes only when $(m+n)$ equal to zero but the case is totally change for $\alpha_n^-$ i.e. two Virasoro operators $L_m^\perp$ and $L_n^\perp$ never commute for $(m \neq n)$ . Now to find the commutation relations for

Virasoro operators lets begin from the commutation relation of Virasoro operator and oscillator, which is

$$\left[L_m^\perp, \alpha_n^j\right] = \frac{1}{2}\sum_{p\in Z}\left[\alpha_{n-p}^l\alpha_p^l, \alpha_n^j\right]$$

$$= \frac{1}{2}(-n\alpha_{m+n}^j - n\alpha_{m+n}^j)$$

So we find

$$\left[L_m^\perp, \alpha_n^j\right] = -n\alpha_{m+n}^j \tag{2.92}$$

This commutation relation is also valid for $m = 0$ . Similarly

$$\left[L_m^\perp, x_0^l\right] = -i\sqrt{2\alpha'}\alpha_m^l \tag{2.93}$$

Now let us consider the two Virasoro operators $L_m^\perp$ and $L_n^\perp$, and the commutation relation between them is not quit easy. So let the sum split as

$$L_m^\perp = \frac{1}{2}\sum_{k\geq 0}\alpha_{m-k}^l\alpha_k^l + \frac{1}{2}\sum_{k<0}\alpha_k^l\alpha_{m-k}^l$$

We made the above $L_m^\perp$ as normal ordered for any value of $m$. So we can now evaluate the commutation relation like

$$\left[L_m^\perp, L_n^\perp\right] = \frac{1}{2}\sum_{k\geq 0}\left[\alpha_{m-k}^l\alpha_k^l, L_n^\perp\right] + \frac{1}{2}\sum_{k<0}\left[\alpha_k^l\alpha_{m-k}^l, L_n^\perp\right]$$

Now evaluating then we get

$$\begin{aligned}\left[L_m^\perp, L_n^\perp\right] = {} & \frac{1}{2}\sum_{k\geq 0}(m-k)\alpha_{m+n-k}^l\alpha_k^l + \frac{1}{2}\sum_{k<0}(m-k)\alpha_k^l\alpha_{m+n-k}^l \\ & + \frac{1}{2}\sum_{k\geq 0}k\alpha_{m-k}^l\alpha_{k+n}^l + \frac{1}{2}\sum_{k<0}k\alpha_{k+n}^l\alpha_{m-k}^l\end{aligned} \tag{2.94}$$

Here we have now deal with two different cases $m+n \neq 0$, $m+n = 0$ and . If $m+n \neq 0$ , the two in each term commute then their order is irrelevant so in that case, we will switch the order in the last two terms of equation (2.94) and then replace the variable k by $k - n$. So we get

$$\left[L_m^\perp, L_n^\perp\right] = (m-n)L_{m+n}^\perp \ , \ m+n \neq 0 \tag{2.95}$$

A mathematical set of operators $L_m^{\perp}$ with $n \in Z$, satisfying the above equation (2.95) defines the Lie algebra. This algebra is called the Witt algebra or the Virasoro algebra without central extension. Now the second case $m+n=0$, let $n=-m$ in equation (2.94) then an extra contribution arises due to insisting the normal order. Such that

$$\begin{aligned}\left[L_m^{\perp}, L_n^{\perp}\right] &= \frac{1}{2}\sum_{k=0}(m-k)\alpha_{-k}^{l}\alpha_{k}^{l} + \frac{1}{2}\sum_{k<0}(m-k)\alpha_{k}^{l}\alpha_{-k}^{l} \\ &+ \frac{1}{2}\sum_{k=0}k\alpha_{m-k}^{l}\alpha_{k-m}^{l} + \frac{1}{2}\sum_{k<0}k\alpha_{k-m}^{l}\alpha_{m-k}^{l}\end{aligned} \tag{2.96}$$

For normal ordering, let replace $k \to -k$ in the second terms, $k \to m+k$ in the 3rd terms, and $k \to m-k$ in the last terms

$$\begin{aligned}\left[L_m^{\perp}, L_n^{\perp}\right] &= \frac{1}{2}\sum_{k=0}(m-k)\alpha_{-k}^{l}\alpha_{k}^{l} + \frac{1}{2}\sum_{k=1}(m+k)\alpha_{-k}^{l}\alpha_{k}^{l} \\ &+ \frac{1}{2}\sum_{k=-m}(m+k)\alpha_{-k}^{l}\alpha_{k}^{l} + \frac{1}{2}\sum_{k=m+1}(m-k)\alpha_{-k}^{l}\alpha_{k}^{l}\end{aligned} \tag{2.97}$$

Let assume that $m>0$, (this argument goes the same way as in the other case) by this all the terms are now normal ordered except the 3rd term for which $-m \leq k \leq 0$. Splitting the summation of 3rd term in above equation (2.97) and then simplifying, we get

$$\left[L_m^{\perp}, L_n^{\perp}\right] = \frac{1}{2}\sum_{k=0}^{\infty}(m-k)\alpha_{-k}^{l}\alpha_{k}^{l} + \frac{1}{2}\sum_{k=1}^{\infty}(m+k)\alpha_{-k}^{l}\alpha_{k}^{l} + (D-2)A(m) \tag{2.98}$$

Where $A(m)$ is

$$A(m) = \frac{1}{2}\sum_{k=0}^{m}k(m-k) = \frac{1}{2}m\sum_{k=1}^{m}k - \frac{1}{2}\sum_{k=1}^{m}k^2 \tag{2.99}$$

By using the Mathematical induction and Zeta functions. We get

$$A(m) = \frac{1}{12}(m^3-m) \tag{2.100}$$

Now putting the value of $A(m)$ from equation (2.100) into equation (2.98) and solving, we find

$$\left[L_m^\perp, L_n^\perp\right] = 2m\ L_0^\perp + \frac{1}{12}(D-2)(m^3 - m)$$

This completes the commutation relation of Virasoro operators for $m + n = 0$. Now generalizing these two cases, we get

$$\left[L_m^\perp, L_n^\perp\right] = 2m\ L_{m+n}^\perp + \frac{1}{12}(D-2)(m^3 - m)\delta_{m+n,0} \tag{2.101}$$

The second term of right-hand side of the above equation (2.101) is called the central extension and a mathematical set of operators $L_m^\perp$ with $n \in Z$, which satisfying the above equation (2.101) defines the centrally extended Virasoro algebra. As the term is said to be central because it is a constant and commute with all other operators in the algebra. There is no central term for $m = 0$ and $m = \pm 1$.

| $[X, Y]$ | $x^i$ | $x^-$ | $p^i$ | $p^+$ | $p^-$ | $\alpha_m^i$ | $\alpha_m^-$ | $X \rightarrow$ |
|---|---|---|---|---|---|---|---|---|
| $x^j$ | 0 | 0 | $-i\delta^{ij}$ | 0 | $-i\,p^j/p^+$ | $-i\,\delta^{ij}\delta_m$ | $-i\,\alpha_m^j/p^+$ | |
| $x^-$ | 0 | 0 | 0 | $i$ | $-ip^-/p^+$ | 0 | $-i\,\alpha_m^-/p^+$ | |
| $p^j$ | $i\,\delta^{ij}$ | 0 | 0 | 0 | 0 | 0 | 0 | |
| $p^+$ | 0 | $-i$ | 0 | 0 | 0 | 0 | 0 | |
| $p^-$ | $ip^i/p^+$ | $ip^-/p^+$ | 0 | 0 | 0 | $m\,\alpha_m^i/p^+$ | $m\,\alpha_m^-/p^+$ | |
| $\alpha_n^j$ | $i\,\delta^{ij}\delta_n$ | 0 | 0 | 0 | $-n\,\alpha_n^j/p^+$ | $m\,\delta^{ij}\delta_{m+n}$ | $-n\,\alpha_{m+n}^j/p^+$ | |
| $\alpha_n^-$ | $i\,\alpha_n^i/p^+$ | $i\,\alpha_n^-/p^+$ | 0 | 0 | $-n\,\alpha_n^-/p^+$ | $m\,\alpha_{m+n}^i/p^+$ | $\star\star$ | |
| $Y \downarrow$ | | $\star\star = \frac{m-n}{p^+}\alpha_{m+n}^- + \left(\frac{D-2}{12}m(m^2-1) + 2a\,m\right)\frac{\delta_{m+n}}{(p^+)^2}$ | | | | | | |

These are the complete commutation relations, here $i, j$ are the transverse indices.

Closed strings Virasoro operators and their commutation relations have the same techniques as we have done for open strings [12, 15].

## 2.8 Lorentz generators and critical dimensions

Lorentz invariance of a string action allows us to find a set of conserve word sheet currents, $M^{\alpha}_{\mu\nu}$ and the resulting word-sheet charges $M_{\mu\nu}$ for open strings with $\sigma \in [0, \pi]$, are

$$M_{\mu\nu} = \int_0^\pi M^{\alpha}_{\mu\nu}(\tau, \sigma) d\sigma = \int_0^\pi (X_\mu P^\tau_\nu - X_\nu P^\tau_\mu) d\sigma \tag{2.102}$$

Now putting the value of $P^\tau_\mu$ from equation (2.15) so we get

$$M_{\mu\nu} = \frac{1}{2\pi\alpha'} \int_0^\pi (X_\mu \dot{X}_\nu - X_\nu \dot{X}_\mu) d\sigma \tag{2.103}$$

Now using the explicit mode expansion for and from equation (2.26), by simplifying we get

$$M^{\mu\nu} = x_0^\mu p^\nu - x_0^\nu p^\mu - i \sum_{n=1}^{\infty} \frac{1}{n} \alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_{-n} \alpha^\mu_n) \tag{2.104}$$

where the 1st two terms are due to orbital angular momentum while the summation terms are due to the angular momentum due to excited oscillator modes [12]. This equation (2.104) is the classical Lorentz generators of open strings in the terms of oscillators. Similarly for closed strings, the Lorentz generator becomes

$$M^{\mu\nu} = x_0^\mu p^\nu - x_0^\nu p^\mu - i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_{-n} \alpha^\mu_n) - i \sum_{n=1}^{\infty} \frac{1}{n} \bar{\alpha}^\mu_{-n} \bar{\alpha}^\nu_n - \bar{\alpha}^\nu_{-n} \bar{\alpha}^\mu_n) \tag{2.105}$$

$M^{-l}$ is the most delicate quantum Lorentz generators in light cone gauge because, $X^-$ coordinate is a non-trivial function of the transverse coordinates $X^l$ and a consistent $M^{-l}$ should be generates the Lorentz transformations on the strings coordinates. As Lorentz algebra is not generally reproduces by generators , $M^{\mu\nu}$ which implies that the theory is not Lorentz invariant, so to make it invariant, we should[10]

$$\left[M^{-l}, M^{-j}\right] = 0 \tag{2.106}$$

[10]As for point particle $[J^\mu, J^\nu] = 0$, as $J$ is the Lorentz generator for point particle.

Since from above equation (2.104), we have

$$M^{-l} \sim x_0^- p^l - x_0^l p^- - i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_{-n}^- \alpha_n^l - \alpha_{-n}^l \alpha_n^-) \tag{2.107}$$

As we know that the Lorentz generators should be Hermitian and normal ordered but as from equation (2.107) we write

$$(M^{-l})^\dagger \neq M^{-l}$$

So making it Hermitian by writing $\frac{1}{2}(x_0^l p^- + p^- x_0^l)$ instead of $x_0^l p^-$, we find

$$M^{-l} \sim x_0^- p^l - \frac{1}{2}(x_0^l p^- + p^- x_0^l) - i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_{-n}^- \alpha_n^l - \alpha_{-n}^l \alpha_n^-) \tag{2.108}$$

Now it's fully Hermitian and it's also normal ordered because of $\alpha^-$ are normal ordered. To make it complete Lorentz generator, we should put the definition of $P^-$ and $\alpha^-$ from equation (2.87) and (2.84), so our Lorentz charge $M^{-l}$ becomes

$$M^{-l} = x_0^- p^l - \frac{1}{4\alpha' p^+} \left( x_0^l (L_n^\perp + a) + (L_n^\perp + a) x_0^l \right) - \frac{i}{\sqrt{2\alpha'} p^+} \sum_{n=1}^{\infty} \frac{1}{n} (L_{-n}^\perp \alpha_n^l - \alpha_{-n}^l L_n^\perp) \tag{2.109}$$

The above equation (2.109) is our Lorentz charges in light cone gauge, in the order to make it Lorentz invariant, we should calculate the above commutation relation in equation (2.106) and then simplifying, we got

$$\left[M^{-l}, M^{-j}\right] = -\frac{1}{\alpha' p^{+2}} \sum_{m=1}^{\infty} \Delta_m (\alpha_{-m}^l \alpha_m^l - \alpha_{-m}^l \alpha_m^l) \tag{2.110}$$

With $\Delta_m$ which is

$$\Delta_m = \left\{ m \left[ 1 - \frac{1}{24}(D-2) \right] + \frac{1}{m} \left[ \frac{1}{24}(D-2) + a \right] \right\} \tag{2.111}$$

As for Lorentz invariance, the above equation (2.110) should be equal to zero, so this means

$$m \left( 1 - \frac{1}{24}(D-2) \right) + \frac{1}{m} \left( \frac{1}{24}(D-2) + a \right) = 0, \forall\, m \in Z^+$$

By solving this we get

$$D = 26$$

And $a = -1$. This shows that the relativistic strings can be properly quantized in the flat Minkowski space if the number of space-time dimensions is 26 [3, 13, 16, 17, 18]. A similar calculation can fix the dimensionality of space time to the value $D = 10$ in super strings.

## 2.9 State space and mass spectrum

The mass of relativistic string (that perform an arbitrary motion) can be calculated from the mass operators, which is the relativistic equation in light cone coordinates, as

$$M^2 = -p^2 = 2p^+p^- - p^lp^l$$

Now writing the mass operator in the term of Virasoro operators (open strings), as from equation (2.55) putting the value of $2p^+p^-$ , with normal ordering constant $(a = -1)$ we find

$$M^2 = \frac{1}{\alpha'}\left(-1 + \sum_{n=1}^{\infty} na_n^{l\dagger}a_n^l\right) \tag{2.112}$$

As the sum inside the bracket in the Number operator, like

$$N^{\perp} \equiv \sum_{n=1}^{\infty} na_n^{l\dagger}a_n^l \tag{2.113}$$

So the mass operator then becomes

$$M^2 = \frac{1}{\alpha'}(-1 + N^{\perp}) \tag{2.114}$$

And we can easily calculate the commutation relations of number operator with oscillators, which are

$$\left[N^{\perp}, a_n^{l\dagger}\right] = na_n^{l\dagger} \ , \ \left[N^{\perp}, a_n^l\right] = -na_n^l \tag{2.115}$$

Now let us define our ground state of quantum strings. We have started the quantization from our basic operators as in equation (2.63) and from this we have canonical pairs $(x_0^l, p^l)$ and $(x_0^-, p^+)$ so we can define the ground state from this pairs, as it is usually convenience to work in momentum space so the ground states is

$$\left|p^+, \vec{p}_T\right\rangle \tag{2.116}$$

These are the ground states of string for all values of momenta indicated by the labels and also called the vacuum states for oscillators in string theory. We can create states from the ground states by simply acting the creation operator on the ground states. As we have an infinite numbers of creations operator, for which we can write the general basis state $|\lambda\rangle$ of the state space, so

$$|\lambda\rangle = \prod_{n=1}^{\infty}\prod_{l=2}^{25}\left(a_n^{l\dagger}\right)^{\lambda_{n,l}}\left|p^+, \vec{p}_T\right\rangle \tag{2.117}$$

Here $\lambda_{n,l}$ is the positive integer and is define as the number of times that $a_n^{l\dagger}$ (creation operators) appears. And as the number operator acts on the basis state, so their Eigen value will be

$$N^\perp\left|\lambda\right\rangle = N_\lambda^\perp\left|\lambda\right\rangle, \text{ with } N_\lambda^\perp = \prod_{n=1}^{\infty}\prod_{l=2}^{25} n\lambda_{n,l}$$

Similarly for closed strings, the states becomes

$$\left|\lambda, \bar{\lambda}\right\rangle = \prod_{n=1}^{\infty}\prod_{l=2}^{25}\left(a_n^{l\dagger}\right)^{\lambda_{n,l}}\left|p^+, \vec{p}_T\right\rangle \times \prod_{m=1}^{\infty}\prod_{j=2}^{25}\left(\bar{a}_m^{j\dagger}\right)^{\bar{\lambda}_{m,j}}\left|p^+, \vec{p}_T\right\rangle \tag{2.118}$$

And the number operators are

$$N^\perp = \sum_{n=1}^{\infty} n a_n^{l\dagger} a_n^l \text{ And } \bar{N}^\perp = \sum_{m=1}^{\infty} m a_m^{j\dagger} a_m^j \tag{2.119}$$

From the zero modes of transverse Virasoro operators, the level matching condition is $N^\perp = \bar{N}^\perp$. Some of states and the mass spectrum of both open and closed strings are given in the below tables.

List of some open string states

| $N^{\perp}$ | $\lvert\lambda\rangle$ | $\alpha' M^2$ | Number of states | Wave function | Associated fields |
|---|---|---|---|---|---|
| 0 | $\lvert p^+, \vec{p}_T\rangle$ | -1 | 1 | $\psi(\tau, p^+, \vec{p}_T)$ | Tachyon |
| 1 | $a_1^{I\dagger}\lvert p^+, \vec{p}_T\rangle$ | 0 | $(D-2)$ | $\psi_I(\tau, p^+, \vec{p}_T)$ | Photon |
| 2 | $a_1^{I\dagger} a_1^{J\dagger}\lvert p^+, \vec{p}_T\rangle$<br>$a_2^{I\dagger}\lvert p^+, \vec{p}_T\rangle$ | 1 | $\frac{1}{2}(D-2)(D+1)$ | $\psi_{IJ}(\tau, p^+, \vec{p}_T)$<br>$\psi_J(\tau, p^+, \vec{p}_T)$ | Massive tensors |

List of some closed string states

| $\lvert N^{\perp}, \bar{N}^{\perp}$ | $\lvert\lambda, \bar{\lambda}\rangle$ | $\frac{1}{2}\alpha' M^2$ | Number of states | Wave function | Associated fields |
|---|---|---|---|---|---|
| 0,0 | $\lvert p^+, \vec{p}_T\rangle$ | -2 | 1 | $\psi(\tau, p^+, \vec{p}_T)$ | Tachyon |
| 1,1 | $a_1^{I\dagger}\bar{a}_1^{J\dagger}\lvert p^+, \vec{p}_T\rangle$ | 0 | $(D-2)$ | $\psi_{IJ}(\tau, p^+, \vec{p}_T)$ | Kalb-Ramond field, Graviton, dilaton |

Since from the closed string spectrum, we get a particle named graviton which shows that the gravity emerge into string theory [3].

# SUPERSTRINGS

The bosonic string theory has some unsatisfactory features, like the spectrum of the closed-string contains a tachyon particle. Open strings spectrum also contains tachyons which are unphysical because of implying instability of the vacuum. As the eliminations of the open string tachyons has been well understood in the term of D-branes's decay, while, the closed-strings tachyon has not been understood yet [7].

And also the spectrum of the open as well as closed strings does not contain fermions. With out fermions, the theory is unrealistic. To make a realistic theory which describes the nature and all particles (Bosons and Fermions), we required a supersymmetry, which relates the bosons and fermions, and the resultant theory is called superstring theory. There are two basic approaches to develop the super string theories [3].

1. The Ramond-Neveu-Schwarz (RNS) formalism, which is the supersymmetric on the string world sheet.

2. The Green-Schwarz (GS) formalism, which is supersymmetric in ten-dimensional Minkowski space-time.

These approaches are equivalent at least for ten-dimensional Minkowski spacetime. This chapter will describe the RNS formalism.

This version of string theory will be constructed along the same line as for the bosonic theory i.e. we will consider the classical action, the solutions to the equation of motion and their mode expansions and then we will apply the canonical and light cone gauge quantization procedures [19].

## 3.1 The super string Action

To get fermions in our theory we introduce a new dynamical world-sheet variables $\psi_1^\mu(\tau,\sigma)$ , and $\psi_2^\mu(\tau,\sigma)$ as like for bosons the dynamical word-sheet variable is $X^\mu(\tau,\sigma)$. These classical variables are $\psi_\alpha^\mu(\tau,\sigma)$ with $(\alpha = 1,2)$ are anti-commuting variables, rather than commuting variables.
In light cone gauge we set $X^+$ proportional to $\tau$ and $X^-$ was solved for, in terms of other quantities. While in the case of superstrings, this remains the same but in additionally the light cone gauge condition also set $\psi_\alpha^+ = 0$ and then allow us to solve for $\psi_\alpha^-$ . As $X^-$ and $\psi_\alpha^-$, both gets contributions from transverse $X^l$ and $\psi_\alpha^l$, So this means that both contributes into the light cone Lorentz Generator $M^{-l}$ , And from this we can fixed the space time dimensions, which is $D = 10$ for super string.
Now let us start from the classical action that describes the full set of degree of freedom. As we will use the light cone gauge which concern with transverse field, so we can write the action in the term of transverse field

$$S = \frac{1}{4\pi\alpha'}\int d\tau \int_0^\pi d\sigma(\dot{X}^l\dot{X}^l - X'^l X'^l) + S_\psi \tag{3.1}$$

With

$$S_\psi = \frac{1}{2\pi}\int d\tau \int_0^\pi d\sigma \left[\psi_1^l(\partial_\tau + \partial_\sigma)\psi_1^l + \psi_2^l(\partial_\tau - \partial_\sigma)\psi_2^l\right] \tag{3.2}$$

Here $S_\psi$ action is the Dirac action for fermions, which live in the two dimension world-sheet [7, 20].

### 3.1.1 Equations of motion and Boundary conditions

In the order to find the equations of motion and the boundary conditions, we will vary the fields $\psi_\alpha^\mu$ in action $S_\psi$. So we have

$$\delta S_\psi = \frac{1}{2\pi}\int d\tau \int_0^\pi d\sigma \left[\delta\psi_1^l(\partial_\tau + \partial_\sigma)\psi_1^l + \psi_1^l(\partial_\tau + \partial_\sigma)\delta\psi_1^l\right.$$

$$+\,\delta\psi_2^l(\partial_\tau-\partial_\sigma)\psi_2^l+\psi_2^l(\partial_\tau-\partial_\sigma)\delta\psi_2^l] \tag{3.3}$$

By simplifying we can get the equations of motion and the boundary conditions, as $\delta S_\psi = 0$ which gives the equations of motion

$$(\partial_\tau+\partial_\sigma)\psi_1^l = 0,\ (\partial_\tau-\partial_\sigma)\psi_2^l = 0 \tag{3.4}$$

This implies that $\psi_1^l$ is the right moving while $\psi_2^l$ is left moving such as

$$\begin{aligned}\psi_1^l(\tau,\sigma) &= \psi_1^1(\tau-\sigma)\\ \psi_2^l(\tau,\sigma) &= \psi_2^1(\tau+\sigma)\end{aligned} \tag{3.5}$$

Similarly from $\delta S_\psi = 0$, the boundary conditions becomes

$$\psi_1^l(\tau,\sigma_*)\delta\psi_1^l(\tau,\sigma_*) - \ \psi_2^l(\tau,\sigma_*)\delta\psi_2^l(\tau,\sigma_*) = 0 \tag{3.6}$$

This should hold for both the end points, $\sigma_* = 0$ and $\sigma_* = \pi$ for all $\tau$. As from above equation (3.6) we can also write $\psi_1^l(\tau,\sigma_*) = \pm\ \psi_2^l(\tau,\sigma_*)$ for each end point and by this, with out loss of generality, we take

$$\psi_1^l(\tau,0) = \psi_2^l(\tau,0) \tag{3.7}$$

And the relative sign between $\psi_1^l$ and $\psi_2^l$ become important for other end of string. So we have

$$\psi_1^l(\tau,\pi) = \pm\psi_2^l(\tau,\pi) \tag{3.8}$$

Let we have a fermion field, $\psi^l$ which is define over an interval $\sigma \in [-\pi,\pi]$ then the boundary conditions can be written as

$$\psi^l \equiv \begin{cases}\psi_1^l(\tau,\sigma)\ \sigma\in[0,\pi]\\ \psi_2^l(\tau,-\sigma)\ \sigma\in[-\pi,0].\end{cases} \tag{3.9}$$

And finally, from the relative sign, we have the two conditions i.e. periodic and anti periodic fermions $\psi^l$. which are

$$\psi^l(\tau,\pi) = +\psi^l(\tau,-\pi) \tag{3.10}$$

$$\psi^l(\tau,\pi) = -\psi^l(\tau,-\pi) \tag{3.11}$$

These are the two sectors or boundary conditions for fermion field and we will call them, Ramond (R) sector for periodic as in equation (3.10) and Neveu Schwarz (NS) sector for anti periodic fermion $\psi^l$ as in equation (3.11).

## 3.2 Neveu Schwarz sector

As Neveu Schwarz fermion is the function of $\tau - \sigma$ and has changed the sign when $\sigma \to \sigma + 2\pi$, so the mode expansion can be written as

$$\psi^l(\tau,\sigma) \;\sim \sum_{r\in Z+1/2} b^l_r e^{-ir(\tau-\sigma)} \tag{3.12}$$

Here $\psi^l$ is an anti periodic, for any $r = n + \frac{1}{2}$ with $n$ is an integer.
As $\psi^l$ is an anti commuting 'field which means that the expansion coefficients $b^l_r$ will be then anti commuting operators, and for the negatively $r$ like $b^l_{-1/2}$ , $b^l_{-3/2}$ , $b^l_{-5/2}$ ,..., are the creation operators, while for positively like $b^l_{1/2}$, $b^l_{3/2}$, $b^l_{5/2}$ ,..., are the annihilation operators. And these operators will also satisfy the anti commuting relation, like

$$\left\{b^l_r, b^j_s\right\} = \delta_{r+s,0}\ \delta^{lj} \tag{3.13}$$

Similarly these operators will act on the ground or vacuum which we called Neveu Schwarz vacuum or simply $|\mathrm{NS}\rangle$. And the states in the Neveu Schwarz sectors are

$$|\lambda\rangle = \prod_{l=2}^{9}\prod_{n=1}^{\infty}(\alpha^l_{-n})^{\lambda_{n,l}}\prod_{j=2}^{9}\prod_{r=1/2,3/2..}^{\infty}(b^j_{-r})^{\rho_{r,j}}\,|\mathrm{NS}\rangle \;\otimes\; \left|p^+,\vec{p}_T\right\rangle \tag{3.14}$$

This is the full ground state which is in the product, $\otimes$ of the ground state $|p^+,\vec{p}_T\rangle$ for $\alpha^l_{-n}$ and $|\mathrm{NS}\rangle$ for $b^j_{-r}$ operators.
For the NS sector the mass squared operator with out normal ordering, is given by

$$M^2 = \frac{1}{\alpha'}\left(\frac{1}{2}\sum_{p\neq 0}\alpha^l_{-p}\alpha^l_p + \frac{1}{2}\sum_{r\in Z+1/2} r b^l_{-r}b^l_r\right) \tag{3.15}$$

Now to find the normal ordering constant, we use the same procedure as we have done for Bosonic string theory. As for bosonic oscillators, $\alpha^l$ each coordinate contribute $-\frac{1}{24}$ to the normal ordering constant, " $a$ ". So we take this as

$$a_b = -\frac{1}{24}$$

Similarly for NS sector fermions the contribution term in mass squared operator $M^2$ will be

$$\frac{1}{2} \sum_{r=-1/2,-3/2,..} r b^l_{-r} b^l_r \ = \ \frac{1}{2} \sum_{r=1/2,3/2,..} r b^l_{-r} b^l_r - (D-2) \sum_{r\in Z^+ + 1/2} r \tag{3.16}$$

And as

$$\sum_{r=1}^{\infty} r = \sum_{r\in Z^+_{odd}} r + \sum_{r\in Z^+_{Even}} r = \sum_{r\in Z^+_{odd}} r + 2\sum_{r=1}^{\infty} r$$

From this we write

$$\sum_{r\in Z^+_{odd}+1/2} r = -\frac{1}{2}\sum_{r=1}^{\infty} r = \frac{1}{24} \tag{3.17}$$

So the above equation (3.16) then becomes

$$\frac{1}{2} \sum_{r=-1/2,-3/2,..} r b^l_{-r} b^l_r \ = \ \frac{1}{2} \sum_{r=1/2,3/2,..} r b^l_{-r} b^l_r - \frac{1}{48}(D-2) \tag{3.18}$$

above equation (3.18) gives the idea that the contribution term for NS fermions is

$$a_{NS} = -\frac{1}{48}$$

And the full normal ordering constant for $M^2$ is written as

$$a = (D-2)(a_B + a_{NS}) = -(D-2)\frac{1}{16} \tag{3.19}$$

For $D = 10$ then from above equation (3.19), $a = -\frac{1}{2}$ and hence the mass squared operator becomes

$$M^2 = \frac{1}{\alpha'}\left(-\frac{1}{2} + N^{\perp}\right), \text{ with } N^{\perp} = \sum_{p=1}^{\infty} \alpha^l_{-p}\alpha^l_p + \sum_{r=1/2,3/2,..} r b^l_{-r} b^l_r \tag{3.20}$$

As the fermionic oscillators are contributing half integers $N^{\perp}$ to so by this we get some of the first few states in this NS sector are as

$$\begin{aligned}
\alpha' M^2 = -\frac{1}{2}\ ,\ N^{\perp} = 0 :\ |\mathrm{NS}\rangle\ \otimes\ \left|p^+, \vec{p}_T\right\rangle \\
\alpha' M^2 = 0\ ,\ N^{\perp} = \frac{1}{2} :\ b^l_{-1/2}\,|\mathrm{NS}\rangle\ \otimes\ \left|p^+, \vec{p}_T\right\rangle \\
\alpha' M^2 = \frac{1}{2}\ ,\ N^{\perp} = 1 :\ \left\{\alpha^l_{-1}, b^l_{-1/2} b^j_{-1/2}\right\}|\mathrm{NS}\rangle\ \otimes\ \left|p^+, \vec{p}_T\right\rangle \\
\alpha' M^2 = 1\ ,\ N^{\perp} = \frac{3}{2} :\ \left\{\alpha^l_{-1} b^j_{-1/2},\ b^l_{-3/2},\ b^l_{-1/2} b^j_{-1/2} b^k_{-1/2}\right\}|\mathrm{NS}\rangle\ \otimes\ \left|p^+, \vec{p}_T\right\rangle
\end{aligned} \tag{3.21}$$

The ground state ($N^{\perp} = 0$) of the NS sector is a tachyon. The first excited states ($N^{\perp} = \frac{1}{2}$) are massless and are eight states labeled by transverse index $l$.
As from this list we have both boson as well as fermions. As from above list (3.21) States which have an even fermion number are bosonic while odd fermion number are fermionic. So let us define an operator which distinguishes bosons and fermions from one another. This operator will be called as $(-1)^F$ , where $F$ stand for fermion number. The operator has plus one value on the bosonic state while minus one value for fermionic state. The fermions ground state is then written as

$$(-1)^F\,|\mathrm{NS}\rangle\ \otimes\ \left|p^+, \vec{p}_T\right\rangle = -\,|\mathrm{NS}\rangle\ \otimes\ \left|p^+, \vec{p}_T\right\rangle \tag{3.22}$$

Similarly the operator $(-1)^F$ acts on the state, as in equation (3.14)

$$(-1)^F\,|\lambda\rangle = -(-)^{\sum_{r,j}\ \rho_{r,j}}\,|\lambda\rangle \tag{3.23}$$

Hence this result allow us to take $(-1)^F$ operator as an anti commuting with all other fermionic operators, like

$$\left\{(-1)^F, b^l_r\right\} = 0 \tag{3.24}$$

The fermionic or bosonic character of the states is so far restricted to the $(\tau, \sigma)$ world-sheet [7, 21]. We will discuss later that these states are fermions or bosons in spacetime.

## 3.3 Ramond sector

Now the Ramond boundary conditions (3.10) in which field $\psi^l$ is a periodic and can be expanded in the terms of oscillators

$$\psi^l(\tau,\sigma) \sim \sum_{n\in Z} d_n^l e^{-ir(\tau-\sigma)} \tag{3.25}$$

As $\psi^l$ is an anti commuting, so the Ramond oscillators $d_n^l$ will be also anti commuting operators. And the negatively like $d_{-1}^l, d_{-2}^l, d_{-3}^l$ ,..., are the creation operators, while for positively like $d_1^l, d_2^l, d_3^l$ ,..., are the annihilation operators. Similarly the Ramond oscillators $d_n^l$ will satisfy the anti commutation relations

$$\left\{d_m^l, d_n^j\right\} = \delta_{m+n}\delta^{l\,j} \tag{3.26}$$

As from this (3.26) the zero modes, $d_0^l$ should be treated with care. It turns out the idea that these eight operators will have to organize by the simple linear combination of the four creation and four annihilation operators. Let us we have the four creation operators out of these eight operators

$$\xi_1, \xi_2, \xi_3, \xi_4 \tag{3.27}$$

As being the zero modes, these creation operators will act with out changing the mass squared state. Let we have a vacuum $|0\rangle$, since the creation operators in (3.27) can give the $2^4$ degenerate Ramond ground states. Since we have total 16 ground states in which eight of them will have an even number of $\xi s$ acting on $|0\rangle$, while the other eight will have an odd number of $\xi s$ acting on $|0\rangle$. Let the eight states which have an even number of creation operators is $|R_a\rangle$ with $a = 1, 2, ..., 8$ then

$$|R_a\rangle \;:\; \begin{cases} |0\rangle\,, \\ \xi_1\xi_2\,|0\rangle\,, \xi_1\xi_3\,|0\rangle\,, \xi_1\xi_4\,|0\rangle\,, \xi_2\xi_3\,|0\rangle\,, \xi_2\xi_4\,|0\rangle\,, \xi_3\xi_4\,|0\rangle\,, \\ \xi_1\xi_2\xi_3\xi_4\,|0\rangle\,. \end{cases} \tag{3.28}$$

Similarly the other eight states which have an odd number of creation operators is $|R_{\bar{a}}\rangle$ with $\bar{a} = \bar{1}, \bar{2}, ..., \bar{8}$ then

$$|R_{\bar{a}}\rangle \ : \ \begin{cases} \xi_1 |0\rangle \, , \xi_2 |0\rangle \, , \xi_3 |0\rangle \, , \xi_4 |0\rangle \, , \\ \xi_1\xi_2\xi_3 |0\rangle \, , \xi_1\xi_2\xi_4 |0\rangle \, , \xi_1\xi_3\xi_4 |0\rangle \, , \xi_2\xi_3\xi_4 |0\rangle \, . \end{cases} \tag{3.29}$$

Hence $|R_a\rangle$and $|R_{\bar{a}}\rangle$ states makes the complete set of the degenerate Ramond ground states and can be denoted as $|R_A\rangle$ with $A = 1, 2, ..., 16$ . The state space of Ramond sector can be written as

$$|\lambda\rangle = \prod_{l=2}^{9} \prod_{n=1}^{\infty} (\alpha_{-n}^{l})^{\lambda_{n,l}} \prod_{j=2}^{9} \prod_{r=1/2,3/2..}^{\infty} (d_{-m}^{j})^{\rho_{m,j}} |R_A\rangle \ \otimes \ \left|p^+, \vec{p}_T\right\rangle \tag{3.30}$$

Just like as in NS sector, the Ramond sector has also an operator,$(-1)^F$ which is anti commuting with all the other fermionic oscillators, including the zero modes

$$\left\{(-1)^F, d_n^l\right\} = 0 \tag{3.31}$$

Conventionally, we declare $|0\rangle$ to be fermionic, like

$$(-1)^F |0\rangle = - |0\rangle \tag{3.32}$$

The operator gives the idea that $(-1)^F$ states are fermionic while $|R_a\rangle$ states are bosonic. In the $R$ sector $|R_{\bar{a}}\rangle$, the mass squared operator without normal ordering can be written as

$$M^2 = \frac{1}{\alpha'} \left( \frac{1}{2} \sum_{p\neq 0} \alpha_{-p}^{l} \alpha_{p}^{l} + \frac{1}{2} \sum_{n\in Z} n d_{-n}^{l} d_{n}^{l} \right) \tag{3.33}$$

Similarly for $R$ sector, the contribution term in $M^2$ is

$$\frac{1}{2} \sum_{n=-1,-2,..} n d_{-n}^{l} d_{n}^{l} = \frac{1}{2} \sum_{n=1,2,..} n d_{-n}^{l} d_{n}^{l} + \frac{1}{24}(D-2) \tag{3.34}$$

Since the above equation (3.34) gives the idea that the contribution term for $R$ fermions is

$$a_R = \frac{1}{24}$$

Hence the contribution term for $R$ sector $a_R$ is equal (with opposite sign) to the contribution term for bosonic $a_B$. So the total normal ordering constant becomes zero, thus we can write the mass squared operators as

$$M^2 = \frac{1}{\alpha'} \sum_{n \geq 1} \left( \alpha^l_{-p} \alpha^l_p + n d^l_{-n} d^l_n \right) \tag{3.35}$$

This implies that the Ramond ground states are massless and some of few excited states in this $R$ sector are as

$$\alpha' M^2 = 0 \quad |R_a\rangle \quad \| \quad |R_{\bar{a}}\rangle$$

$$\alpha' M^2 = 1 \quad \alpha^l_{-1} |R_a\rangle , d^l_{-1} |R_{\bar{a}}\rangle \quad \| \; \alpha^l_{-1} |R_{\bar{a}}\rangle , d^l_{-1} |R_a\rangle$$

$$\alpha' M^2 = 2 \quad \alpha^l_{-2}, \alpha^l_{-1}\alpha^j_{-1}, d^l_{-1}d^j_{-1} |R_a\rangle , \; \| \; \{\alpha^l_{-2}, \alpha^l_{-1}\alpha^j_{-1}, d^l_{-1}d^j_{-1}\} |R_{\bar{a}}\rangle ,$$

$$\{\alpha^l_{-1}d^j_{-1}, d^l_{-2}\} \; |R_{\bar{a}}\rangle \quad \| \; \{\alpha^l_{-1}d^j_{-1}, d^l_{-2}\} \; |R_a\rangle \tag{3.36}$$

As we have separated the states into the two groups which have an identical number of states, the left of the bars states gives $(-1)^F = -1$ (fermionic states). While the right of the bars states gives the bosonic states i.e. $(-1)^F = +1$ . And hence for each fermionic state, there is a corresponding bosonic state which is the supersymmetry [7, 21]. But however, this supersymmetry is on the world sheet and the space time supersymmetry will be arise by combining the states from both, Ramond and Neveu Schwarz sectors.

## 3.4 Super transverse Virasoro operators

For quantization of superstring theory, we needed the super Transverse Virasoro operators which are the generalization of the Transverse Virasoro operators. As before we find the transverse Virasoro operators for bosonic string theory, the similar calculation will be required for the superstring [13, 22]. By including the supersymmetry, the Virasoro operators will become the super Virasoro operators and can be written as

$$L_n^{\perp} = L_n^{\perp(a)} + L_n^{\perp(b)} \text{ NS sector}$$

$$L_n^{\perp} = L_n^{\perp(a)} + L_n^{\perp(d)} \text{ R sector}$$

Here $L_n^{\perp(a)}$ is the transverse Virasoro operator which can be written as

$$L_n^{\perp(a)} = \frac{1}{2}\sum_{p\in Z} \alpha_{n-p}^l \alpha_p^l$$

And the $L_n^{\perp(b)}$ is terms is due to NS sector while $L_n^{\perp(d)}$ term is for R sector, which can be written as

$$L_n^{\perp(b)} = \tfrac{1}{2}\sum_{r\in Z+1/2}(r+\tfrac{n}{2})b_{-r}^l b_{n+r}^l \text{ and } L_n^{\perp(d)} = \tfrac{1}{2}\sum_{m\in Z}(m+\tfrac{n}{2})d_{-n}^l d_{n+m}^l$$

These are helpful in quantizing the theory.

## 3.5 Counting states

In the order to count the number of states at any given mass squared operator. For this we need a generating function which contains the information about the number of states. As for bosonic string theory (open string), we have the oscillators $(a_1^{\dagger}, a_2^{\dagger}, a_3^{\dagger}, ..., a_n^{\dagger})$, then the generating function can be written as

$$f(x) \;=\; \prod_{n=1}^{\infty} \frac{1}{1-x^n} \tag{3.37}$$

As there are 24 transverse light cone directions for each oscillator, since each will have a generating function. So the complete generating function for bosonic string theory can be written as

$$f(x) \;=\; \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^{24}} \tag{3.38}$$

This is the generating function for the bosonic open string or simply for $N^{\perp}$. Since the mass squared $\alpha^{/}M^2$, is more physical than the number operator $N^{\perp}$. So we need to define the generating function by using $\alpha^{/}M^2$ instead of $N^{\perp}$. As for bosonic open string $\alpha^{/}M^2 = N^{\perp} - 1$ then for this the generating function will be obtain by dividing the above generating function (3.38) by the one power of $x$. Hence the generating function for the bosonic open string theory, $f_{os}(x)$ can be written as

$$f_{os}(x) = \frac{1}{x}\prod_{n=1}^{\infty} \frac{1}{(1-x^n)^{24}} \tag{3.39}$$

We can also expand it as

$$f_{os}(x) = \frac{1}{x} + 24 + 324x + 3200x^2 + 25650x^3 + .... \tag{3.40}$$

These are the states of the mass squared operators i.e. 24 massless states of photon and so on.

Similarly the generating function for Neveu Schwarz is written as

$$f_r(x) = 1 + x^r$$

As there are 8 transverse light cone directions for each oscillators $b^l_{-1/2}, b^l_{-3/2}, \ldots$ , since each will have a generating function, and also the $\alpha' M^2 = N^{\perp} - \frac{1}{2}$ for NS sector, so the complete generating function for NS sector can be written as

$$f_{NS}(x) = \frac{1}{\sqrt{x}}\prod_{n=1}^{\infty} \frac{(1+x^{n-\frac{1}{2}})^8}{(1-x^n)^8} \tag{3.41}$$

And by expanding we find

$$f_{NS}(x) = \frac{1}{\sqrt{x}} + 8 + 36\sqrt{x} + 128x + 402x\sqrt{x} + ... \tag{3.42}$$

These are the states of the mass squared operators i.e. 8 massless states.

Now for Ramond sector, as $\alpha' M^2 = N^{\perp}$, since the generating function becomes

$$f_R(x) = 16 \prod_{n=1}^{\infty} \frac{(1+x^n)^8}{(1-x^n)^8} \tag{3.43}$$

The overall 16 factor is due to each combination of the Ramond oscillator which gives rise to the 16 states as by acting on each ground state. By expanding, we find

$$f_R(x) = 16 + 256x + 2304x^2 + ... \tag{3.44}$$

We notice that the Ramond coefficients (3.44) are actually double to the corresponding Neveu Schwarz coefficients (3.42). We will discuss this later.

## 3.6 Open superstrings and the GSO projection

As from the Ramond sector, we see that the word sheet has a supersymmetry i.e. the ground state which is divided into two groups of $|R_a\rangle$ and $|R_{\bar{a}}\rangle$, which built from the zero modes $d_0^l$, and has an equal number of fermionic and bosonic states having the opposite value of $(-1)^F$.

As the zero modes, $d_0^l$ carry the Lorentz index vector and can transform under Lorentz transformation but the Ramond ground states $(|R_a\rangle, |R_{\bar{a}}\rangle)$ do not transform like a vectors but transform as a spinors. Which shows that the index $a, \bar{a}$ are the spinor indices but are different spinors (and different fermions).

This means that there are two types of fermions in $R$ sector, but both of them gives the opposite values of $(-1)^F$ and with two types of fermions we cannot get any space time supersymmetry because we identified $|R_{\bar{a}}\rangle$ as a space time bosons, but bosons cannot carry any kind of spinor index.

Now to solve this issue as, by projecting the spectrum of $R$ sector in a specific way in which we choose only those fermions which have $(-1)^F = -1$, by this we can get space time fermions, and this projection is called Gliozzi, Scherk, and Olive (GSO) projection [3, 7, 21]. Hence the resultant projected $R$ sector is then called the $R-$ sector ($R$ minus) which have $(-1)^F = -1$. Similarly, the , is the set of states for which $(-1)^F = +1$. So the generating function for $R-$ sector then becomes as

$$f_{R-}(x) = 8 \prod_{n=1}^{\infty} \frac{(1+x^n)^8}{(1-x^n)^8} \tag{3.45}$$

By expanding, the power series of (3.45) is then

$$f_{R-}(x) = 8 + 128x + 1152x^2 + ... \tag{3.46}$$

So we see that there are eight fermionic massless states in $R-$ sector. Similarly doing the same thing for $NS$ sector, hence, we find sector states for which $(-1)^F = -1$ and $NS+$ sector which have $(-1)^F = +1$ . Since the $NS-$ sector contain a tachyon, hence it is useful to take the full open superstring by combining the set of states from the $R-$

sector and $NS+$ sector.

Now to find the generating function for $NS+$ sector $f_{NS+}(x)$ by taking account of GSO projection which eliminate the contribution of even number of fermions, so we get the generating function for $NS+$ sector as

$$f_{NS+}(x) = \frac{1}{2\sqrt{x}}\left[\prod_{n=1}^{\infty}\frac{(1+x^{n-\frac{1}{2}})^8}{(1-x^n)^8} - \prod_{n=1}^{\infty}\frac{(1-x^{n-\frac{1}{2}})^8}{(1-x^n)^8}\right] \tag{3.47}$$

For supersymmetry, we need to have $f_{NS+}(x) = f_{R-}(x)$ so this means that

$$\frac{1}{2\sqrt{x}}\left(\prod_{n=1}^{\infty}\frac{(1+x^{n-\frac{1}{2}})^8}{(1-x^n)^8} - \prod_{n=1}^{\infty}\frac{(1-x^{n-\frac{1}{2}})^8}{(1-x^n)^8}\right) = 8\prod_{n=1}^{\infty}\frac{(1+x^n)^8}{(1-x^n)^8} \tag{3.48}$$

This identity is being proved by Carl Gustav Jacob Jacobi in his work on elliptic functions, which is published in 1829. While in our case, for the basis of supersymmetric, it is a key equation.

## 3.7 Closed string theories

As we know that the closed (bosonic) strings can be obtained by the combining the left moving, and the right moving copies of the open strings. Similarly the closed superstrings can be obtained by combining the open superstrings. Since there are two sectors (*NS* and *R*) for open superstring, so there are four possibilities for the closed string sectors, like

$$\text{(NS, NS)}, \text{(NS, R)}, \text{(R, NS)}, \text{(R, R)} \tag{3.49}$$

As in the case of open superstrings, the space time bosons arises from *NS* sector while the space time fermions arises from *R* sector. Similarly, in the case of closed superstring, we can get the space time bosons from (NS, NS) as well as from the (R, R)[1] sectors while the space time fermions from (R, NS) and (NS, R) sectors.

[1]The (R,R) sectors are "doubly" fermionic and thus the spacetime bosons

### 3.7.1 Type IIA Superstring Theory

In the order to get the closed superstring theory with a supersymmetry, for this we need to truncate the above four sectors in equation (3.49). Since we can truncate as

$$\text{Left sector} : \left\{ \begin{array}{c} \text{NS} + \\ \text{R} - \end{array} \right\} \text{Right sector} : \left\{ \begin{array}{c} \text{NS} + \\ \text{R} + \end{array} \right\} \tag{3.50}$$

By this we find the four sectors which is called the type IIA superstring and these sectors are

$$(\text{NS} + , \text{NS} + ), (\text{NS} + , \text{R} + ), (\text{R} - , \text{NS} + ), (\text{R} - , \text{R} + ) \tag{3.51}$$

This is the type IIA superstring and for this, the mass squared is written as

$$\frac{1}{2}\alpha' M^2 = \alpha' M_L^2 + \alpha' M_R^2 \tag{3.52}$$

Here $M_L^2$ and $M_R^2$ are denoting the mass squared operators of the left and right sectors respectively. Now listing some of the massless states of the varies sectors, which are

$$(\text{NS} + , \text{NS} + ) : \bar{b}^l_{-1/2} \left|\text{NS}\right\rangle_L \;\otimes\; b^j_{-1/2} \left|\text{NS}\right\rangle_{\text{R}} \;\otimes\; \left|p^+, \vec{p}_T\right\rangle \tag{3.53}$$

$$(\text{NS} + , \text{R} + ) : \bar{b}^l_{-1/2} \left|\text{NS}\right\rangle_L \;\otimes\; \left|\text{R}_{\bar{b}}\right\rangle_{\text{R}} \;\otimes\; \left|p^+, \vec{p}_T\right\rangle \tag{3.54}$$

$$(\text{R} - , \text{NS} + ) : \; \left|\text{R}_{\text{a}}\right\rangle_L \;\otimes\; b^l_{-1/2} \left|\text{NS}\right\rangle_{\text{R}} \;\otimes\; \left|p^+, \vec{p}_T\right\rangle \tag{3.55}$$

$$(\text{R} - , \text{R} + ) : \; \left|\text{R}_{\text{a}}\right\rangle_L \;\otimes\; \left|\text{R}_{\bar{b}}\right\rangle_{\text{R}} \;\otimes\; \left|p^+, \vec{p}_T\right\rangle \tag{3.56}$$

Hence, there are 64 bosonic states in (3.53) due to the index and having eight values. These are just like the bosonic closed string theory massless states and carry the two indices. So we get the graviton (35 states), the Kalb-Ramond field (28 states), and the dilaton (one state).

$$(\text{NS} + , \text{NS} + ) \text{ massless fields are } : g_{\mu\nu}, B_{\mu\nu}, \phi \tag{3.57}$$

There are 64 fermionic states in each of the states (3.54) and (3.55) due to Both the states given in (3.54) and (3.55) have included a Ramond ground state, so these states are space time fermions. And give the total of 2 x 8 x 8 = 128 fermionic states. Similarly, the states given (3.56) having the two R ground states, which have doubly fermionic, so these will be the space time bosons which have 8 x 8 = 64 massless bosonic states and together with the bosonic states in (3.53) gives the total massless, bosonic states as 64 + 64 = 128 of the type IIA superstring. The space time bosonic states match with the space time fermionic states, which is the supersymmetry of the theory.

### 3.7.2 Type IIB Superstring Theory

A different superstring theory arises by truncating the four sectors which is given in (3.49) such that

$$\text{Left sector}: \left\{ \begin{array}{c} \text{NS}+ \\ \text{R}- \end{array} \right\} \text{Right sector}: \left\{ \begin{array}{c} \text{NS}+ \\ \text{R}- \end{array} \right\} \tag{3.58}$$

By this we find the four sectors which is called the type IIB superstring and these sectors are

$$(\text{NS}+,\text{NS}+),(\text{NS}+,\text{R}-),(\text{R}-,\text{NS}+),(\text{R}-,\text{R}-) \tag{3.59}$$

Now listing the massless states of this type IIB superstring theory, which are

$$(\text{NS}+,\text{NS}+): \bar{b}^{l}_{-1/2}\,|\text{NS}\rangle_L \otimes b^{j}_{-1/2}\,|\text{NS}\rangle_\text{R} \otimes \left|p^+,\vec{p}_T\right\rangle \tag{3.60}$$

$$(\text{NS}+,\text{R}-): \bar{b}^{l}_{-1/2}\,|\text{NS}\rangle_L \otimes |\text{R}_\text{b}\rangle_\text{R} \otimes \left|p^+,\vec{p}_T\right\rangle \tag{3.61}$$

$$(\text{R}-,\text{NS}+): |\text{R}_\text{a}\rangle_L \otimes b^{l}_{-1/2}\,|\text{NS}\rangle_\text{R} \otimes \left|p^+,\vec{p}_T\right\rangle \tag{3.62}$$

$$(\text{R}-,\text{R}-): |\text{R}_\text{a}\rangle_L \otimes |\text{R}_\text{b}\rangle_\text{R} \otimes \left|p^+,\vec{p}_T\right\rangle \tag{3.63}$$

(NS - NS ) sector: This sector is same for both the superstrings, type IIA and type IIB. (NS - R ) and (R - NS ) sectors: these sectors give the space time fermions and contain a spin 3/2 gravitino (56 states) and a spin 1/2 fermion called the dilatino (eight states). In the case of type IIB, the two gravitinos will have the same chirality while for the type IIA superstring, they will have opposite chirality.

(R - R ) sector: This sector gives the space time bosons and in the case of type IIA superstring theory, the massless bosons contain the Maxwell field (eight states) and the anti-symmetric gauge field with three index (56 states), while the type IIB superstring theory, the massless bosons contain the scalar field (one state), the Kalb Ramond field (28 states), and the totally anti-symmetric gauge field with four index (35 states) [3, 7].

$$\text{(R - R) the massless fields of type IIA} : A_\mu, A_{\mu\nu\rho} \tag{3.64}$$

$$\text{(R - R) the massless fields of type IIB} : A, A_{\mu\nu}, A_{\mu\nu\rho\sigma} \tag{3.65}$$

These are the some massless field of R sectors.

### 3.7.3 Heterotic superstring theories

There are two types of Heterotic superstring theories. These are the closed superstring theories. in the Heterotic string, we will combine the left moving bosonic open string with the right moving open superstring as like the type II closed superstring theories which arises by the combination of both left and right moving copies of the open superstrings. As there are 26 space time dimension of open bosonic string theory, ten of them are matched by the right moving open bosonic coordinates with the open superstring [3]. The extra 16 left moving coordinates can be described a torus with a very special properties to gives a consistent superstring theory. There are precisely two distinct tori which have the require properties and they will corresponds to a Lie algebras $SO(32)$[2] and $E_8 \times E_8$[3]. By this we can get a consistent theory which lives in a 10 dimensional space time. Heterotic superstrings theory comes into two versions, $E_8 \times E_8$ type and $SO(32)$ type. These groups characterize the symmetries which exist in the theories.

[2]The group SO(32) is the group generated by 32-by-32 matrices which are the orthogonal and having a unit determinant.

[3]$E_s$ is the largest exceptional group, here E is for the exceptional.

## 3.8 Type I

We have discussed the oriented string theory [4] in which the operator $X^l(\tau,\sigma)$ involves a parameter $\sigma \in [0,\pi]$ i.e. both the type II superstring theories and the Heterotic superstring theories are the theories of oriented closed strings. We can also construct a theory that will be an unoriented string theory. For this, we define an operator $\Omega$ which can reverse the orientation of the strings [3, 7]. The unoriented strings can be obtained by restricting the oriented strings spectrum to the set of the states which are invariant under the action of $\Omega$. Unoriented strings are not the strings with out the orientation i.e. they viewed like the quantum superposition of states which as a whole are invariant under the action of $\Omega$. Hence, we can imagine the unoriented state as a superposition of a string states and the same states with opposite orientation. A supersymmetric theory of both open and closed unoriented strings is called the Type I superstring theory. These are the five different superstrings theories which can be relate by the dualities.

## 3.9 Critical dimensions

For bosonic string theory, we have fixed the space time dimension by using the commutation relation of the Lorentz generators, $M^{-l}$ as

$$\left[M^{-l}, M^{-j}\right] = 0$$

The same idea will be used here as we used before in bosonic theory [7]. Now will take the super Generators, by the same phenomenology we can construct the super Lorentz generators. For NS sector the super Lorentz generator becomes

$$M^{-l} = x_\circ^- p^l - \frac{1}{4\alpha' p^+}\left(x_\circ^l(L_0^\perp + a) + ((L_0^\perp + a)x_\circ^l\right) - \frac{i}{\sqrt{2\alpha'}p^+}\sum_{n=1}^{\infty}\frac{1}{n}(L_{-n}^\perp\alpha_n^l - \alpha_{-n}^l L_n^\perp)$$

$$-\frac{i}{\sqrt{2\alpha'}p^+}\sum_{r=1/2}^{\infty} r(L_{-r}^\perp b_r^l - b_{-r}^l L_r^\perp)$$

[4]The orientation is defined as the direction of increasing of $\sigma$.

And similarly, the super Lorentz generator for $R$ sector can be written as

$$M^{-l} = x_\circ^- p^l - \frac{1}{4\alpha' p^+}\left(x_\circ^l (L_0^\perp + a) + ((L_0^\perp + a)x_\circ^l\right) - \frac{i}{\sqrt{2\alpha'}p^+}\sum_{n=1}^{\infty}\frac{1}{n}(L_{-n}^\perp \alpha_n^l - \alpha_{-n}^l L_n^\perp)$$

$$-\frac{i}{\sqrt{2\alpha'}p^+}\sum_{m=1}^{\infty} m(L_{-m}^\perp d_m^l - d_{-m}^l L_m^\perp)$$

By calculating the above commutation relation for $NS$ sector the super Lorentz generator, we get the non vanishing result which is

$$\left[M^{-l}, M^{-j}\right] = -\frac{1}{\alpha' p^{+2}}\sum_{m=1}^{\infty}\Delta_m(\alpha_{-m}^l \alpha_m^l - \alpha_{-m}^l \alpha_m^l)$$

with $\Delta_m$ is

$$\Delta_m = \left\{ m\left[1 - \frac{1}{8}(D-2)\right] + \frac{1}{m}\left[-\frac{1}{8}(D-2) - 2a\right]\right\}$$

As for Lorentz invariance, the above equation should be equal to zero and for this the dimension of space time, $D = 10$ and the normal ordering constant, NS sector $a_{NS} = -\frac{1}{2}$. and by the similarly method, for the super Lorentz generator $R$ sector $a_R = 0$.

# D-BRANES

A Dp-brane (with $p$ spatial dimensions) is an extended object. The letter D stands for Dirichlet. When $p$ is equal to the total number of spatial dimensions, then this type of Dp-brane is called space filling brane. In bosonic string theory, D25-brane is a space filling brane because $p$ is equal to the total number of spatial dimensions. The endpoints of open strings must lie on the D-brane.
Similarly, when the Dirichlet conditions applied to the subset of the d indices of $X^\mu$, i.e. to the indices $p+1, ..., d$, this means that the endpoints of the string are restricted to move on a p-dimensional hyper plane. This higher dimensional object is then calling a Dp-brane, where $p$ indicates its (spatial) dimensions and D stands for Dirichlet.
It turns out that these objects should be considered to be dynamical as well.

## 4.1 Tachyons and D-brane decay

In bosonic string theory there is a particle having imaginary mass, called tachyon. The open string is attached to a D-brane. The tachyons make the D-brane unstable [7, 23]. In the order to study the tachyons and D-brane decay, let start from the scalar field because the field associated with the tachyons are the scalar field and the Lagrangian for a scalar field can be written as

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi \; \partial_\nu\phi - V(\phi)$$

Here $V(\phi)$ is the potential of scalar field. From scalar field Lagrangian, we can find the equation of motion

$$-\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi - V'(\phi) = 0$$

As the scalar field potential can be written in the form of mass, as

$$V(\phi) = \frac{1}{2}M^2\phi^2$$

As when $M^2 > 0$ then the potential $V(\phi)$ will have the stable minimum at $\phi = 0$ while when $M^2 < 0$ then the potential $V(\phi)$ will have the unstable maximum at $\phi = 0$. For simplicity, let us consider the field only depends on time, then the equation of motion becomes as

$$\frac{\partial^2\phi(t)}{\partial t^2} + M^2\phi(t) = 0$$

Now when the mass squared is greater than zero then the solution of the above equation become

$$\phi = A\sin(Mt + a_0)$$

As due to the stable point, the scalar field could be sitting at $\phi = 0$ forever and will be simply oscillate whenever it is displaced [24]. Now consider the imaginary mass squared, like $M^2 = -\beta^2$, where $\beta$ is positive, in this case the equation of motion becomes as

$$\frac{\partial^2\phi(t)}{\partial t^2} - \beta^2\phi(t) = 0$$

The solution of this equation become as

$$\phi(t) = A\cosh(\beta t) + B\sinh(\beta t)$$

Let the solution $\phi(t) = \sinh(\beta t)$. As when time is zero, $\phi(0) = 0$ but when time goes to infinite then $\phi(\infty) = \infty$. This can be imagining as the field $\phi$ is rolled up. By using the trivial solution, we can say that the tachyon will stay at $\phi = 0$ but any small perturbation could make it to roll-off. The point $\phi = 0$ is an unstable point for tachyon and tachyon cannot stay here for some definite time, which shows the instability. As the mass squared of tachyon in an open string theory is

$$M^2 = -\frac{1}{\alpha'}$$

Since the potential of the free tachyon becomes

$$V_{tach}^{free}(\phi) = -\frac{1}{2\alpha'}\phi^2$$

As the presence of the tachyon is the signal of instability of the open string theory. As the open string is attached to the D-brane, since we can say that there is instability in the background of space filling D-branes. In bosonic open string theory, the space filling D-brane is D25-brane. As it is a physical object and its have some energy density $T_{25}$[7]. Tachyons are the states of an open string which is attached to D-brane this means that the tachyons are the excited states of D-brane and by this, we can say the tachyons states will lower the energy of D-brane. Explicitly, the existence of tachyons means that the space filling brane, D25-brane is unstable.

## 4.2 Quantization of open strings in the presence of various kinds of D-branes

### 4.2.1 Dp-branes and boundary conditions

Considering a Dp-brane and let introduces the spacetime coordinates $x^\mu$, with $\mu = 0, 1, 2, .., 25$ for bosonic string theory, splitting these coordinates into two groups i.e. tangent to the brane and normal to the brane. As the tangential coordinates includes the time coordinate as well as let some spatial coordinates $p$ , then the normal to the brane includes $(D-p)$ coordinates. We can write it as

$$\underbrace{x^0, x^1, x^2, .., x^p}_{D_P \text{ tangential coordinates}} \quad , \quad \underbrace{x^{p+1}, x^{p+2}, x^{p+3}, .., x^d}_{D_P \text{ normal coordinates}} \tag{4.1}$$

For simplicity we let $x^a = \bar{x}^a$, with $(a = p+1, ..., d)$. We can also write the above equation (4.1) for string coordinates $X^\mu(\tau, \sigma)$ in similar fashion, like

$$\underbrace{X^0, X^1, X^2, .., X^p}_{D_P \text{ tangential coordinates}} \; , \; \underbrace{X^{p+1}, X^{p+2}, X^{p+3}, .., X^d}_{D_P \text{ normal coordinates}} \tag{4.2}$$

Since the open string attached with the Dp-brane and the end points must lies on this Dp-brane then the normal coordinates of string to the brane will satisfy the Dirichlet boundary conditions such that

$$X^a(\tau,\sigma)|_{\sigma=0} = X^a(\tau,\sigma)|_{\sigma=\pi} = \bar{x}^a \ , \ a = p+1,...,d \tag{4.3}$$

The coordinates $X^\alpha$ will call DD coordinates due to these satisfying the Dirichlet boundary condition on both the end points. The end points of the open string can move everywhere along the tangent direction to the brane. By this the open string coordinates which is tangent to the D-brane will satisfy the Neumann boundary conditions, as

$$X'^m(\tau,\sigma)|_{\sigma=0} = X'^m(\tau,\sigma)|_{\sigma=\pi} = 0 \ , \ m = 0,1,...,p \tag{4.4}$$

These call NN coordinates due to both of the open string endpoints satisfying the Neumann boundary condition. Hence we can summarize this as

$$\underbrace{X^0, X^1, X^2, .., X^p}_{\text{NN coordinates}}, \underbrace{X^{p+1}, X^{p+2}, X^{p+3}, .., X^d}_{\text{DD coordinates}} \tag{4.5}$$

Now in the order to use the light cone coordinates we need to have at least one of the spatial NN coordinate from which we define $X^\pm$ coordinates. So this means that we will assume $p \geq 1$ and our analysis will not apply to the strings which will attach to a D0-brane. Hence in light cone coordinates, we may write

$$\underbrace{X^+, X^-, \{X^i\}}_{\text{NN}} \ , \ \underbrace{\{X^a\}}_{\text{DD}} \ , \ i = 2,...,p \text{ and } a = p+1,...,d \tag{4.6}$$

### 4.2.2 Quantizing open strings on Dp-branes

As before we have quantized the open string in the presence of the space filling D-brane and now we will quantize the open string in the presence of the Dp-brane and to study the effects on string spectrum states due to the presence of the Dp-brane. As the NN coordinates $X^i(\tau,\sigma)$ will also satisfy the same conditions that were satisfied by the light-cone coordinates $X^i(\tau,\sigma)$ of the open strings which were attached to the

D25-brane. Since our previous results are useful for the coordinates $X^i$. Let start from equation (2.47) and let $l \to (i, a)$ then we finds

$$\dot{X}^- \pm X'^- = \frac{1}{2\alpha'}\frac{1}{2P^+}\left((\dot{X}^i \pm X'^i)^2 + (\dot{X}^a \pm X'^a)^2\right) \tag{4.7}$$

As $X^i(\tau, \sigma)$ are satisfying exactly the same conditions which satisfied by $X^l(\tau, \sigma)$ so by this we can write

$$\dot{X}^i \pm X'^i = \sqrt{2\alpha'}\sum_{n\in Z}\alpha_n^i e^{-in(\tau\pm\sigma)} \tag{4.8}$$

Now to investigate the normal coordinates $X^i(\tau, \sigma)$, as these coordinates are normal to the brane will satisfy the wave equation. Now solving the wave equation for $X^\alpha$, we get[1]

$$X^a(\tau, \sigma) = \bar{x}^a + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{\mathrm{n}}\alpha_n^a e^{-in\tau}\sin n\sigma \tag{4.9}$$

And by this we can easily finds

$$X'^a \pm \dot{X}^a = \sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^a e^{-in(\tau\pm\sigma)} \tag{4.10}$$

This is same as equation (4.8) but differ by a minus sign and a zero mode which is absent here in equation (4.10).

We can easily quantize this string which is attached with a Dp-Brane. As $P^{\tau a} = \frac{1}{2\pi\alpha'}\dot{X}^a$ then the non vanishing commutation relations are

$$\left[X^a(\tau, \sigma), \dot{X}^b(\tau, \sigma')\right] = i2\pi\alpha'\eta^{ab}\delta(\sigma - \sigma') \tag{4.11}$$

Similarly we can easily find that

$$\left[\alpha_m^a, \alpha_n^b\right] = m\ \eta^{ab}\delta_{m+n,0} \tag{4.12}$$

The mass squared operators becomes as

$$M^2 = \frac{1}{\alpha'}\left(-1 + \sum_{n=1}^{\infty} n a_n^{i\dagger} a_n^i + \sum_{m=1}^{\infty} m a_m^{a\dagger} a_m^a\right) \tag{4.13}$$

[1]Here $\bar{x}^a$ is not an operator but a number.

Here the repeated index will be summed, the normal ordering constant is ($a = -1$), and the critical dimensions are same as for D25-brane. As $l \rightarrow (i, a)$ since our ground state will not remain the same as D25-brane. Since for Dp-brane, the ground state will be labeled by $p^+$ , $p^i$ and then becomes $|p^+, \vec{p}\rangle$as with $\vec{p} = (p^2, ..p^p)$. Now the state space are written as

$$|\lambda\rangle = \left[\prod_{n=1}^{\infty}\prod_{i=2}^{p}\left(a_n^{i\dagger}\right)^{\lambda_{n,i}}\right]\left[\prod_{m=1}^{\infty}\prod_{a=p+1}^{d}\left(a_m^{a\dagger}\right)^{\lambda_{m,a}}\right]|p^+, \vec{p}\rangle \tag{4.14}$$

The associated wave functions take the form

$$\psi_{i_1....i_p a_1....a_q.}(\tau, p^+, \vec{p})$$

Since some of the fields that satisfy $M^2 \leq 0$ associated with Dp-branes are, the scalar field (tachyons), the massless Maxwell field (photons) $a_1^{i\dagger}|p^+, \vec{p}\rangle$ and similarly the oscillator that acts on the ground states from those coordinates normal to the Dp-branes are $a_1^{i\dagger}|p^+, \vec{p}\rangle$, these are the massless the Lorentz scalar (due to the index ) which are normal to the branes.

### 4.2.3 Open string between parallel Dp-branes

Now considering the open strings which is extends between the two parallel Dp-branes. Let the first Dp-brane is located at $x^a = \bar{x}_1^a$ 'while the second one is at $x^a = \bar{x}_2^a$. The classes of the open string which are supported on the particular configurations of the D-branes are called sectors. In the presence of a two parallel Dp-branes, an open string can begin from one brane and end on the other brane. There are four sectors in our case of two parallel Dp-branes.

Consider the sector, in which open strings begins on the brane one and end on the brane two. For this, the boundary conditions for the DD strings coordinates are

$$X^a(\tau, \sigma)\big|_{\sigma=0} = \bar{x}_1^a\ ,\ \ X^a(\tau, \sigma)\big|_{\sigma=\pi} = \bar{x}_2^a\ ,\ a = p+1, ..., d \tag{4.15}$$

Similarly for $X^a$which are normal to the branes and will satisfy the wave equation. By solving the wave equation for $X^a$, we find

$$X^a(\tau,\sigma) = \bar{x}_1^a + (\bar{x}_2^a - \bar{x}_1^a)\frac{\sigma}{\pi} + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^a e^{-in\tau}\sin n\sigma \tag{4.16}$$

From this we get

$$X'^a \pm \dot{X}^a = \sqrt{2\alpha'}\sum_{n\in Z}\alpha_n^a e^{-in(\tau\pm\sigma)} \text{ and } \sqrt{2\alpha'}\alpha_0^a = \frac{1}{\pi}(\bar{x}_2^a - \bar{x}_1^a) \tag{4.17}$$

Similarly the mass squared will becomes

$$M^2 = \left(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'}\right)^2 + \frac{1}{\alpha'}\left(-1 + N^{\perp}\right) \tag{4.18}$$

Where

$$N^{\perp} = \sum_{n=1}^{\infty}\sum_{i=2}^{p} n a_n^{i\dagger} a_n^i + \sum_{m=1}^{\infty}\sum_{a=p+1}^{d} m a_m^{a\dagger} a_m^a \tag{4.19}$$

To develop the ground state for our parallel Dp-branes, we need additional labels which distinguish the various sectors [7]. These additional labels will be two numbers. The first number will denote the brane on which the open string end point $\sigma = 0$ lies while the second number will denote the brane on which the other end point $\sigma = \pi$ lies. Since the ground states can be written as $|p^+, \vec{p}; [ij]\rangle$ and the four sectors are

$$\left|p^+, \vec{p}; [11]\right\rangle, \left|p^+, \vec{p}; [22]\right\rangle, \left|p^+, \vec{p}; [12]\right\rangle, \left|p^+, \vec{p}; [21]\right\rangle. \tag{4.20}$$

Here the first two sectors are those which we discuss earlier i.e. $\bar{x}_2^a = \bar{x}_1^a$ . Let us first discuss the [12] sector, the mass squared operators for the lowest value of number operators becomes

$$\left|p^+, \vec{p}; [12]\right\rangle \ , M^2 = -\frac{1}{\alpha'} + \left(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'}\right)^2$$

When the separations between the two branes vanished then the field associated is a scalar tachyonic field. From this we can find the critical separation which is

$$|\bar{x}_2^a - \bar{x}_1^a| = 2\pi\sqrt{\alpha'}$$

Hence the ground states represents the massless scalar field for the zero separation while for large separation, the ground states represents the massive scalar fields. The next states (due to the number operators) are

$$a^{a\dagger}\big|p^+, \vec{p}; [12]\big\rangle \ , \ M^2 = -\frac{1}{\alpha'} + \left(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'}\right)^2$$

These are the $(d-p)$ Lorentz scalar which is normal to the branes.

$$a^{i\dagger}\big|p^+, \vec{p}; [12]\big\rangle \ , \ M^2 = -\frac{1}{\alpha'} + \left(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'}\right)^2$$

These are the $(p+1) - 2 = p+1$ massive states. As we know that the massive gauge fields has more degree of freedom than the massless gauge field. Hence from this we have one massive vector as well as $(d-p-1)$ massive scalars.

We have obtained a very interesting situation, as the separation between the Dp-branes goes to zero and coincide but still they are distinguishable and we have four open string sectors. The massless string states that represents a strings extending from Dp-brane one to Dp-brane two, includes the massless gauge field and $(d-p)$ states of a massless scalars. This is same content as that of a sector in which strings begin and end on the same Dp-brane. Whenever the two D-branes coincides, we get four massless gauge fields. From the world-volume of two coincide D-branes we indeed to get a U(2) Yang-Mills theory.

Let we have $N$ Dp-branes, for this the sectors will be labeled by the pairs $[i, j]$[2] , here $i = 1, 2, ..N$. The [i,j] sector will consist of the open strings which will start on ith and end on jth brane and we have $N^2$ sectors. The interaction of strings can be visualized as The first open string will join with a second open string and will form another open string. For this, the end point of first string ($\sigma = \pi$) will join the beginning point of the second string ($\sigma = 0$). And the new string (became from these two) will begin at the beginning point of the first string and will end on the end point of the second string [7, 25]. If the strings stretched between the D-branes, then the first string will from

[2]The discrete labels $i$, $j$ are used to label the branes, and the various open string sectors are sometimes called Chan-Paton indices.

the sector will have to join by the second string from sector to gives the product open string in the sector. The interaction can be written as

$$[i,j]*[j,k] = [i,k], \text{ here } j \text{ is not summed}$$

If the $N$ Dp-branes are coincides, then the $N^2$ sectors results in the $N^2$ interacting massless gauge fields. Since from this $N$ coinciding D-branes will carry U(N) massless gauge fields.

### 4.2.4 Strings between parallel Dp and Dq-branes

Now considering the two parallel D-branes, but having different dimension. Let we have two D-branes which are Dp-brane and Dq-brane and assume that $p > q$. From this, there will be a p-dimensional hyperplane parallel to the Dp-brane which will contain the Dq-brane. This means that we will have some common tangent as well as normal directions for both branes. For strings coordinates $X^\mu$, we can write for parallel Dp-brane and Dq-brane with $p > q$

$$\underbrace{X^0, X^1, .., X^q}_{\text{common tangential coordinates}}, \underbrace{X^{q+1}, X^{q+2}, .., X^p}_{\text{mixed coordinates}}, \underbrace{X^{p+1}, X^{p+2}, .., X^d}_{\text{common normal coordinates}} \tag{4.21}$$

Here the mixed coordinates will satisfy the Neumann boundary conditions on the starting Dp-brane while the Dirichlet boundary conditions on the ending of the Dq-brane and these coordinates will call ND coordinates. Similarly for the strings coordinates, we can also write

$$\underbrace{X^0, X^1, .., X^q}_{\text{NN coordinates}}, \underbrace{X^{q+1}, X^{q+2}, .., X^p}_{\text{ND coordinates}}, \underbrace{X^{p+1}, X^{p+2}, .., X^d}_{\text{DD coordinates}} \tag{4.22}$$

In light cone coordinates, we can write these as

$$\underbrace{X^+, X^-, \{X^i\}}_{\text{NN}}, \underbrace{\{X^r\}}_{\text{ND}}, \underbrace{\{X^a\}}_{\text{DD}} \tag{4.23}$$

Where $i = 2, ..., q$ , $r = q+1, ..., p$ and $a = p+1, ..., d$ . Now let for the position of the Dp-brane is while for position of Dq-brane is $\bar{x}_2^r$ and $\bar{x}_2^a$. The boundary conditions for the ND coordinates $X^r$ are

$$X'^r(\tau,\sigma)\big|_{\sigma=0} = 0\ ,\ \ X^r(\tau,\sigma)\big|_{\sigma=\pi} = \bar{x}_2^r \tag{4.24}$$

The expansion of $X^r(\tau,\sigma)$, by using the same procedure, can be written as

$$X^r(\tau,\sigma) = \bar{x}_2^r + i\sqrt{2\alpha'}\sum_{n\in Z_{\rm odd}} \frac{2}{n}\alpha^r_{\frac{n}{2}} e^{-i\tau\frac{n}{2}}\cos\left(\frac{n\sigma}{2}\right) \tag{4.25}$$

The Hermiticity of the above expansion, $X^r$ gives $(\alpha^r_{\frac{n}{2}})^\dagger = \alpha^r_{-\frac{n}{2}}$. And by the above expansion (4.16) we can find

$$\dot{X}^r \pm X'^r = \sqrt{2\alpha'}\sum_{n\in Z_{\rm odd}} \alpha^r_{\frac{n}{2}} e^{-i\frac{n}{2}(\tau\pm\sigma)} \tag{4.26}$$

The non vanishing commutation relations are

$$\Big[X^r(\tau,\sigma), \dot{X}^s(\tau,\sigma')\Big] = i(2\pi\alpha')\delta(\sigma-\sigma')\delta^{rs} \tag{4.27}$$

And by the similar method as we have done for space filling D-Brane, we can also find as

$$\Big[\alpha^r_{\frac{m}{2}}, \alpha^s_{\frac{n}{2}}\Big] = \frac{m}{2}\delta^{rs}\delta_{m+n,0} \tag{4.28}$$

Now as we have three types of coordinates like NN, DD, and ND or DN. The normal ordering contribution term for both NN and DD coordinates is same because of all oscillators are integrally moded

$$a_{\rm DD} = a_{\rm NN} = -\frac{1}{24} \tag{4.29}$$

However, for the ND or DN coordinates, the contribution term can be calculated as

$$\frac{1}{2}\sum_{m\in Z_{\rm odd}} \alpha^r_{-\frac{m}{2}}\alpha^r_{\frac{m}{2}} = \frac{1}{2}\sum_{m\in Z^+_{\rm odd}} \alpha^r_{-\frac{m}{2}}\alpha^r_{\frac{m}{2}} + \frac{1}{48}(p-q) \tag{4.30}$$

Since this means that the contribution term for ND or DN is

$$a_{\rm DN} = a_{\rm ND} = \frac{1}{48} \tag{4.31}$$

Similarly we can easily find the total normal order constant, which is

$$\begin{aligned} a &= -\frac{1}{24}[q+25-(p+1)] + \frac{1}{48}(p-q) \\ &= -1 + \frac{1}{16}(p-q) \end{aligned} \tag{4.32}$$

Now the mass squared operator then becomes

$$M^2 = \left(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'}\right)^2 + \frac{1}{\alpha'}\left(N^\perp - 1 + \frac{1}{16}(p-q)\right) \tag{4.33}$$

With

$$N^\perp = \sum_{n=1}^{\infty}\sum_{i=2}^{p} n a_n^{i\dagger} a_n^i + \sum_{k\in Z_{\mathrm{odd}}^+}\sum_{r=q+1}^{p} \frac{k}{2}\alpha_{\frac{k}{2}}^{r\dagger}\alpha_{\frac{k}{2}}^{r} + \sum_{m=1}^{\infty}\sum_{a=p+1}^{d} m a_m^{a\dagger} a_m^a \tag{4.34}$$

And the ground state are now labeled as

$$\left|p^+, \vec{p}; [12]\right\rangle \ , \vec{p} = (p^2, ....., p^q) \tag{4.35}$$

The labels of the ground state indicates that the corresponding fields will be living in a (*q+1*) dimensions of the space time i.e. the fields will live on the Dq-brane world-volume, which has the lower dimensions. The ground states having $N^\perp = 0$ corresponds to a scalar field on Dq-brane. This scalar depends upon the separation of the branes.

## 4.3 String charge and electric charge

### 4.3.1 Fundamental string charge

As we know that, for the point particle, the word line is one dimensional and Maxwell gauge field $A_\mu$ carrying one index and the point particle carries electric charges whenever it interacts with Maxwell field, in which the particle couples to the Maxwell field. Similar idea used for string charge, the string should couples to a new kind of gauge field. This gauge field is the Kalb Ramond anti symmetric tensor (of rank two)$B_{\mu\nu}$ which is the massless field and arisen in closed string. The complete dynamics of the string coupling to Kalb Ramond field can be written in the term of action is

$$S = S_{\mathrm{str}} - \frac{1}{2}\int d\tau d\sigma B_{\mu\nu}(X)\left(\partial_\tau X^\mu \partial_\sigma X^\nu - \partial_\tau X^\nu \partial_\sigma X^\mu\right) - \frac{1}{6k^2}\int d^D x \ H_{\mu\nu\rho}H^{\mu\nu\rho} \tag{4.36}$$

Here $S_{str}$ is the string action the second term is the action term due to string charge while the last term is the action for the Kalb Ramond. And $k$ is the dimensionful constant which makes the action dimensionless i.e. ($[k^2] = M^{6-D}$). The $H_{\mu\nu\rho}$ is the field strength of $B_{\mu\nu}$ which can be written as

$$H_{\mu\nu\rho} \equiv \partial_\mu B_{\nu\rho} + \partial_\nu B_{\mu\rho} + \partial_\rho B_{\mu\nu} \tag{4.37}$$

In the second term of the above action (4.36), $B_{\mu\nu}(X(\tau,\sigma))$ can be written for the general space time $B_{\mu\nu}(X)$ as

$$B_{\mu\nu}(X(\tau,\sigma)) = \int d^D x \delta^D(x - X(\tau,\sigma))\ B_{\mu\nu}(x) \tag{4.38}$$

Now the variation of above action (4.36) gives us

$$\delta S = \int d^D x\ \delta B_{\mu\nu}(x) \left( \frac{1}{k^2} \partial_\rho H^{\mu\nu\rho} - j^{\mu\nu} \right) \tag{4.39}$$

Here $j^{\mu\nu}$ is the (anti symmetric) current

$$j^{\mu\nu} = \frac{1}{2} \int d\tau d\sigma \delta^D(x - X(\tau,\sigma)) \left(\partial_\tau X^\mu \partial_\sigma X^\nu - \partial_\tau X^\nu \partial_\sigma X^\mu\right) \tag{4.40}$$

By the variation principle, we find

$$\frac{1}{k^2} \partial_\rho H^{\mu\nu\rho} = j^{\mu\nu} \tag{4.41}$$

The tensor $j^{\mu\nu}$ is conserved quantity like

$$\frac{\partial j^{\mu\nu}}{\partial x^\mu} = \frac{1}{k^2} \frac{\partial^2 H^{\mu\nu\rho}}{\partial x^\mu \partial x^\rho} = 0 \tag{4.42}$$

As the above equation give the idea of conservation of $j^{\mu\nu}$ but there is a $\nu$ free index, which means that $j^{\mu\nu}$ is the set of conserved current labeled by ($\nu$) a free index. And the charge density, as the zeroth component of $j^{\mu\nu}$, since the charge density ($\vec{j}^0$) of the Kalb Ramond field is then $j^{0k}$, here k will be running over the spatial value. Similarly from the above equation (4.42), we find $\nabla.\vec{j}^0 = 0$ and from the charge density we can easily find the charge of string, $\bar{Q}$ as

$$\vec{Q} = \int d^d x\ \vec{j}^0 \tag{4.43}$$

To understand it more, using the static gauge $X^0 = \tau = t$ and then simplifying the equation (4.40), we get

$$\vec{j}^0(\vec{x}, t) = \frac{1}{2}\int d\sigma\delta\left(\vec{x} - \vec{X}(t,\sigma)\right)\,\vec{X}'(t,\sigma) \tag{4.44}$$

From this, we get the idea that the charge density is tangent to the every point on string and lies along orientation of the string.

In the order to write this in more explicit form that the string density depends upon the orientation, for this let a long static string stretched along the $x^l$, then the string can be describe as

$$X^1(\tau,\sigma) = f(\sigma)\ ,\ X^2 = X^3 = .... = X^d = 0 \tag{4.45}$$

Here $f(\sigma)$ is the function of $\sigma$ with the range of $-\infty$ to $\infty$, this function will be either increasing or decreasing. Now using the above equation (4.45) and solving the equation (4.44), then we get

$$j^{01}(x^1, ..., x^d; t) = \frac{1}{2}\,\mathrm{sgn}(f')\delta(x^2)\delta(x^3)...\delta(x^d) = \frac{1}{2}\,\mathrm{sgn}(f')\delta(x^{\perp}) \tag{4.46}$$

Here $\mathrm{sgn}(f')$ is for the sign of $f'$. This result show explicitly that the charge density $\vec{j}^0$ depends upon the orientation or the sign of $f'(\sigma)$.

### 4.3.2 Visualizing string charge

In the order to visualize the charge of string, let us consider we have a static string for which $j^{ik} = 0$ ( $i$,$k$ are spatial components), since for a static string, the equation (4.41) becomes as

$$\frac{\partial H^{ik\rho}}{\partial x^\rho} = 0 \tag{4.47}$$

By this $H$ are to be time independent so $H^{ijk} = 0$ and the other equation is

$$\frac{\partial H^{okl}}{\partial x^l} = k^2 j^{ok} \tag{4.48}$$

Now let us introduce a new vector $\vec{B}_H$. This is called field strength dual to $H$ and which can be define as

$$H^{0kl} = \varepsilon^{klm}\vec{B}_H \tag{4.49}$$

Here $\varepsilon^{ijk}$ is the anti symmetric Levi-civita symbol. Since the above equation (4.48) then becomes

$$(\nabla \times \vec{B}_H)_k = k^2 j^{0k} \tag{4.50}$$

Now from this equation we can write

$$\nabla \times \vec{B}_H = k^2 \vec{j}^0 \tag{4.51}$$

This equation is the Ampere's equation, where $\vec{B}_H$ is the magnetic field. Integrating the above equation (4.51) over a curve $\Gamma$ of a two dimensional surface S. so we get

$$\frac{1}{k^2} \oint_\Gamma \vec{B}_H . d\vec{l} = \int_S \vec{j}^0 . d\vec{a} \tag{4.52}$$

By this equation, we get the idea that the curve $\Gamma$ is link to a string and the strings number $\mathcal{N}$ which is associated with $\Gamma$ is define as

$$\frac{1}{2}\mathcal{N} = \frac{1}{k^2} \oint_\Gamma \vec{B}_H . d\vec{l} = \int_S \vec{j}^0 . d\vec{a} \tag{4.53}$$

Here $\mathcal{N}$ is the number of strings which is linked by the curve $\Gamma$.

### 4.3.3 Strings ending on D-branes

As by the quantization of closed strings, there arises a massless field called Kalb Ramond field which lives over the all spacetime and the string can couple electrically with it to get charged. Now considering the charged string which attached to D-branes, for this, the action for the string couple to Kalb Ramond field, is written as

$$S_{\mathrm{B}} = -\frac{1}{2} \int d\tau d\sigma \ \varepsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X(\tau, \sigma)) \tag{4.54}$$

Here $\varepsilon^{\alpha\beta}$ is the two dimensional indices $\alpha, \beta = 0, 1$, anti symmetric with $\varepsilon^{01} = 1$. Now using the above action and calculating the variation in $S_B$, which we find that

$$\delta S_{\mathrm{B}} = -\int d\tau d\sigma \ \ (\partial_\tau(\Lambda_\nu \partial_\sigma X^\nu) - \partial_\sigma(\Lambda_\nu \partial_\tau X^\mu)) \tag{4.55}$$

Here we use

$$\delta B_{\mu\nu}(X) = \frac{\partial \Lambda_\nu}{\partial X^\mu} - \frac{\partial \Lambda_\mu}{\partial X^\nu} \tag{4.56}$$

The arguments of $\Lambda$ are the string coordinates $X(\tau,\sigma)$due to the arguments of $B_{\mu\nu}$. As from equation (4.55), the two total derivative in which the time derivative give no contribution, since this means that $\Lambda$ are vanishes at the end of time. For a closed string both the derivative vanishes while for the open string, there introduces a boundary contributions terms, which in not vanishes. In the order to find $\delta B_{\mu\nu}$ for an open string, let the string coordinates along the brane are $X^m$ and the string coordinates normal to the brane are $X^a$. Since we can write this as

$$X^\mu = (X^m, X^a) \ , \ \mu = (m, a) \tag{4.57}$$

If this D-brane is Dp-brane, then $(m = 0, 1, ...p)$ and solving the equation (4.55) for $X^\mu = (X^m, X^a)$ coordinates, so the coordinates $X^a$ are DD for which $\partial_\tau X^a = 0$ and the limits we take the $\sigma \in [0, \pi]$ then we get

$$\delta S_{\mathrm{B}} = \int d\tau \ \Lambda_m \partial_\tau X^m|_{\sigma=\pi} - \int d\tau \ \Lambda_m \partial_\tau X^m|_{\sigma=0} \tag{4.58}$$

Here we get the two boundary terms and hence, the gauge invariance has failed. From this we got the idea that the string charge conservation failed at the end points of the string. In the order to restore the gauge invariance, we should add some terms which give the electric charges to the end points of the string. then the Action can be written as

$$S = S_{\mathrm{B}} + \int d\tau \ \mathrm{A}_m(X) \partial_\tau X^m|_{\sigma=\pi} - \int d\tau \ \mathrm{A}_m(X) \partial_\tau X^m|_{\sigma=0} \tag{4.59}$$

As whenever we vary $\Lambda_\mu$ (i.e. $\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$ ), then we have to also vary Maxwell field $A_\mu$ on D-brane, like wise $\delta \mathrm{A}_m = -\Lambda_m$, and the two terms have opposite sign which show that the end points of string, which lies on Dp-brane, are oppositely charged. This action (4.59) restored the string charge conservation and we got the idea that the string end points have the opposite charge due to the Maxwell gauge field $A_m$ and by this we restore the conservation of charge invariance.

The Maxwell action is also not invariant, as it is proportional to $F^2$, like

$$\delta F_{mn} = \partial_m \delta \mathrm{A}_n - \partial_n \delta \mathrm{A}_m = -\partial_m \Lambda_n + \partial_n \Lambda_m = -\delta B_{mn} \tag{4.60}$$

To make it invariant, we should add $B_{mn}$ into Maxwell field $F_{mn}$ such that

$$\mathcal{F}_{mn} \equiv F_{mn} + B_{mn} \ , \ \delta\mathcal{F}_{mn} = 0 \tag{4.61}$$

This $\mathcal{F}_{mn}$ is called the field strength on D-brane and the Lagrangian density on D-brane is proportional to $-\frac{1}{4}\mathcal{F}^{mn}\mathcal{F}_{mn}$ , so expanding it

$$-\frac{1}{4}\mathcal{F}^{mn}\mathcal{F}_{mn} = -\frac{1}{4}B^{mn}B_{mn} - \frac{1}{4}F^{mn}F_{mn} - \frac{1}{2}F^{mn}B_{mn} \tag{4.62}$$

As the last term has some interesting physical significance, expanding it as

$$-\frac{1}{2}F^{mn}B_{mn} = -F^{0k}B_{0k} + ... \tag{4.63}$$

In the order to understand the physical significance of this term, let as the action for string charge, second term in (4.36), can be written as

$$\equiv -\int d^D x \ B_{\mu\nu}(x) \ j^{\mu\nu}(x) \tag{4.64}$$

This equation tells us that string charge density $j^{0k}$ couples to $B_{0k}$ . Since anything couple to $B_{0k}$ will carry string charge so it means that $F^{0k}$ will represent the string charge on D-brane but as $F^{0k}$ is equal to the electric field ($F^{0k} = E_k$), which gives the idea that the electric field lines will carry the string charge on D-branes.

## 4.4 D-brane charges and stable D-branes in Type II

There are also other extended objects rather than strings that carry charge i.e. Dp-branes. To study the charge of a Dp-branes, let start from our previous concept of charge that the string carry charge when they couple to Kalb Ramond gauge field, so this coupling is written as

$$-\int d\tau \ d\sigma \ \partial_\tau X^\mu \partial_\sigma X^\nu B_{\mu\nu}(X(\tau,\sigma)) \tag{4.65}$$

Now generalizing this idea like, a Dp-brane will be electrically charge if it coupled to a massless anti symmetric tensor field of indices ($p$ + $1$ ). As Dp-brane has ($p$+ $1$) dimensions and is parameterize $\tau$ by and the set of its coordinates $(\sigma^1, \sigma^2, ..., \sigma^p)$. Let the space time coordinates which describe the position of this brane is $X^\mu(\tau, \sigma^1, \sigma^2, ..., \sigma^p)$ with $\mu = 0, 1, ...d$, and the anti symmetric tensor field is describe by $A_{\mu\mu_1\mu_2...\mu_p}(x)$ then the generalize coupling from (4.65) is written as

$$S_p = -\int d\tau\ d\sigma_1..d\sigma_p\ \partial_\tau X^\mu \partial_{\sigma^1} X^{\mu_1}...\partial_{\sigma^p} X^{\mu_p} A_{\mu\mu_1\mu_2...\mu_p}(X^\mu(\tau, \sigma^1, \sigma^2, ..., \sigma^p)) \tag{4.66}$$

Kalb Ramond gauge field is the only massless anti symmetric tensor in bosonic closed string theory and there is no other massless anti symmetric tensor which means that the Dp-branes cannot be charged in bosonic string theory. While the type II superstring theories have some additional anti symmetric massless tensor in (R - R ) sectors, as listed in (3.64) and (3.65)

$$\text{(R - R ) the massless fields of type IIA: } A_\mu, A_{\mu\nu\rho} \tag{4.67}$$

$$\text{(R - R ) the massless fields of type IIA: } A, A_{\mu\nu}, A_{\mu\nu\rho\sigma} \tag{4.68}$$

This means that $A_\mu$ coupled to the D0-branes and $A_{\mu\nu\rho}$ coupled to the D2-branes while, $A_{\mu\nu}$ coupled to the D1-branes and $A_{\mu\nu\rho\sigma}$ coupled to the D3-branes. And the field $A$ does not couple to any D-brane (it coupled electrically to the object called, D-instanton). Summarizing these as

$$TypeIIA : D0, D2, \tag{4.69}$$

$$TypeIIB : D1, D3. \tag{4.70}$$

These branes are the stable charged and cannot decay into closed or open string states, while the bosonic D-branes are unstable due to the existence of tachyons, and carry no charge. Similarly in type IIA superstring theory, the Dp-branes with even are stable but are unstable in type IIB superstring theory while, in type IIB superstring theory, the Dp-branes with odd $p$ are stable but are unstable in type IIA superstring theory.

All the stable D-branes of type II superstring theories are charged and the Dp-branes which is not appeared in the above lists (4.69) and (4.70), like D4, D6, and D8 of type IIA superstring theory while the D5, D7, and D9 of type IIB superstring theory, means that these are carrying the magnetic charges for either (R - R ) gauge fields in listed above in (4.67) and (4.68) or for the other (R - R ) states that we not includes in the discussion [3, 7, 27, 28].

The (electric) charge of the Dp-brane has the simple physical significance, when the $p$ (spatial dimensions) are curled up to a circles and Dp-brane is wrapping around, the resulting compact space. Since here, $p$ compact space directions will lies on D-brane and others space time directions, which defined here, the lower dimensional spacetime, will be normal to the brane. Hence, the lower-dimensional observer which has only access to the non compact directions and will see the brane as a point particle.

Let $(x^1, ...., x^p)$ be the compact directions and $(X^1, ...., X^p)$ be their corresponding coordinates of brane. If compact directions are curled up to the circles of radii $(R^1, ...., R^p)$ and assuming the parameters $\sigma^k \in [0, 2\pi]$ then

$$X^k(\tau, \sigma^1, ..., \sigma^p) = R^k \sigma^k \ , \ k = 1, ..., p \tag{4.71}$$

Here the repeated index $k$ is not summed. This represent the wrapped Dp-branes and the coordinates $X^k$ are running from zero to $2\pi R^k$. Let the non compact dimensions be the $X^m$ with m index is for the non compact directions, then

$$X^m(\tau, \sigma^1, ..., \sigma^p) = x^k(\tau) \tag{4.72}$$

This equation shows that, Dp-brane appeared as the point particle to the observer which has the lower-dimensions. These assumptions made the equation (4.66) as

$$S_p = -\int d\tau \ d\sigma_1..d\sigma_p \ \partial_\tau X^\mu R^1...R^p A_{\mu \ 12...\mathrm{p}}(X(\tau, \sigma^1, \sigma^2, ..., \sigma^p)) \tag{4.73}$$

Since the $\mu$ will take the values over the non compact directions, $\mu = m$, then the above equation (4.73) becomes

$$S_p = -\int d\tau \ d\sigma_1..d\sigma_p \ \partial_\tau X^m R^1...R^p A_{m \ 12..p} \left(X^m(\tau), X^k(\sigma^k)\right) \tag{4.74}$$

Finally, let the part of A...field which is independent of compact coordinates i.e. $A_{m\ 12..p}(x^m(\tau))$, then the above equation (4.74) becomes

$$S_p = -R^1...R^p \int d\tau\ d\sigma_1..d\sigma_p\ \partial_\tau x^m A_{m\ 12..p}(x^m(\tau)) \tag{4.75}$$

By solving this integral, we find

$$S_p = -\frac{V_p}{(\alpha')^{p/2}} \int d\tau\ \partial_\tau x^m \bar{A}_m(x^m(\tau)) = -\frac{V_p}{(l_s)^p} \int \bar{A}_m\ dx^m \tag{4.76}$$

Here $V_p$ is the volume of compact space, $V_p = (2\pi R^1)....(2\pi R^p)$ , And $l_s$ is the string length $l_s = (\alpha')^{1/2}$. And the gauge field $\bar{A}_m$ as $\frac{1}{(\alpha')^{p/2}}\bar{A}_m(x^m(\tau)) = A_{m\ 12...\mathrm{p}}(x^m(\tau))$. The equation (4.76) recognized that the coupling of the point particle to the Maxwell field $\bar{A}_m$ which means that the Dp-brane appeared as a charged point particle. From this the charge $Q$ of the brane is

$$Q = \frac{V_p}{(l_s)^p} \tag{4.77}$$

The charge $Q$ depends upon the volume of the branes [7].

# STRING DUALITIES

Duality is generally used for the relationship between the two systems which have very different descriptions, but identical physics. There are two types of dualities in string theory, T-duality and S-duality.[1] In many cases, the T-duality implies that the two different geometries of the extra dimensions are physically equivalent i.e. the circle of radius ($R$) is equivalent to the circle of ($\alpha' R^{-1}$) radius. As there are five superstring theories which looks like the different theories from one another, but the T-duality relates the two type II superstring theories and the two Heterotic superstring theories. Similarly, S-duality relates, string's coupling constant $g_s$ to ($g_s^{-1}$), in the same ways as like the T-duality. S-duality relates the type I superstring theory to Heterotic SO (32) string theory, and type IIB superstring theory relates to itself.

To deal with the T-duality, we should first discuss the effects on the string when one of the spatial dimensions has curled up to compact space.

## 5.1 T-duality and closed strings

### 5.1.1 5.1.1 Mode expansions for compact dimension

Let us consider one of the spatial dimension be curled up,[2] in free bosonic string theory i.e. the $X^{25}$ dimension is curled up into a circle of radius $R$. Now we are going to check the effects of this on closed string [7, 29], as before we have taken the closed string

[1]Where some author writes T for target 'while some use it for toroidal and S is for Strong coupling.

[2]By identification we can compact a dimension i.e. $x \sim x + 2\pi R$

periodicity condition as

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi) \tag{5.1}$$

This periodicity condition is used whenever the closed string moving in a non compact dimensions. As usual we will use the light cone coordinates. When there is a compact dimension, let say $X^{25} \equiv X$ and the light cone coordinate are

$$X^+ \ , \ X^- \ , \ \underbrace{X^2, X^3, ..., X^{24}}_{X^i} \ , \ X \tag{5.2}$$

Then the periodicity condition becomes as

$$X(\tau, \sigma) = X(\tau, \sigma + 2\pi) + m(2\pi R) \tag{5.3}$$

Here $m$ is the number, called the winding number, and defined as, the number of times that the string winds around the circle of compact dimension and its sign depend upon the direction of winding. For other non compact dimensions, $\mu \neq 25$ the equation (5.1) holds. Let defines the winding $w$ in the terms of the winding number $m$ and the radius of the compact space as

$$w \equiv \frac{mR}{\alpha'} \tag{5.4}$$

By this the above periodicity condition in equation (5.3) becomes as

$$X(\tau, \sigma) = X(\tau, \sigma + 2\pi) + 2\pi\alpha' w \tag{5.5}$$

The mode expansions for the this equation (5.5) can be similarly derived as we have done before, the expansion become as

$$X_L(u) = \frac{1}{2}x_0^L + \sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0 u + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}_n e^{-inu} \tag{5.6}$$

$$X_R(v) = \frac{1}{2}x_0^R + \sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0 v + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}_n e^{-inv} \tag{5.7}$$

And from these two equations we get

$$\bar{\alpha}_0 - \alpha_0 = \sqrt{2\alpha'} \tag{5.8}$$

As we see that $\bar{\alpha}_0$ is equal to $\alpha_0$ for zero winding. By calculating the momentum along the compact direction like

$$P = \frac{1}{2\pi\alpha'} \int_0^{2\pi} (\dot{X}_L + \dot{X}_R) d\sigma = \frac{1}{\sqrt{2\alpha'}} (\bar{\alpha}_0 + \alpha_0) \tag{5.9}$$

Form solving these two equations (5.8) and (5.9), we get

$$\alpha_0 = \sqrt{\frac{\alpha'}{2}} (p - w)$$

$$\bar{\alpha}_0 = \sqrt{\frac{\alpha'}{2}} (p + w) \tag{5.10}$$

Using these new definition by which we get

$$X(\tau, \sigma) = x_0 + \alpha' p\tau + \alpha' w\sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} e^{-in\tau} (\alpha_n e^{in\sigma} + \bar{\alpha}_n e^{-in\sigma}) \tag{5.11}$$

This is the mode expansion of compact dimension and by this we can get easily

$$\dot{X} + X' = \sqrt{2\alpha'} \sum_{n \in Z} \bar{\alpha}_n e^{-in(\tau+\sigma)} \tag{5.12}$$

$$\dot{X} - X' = \sqrt{2\alpha'} \sum_{n \in Z} \alpha_n e^{-in(\tau-\sigma)} \tag{5.13}$$

### 5.1.2 Quantization and commutation relations

For quantization we adopted the similar method like the modes becomes operators and starting from the commutation relation between the momentum $P^\tau(\tau, \sigma)$ and the string compact dimension, $X$ which is

$$[X(\tau, \sigma), P^\tau(\tau, \sigma')] = i\delta(\sigma - \sigma') \tag{5.14}$$

Similarly the other commutation relations, will become by same method as we done before for non compact dimensions, as

$$[\bar{\alpha}_m, \bar{\alpha}_n] = [\alpha_m, \alpha_n] = m\delta_{m+n,0}\ ,\ \ [\alpha_m, \bar{\alpha}_n] = 0 \tag{5.15}$$

Now by the explicit form of $\bar{\alpha}_0$ and $\alpha_0$ , we get

$$[p, w] = 0 \tag{5.16}$$

Since $\bar{\alpha}_0$ and $\alpha_0$ commuting with $\bar{\alpha}_{\rm n}$, $\alpha_{\rm n}$ which means that

$$[p, \bar{\alpha}_n] = [p, \alpha_n] = [w, \alpha_n] = [w, \bar{\alpha}_n] = 0 \tag{5.17}$$

And from equation (5.14) we easily find that

$$[x_0, \alpha_0] = [x_0, \bar{\alpha}_n] = i\sqrt{\frac{\alpha'}{2}} \tag{5.18}$$

And using the explicit form of $\bar{\alpha}_0$ and $\alpha_0$ we find

$$[x_0, p] = i \ , \ \ [x_0, w] = 0 \tag{5.19}$$

As we see that the winding $w$ commutes with all the operators which appear in $X$. This gives the idea that the winding $w$ is a constant or just a constant number. More physically the winding w is an operator which gives the Eigen values of $w$ corresponding to various possible windings of strings. The quantization is only possible for those closed string sectors which has some fixed winding $w$. As along the compact dimension, the string would act as like a point particle that is moving on a circle[3] since the momentum will be then quantize and can be written as

$$p = \frac{n}{R} \ , n \in Z \tag{5.20}$$

Here $n$ is named as Kaluza Klein excitation number. Since both the operators $p$ and $w$ have discrete values.

### 5.1.3 Constraint and mass spectrum

For the mass spectrum, we begin from the Virasoro operators in which the sum over transverse $l$ is splits into $a$ sum over $i$ and a term due to compact dimension, as

$$\bar{L}_0^{\perp} = \frac{1}{2}\bar{\alpha}_0^l\bar{\alpha}_0^l + \bar{N}^{\perp} = \frac{1}{4}p^ip^i + \bar{\alpha}_0\bar{\alpha}_0 + \bar{N}^{\perp}$$

[3]By winding, we loss and gain some states for string, while for particle, we just lost states when we compact a dimension by identification because the particle cannot wrap around circle to gives new states.

$$L_0^{\perp} = \frac{1}{2}\alpha_0^l \alpha_0^l + N^{\perp} = \frac{1}{4}p^i p^i + \alpha_0 \alpha_0 + N^{\perp} \tag{5.21}$$

All the oscillators of $X^l$ and $X$ will contribute to the number operators $\bar{N}^{\perp}$ and $N^{\perp}$ [2]. Thus we find that

$$\begin{aligned} L_0^{\perp} - \bar{L}_0^{\perp} &= \frac{1}{2}(\alpha_0 \alpha_0 - \bar{\alpha}_0 \bar{\alpha}_0) + N^{\perp} - \bar{N}^{\perp} \\ &= -\alpha' p w + N^{\perp} - \bar{N}^{\perp} \end{aligned} \tag{5.22}$$

Since $L_0^{\perp} - \bar{L}_0^{\perp}$ does not vanish here which imposed a constraint that

$$N^{\perp} - \bar{N}^{\perp} = \alpha' p w \tag{5.23}$$

As both the number operators having the numbers Eigen values, so by the quantization of both $p$ and $w$ makes the above equation (5.23) more physical as

$$N^{\perp} - \bar{N}^{\perp} = n\ m \tag{5.24}$$

This gives the level matching condition and now the mass squared operators, $M^2 = 2p^+ p^- - p^i p^i$ becomes as

$$M^2 = p^2 + w^2 + \frac{2}{\alpha'}(N^{\perp} + \bar{N}^{\perp} - 2) \tag{5.25}$$

This means that both the momentum $p$ and the winding $w$ give the contribution to the mass squared operators [3, 7].

### 5.1.4 State Space of compactified closed strings

As there are additional terms in mass squared operators so for the ground states we will add the additional labels such that

$$\left| p^+, \vec{p}_T;\ n, m \right\rangle \ \ , n, m \in Z \tag{5.26}$$

And the state space can be constructed by applying the creation operators on the ground states like

$$\left[\prod_{r=1}^{\infty}\prod_{i=2}^{24}(a_r^{i\dagger})^{\lambda_{i,r}}\right]\left[\prod_{s=1}^{\infty}\prod_{j=2}^{24}(\bar{a}_s^{j\dagger})^{\bar{\lambda}_{j,s}}\right]\left[\prod_{k=1}^{\infty}\left(a_k^{\dagger}\right)^{\lambda_k}\right]\left[\prod_{l=1}^{\infty}\left(\bar{a}_l^{\dagger}\right)^{\bar{\lambda}_l}\right]\left|p^+, \vec{p}_T;\ n, m\right\rangle \tag{5.27}$$

And the number operators which will act on above state space are

$$N^{\perp} = \sum_{r=1}^{\infty}\sum_{i=2}^{24} r\ \lambda_{i,r} + \sum_{k=1}^{\infty} k\ \lambda_k\ ,\ \bar{N}^{\perp} = \sum_{s=1}^{\infty}\sum_{j=2}^{24} s\ \bar{\lambda}_{j,s} + \sum_{l=1}^{\infty} l\ \bar{\lambda}_l \tag{5.28}$$

The mass squared operators can also be written as

$$M^2 = \left(\frac{n}{R}\right)^2 + \left(\frac{mR}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N^{\perp} + \bar{N}^{\perp} - 2) \tag{5.29}$$

This is the mass squared operators for both none zero momentum and winding for the modified level matching condition $N^{\perp} - \bar{N}^{\perp} = n\ m$

## 5.2 T-Duality for Closed Strings

As the mass squared operators depend upon the compactified radius $R$ and there is a remarkable property that if we replace the radius $R$ by a radius $\tilde{R} = \alpha' R^{-1}$ and $n$ by $m$ then the mass squared operators remain the same, this symmetry is called T-duality for the closed string and the radii $R$ and $\tilde{R}$ are dual radii.

$$R\ \leftrightarrow\ \alpha' R^{-1} \equiv \tilde{R} \tag{5.30}$$

Hence the mass squared operators for both the dual radii are same as

$$M^2\left(R;\ n,\ m\right) = M^2\left(\tilde{R};\ m,\ n\right) \tag{5.31}$$

This gives the idea that if we start a theory having a small compactified radius $R$, we can transform to a dual theory which have the large radius $\bar{R}$ [7].now for a complete dual theory, let us define a dual coordinate (compact coordinate) operator

$$\tilde{X} \equiv X_L(\tau + \sigma) - X_R(\tau - \sigma) \tag{5.32}$$

The mode expansions for dual coordinate $\tilde{X}$ are

$$\tilde{X}(\tau, \sigma) = q_0 + \alpha' w\tau + \alpha' p\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}e^{-in\tau}(\bar{\alpha}_n e^{-in\sigma} - \alpha_n e^{in\sigma}) \tag{5.33}$$

By comparing this with equation (5.11), we find that

$$\begin{Bmatrix} x_0 \to q_0 \\ q_0 \to x_0 \end{Bmatrix} \begin{Bmatrix} p \to w \\ w \to p \end{Bmatrix} \begin{Bmatrix} \alpha_n \to -\alpha_n \\ \bar{\alpha}_n \to \ \bar{\alpha}_n \end{Bmatrix} \tag{5.34}$$

The dual momentum $\tilde{\mathcal{P}^{\tau}}$ becomes as

$$\tilde{\mathcal{P}^{\tau}} \equiv \frac{1}{2\pi\alpha'}\partial_\tau \tilde{X} = \frac{1}{2\pi\alpha'}(\dot{X}_L - \dot{X}_R) \tag{5.35}$$

The commutation relations for ($\tilde{X}$ ,$\tilde{\mathcal{P}^{\tau}}$ ) can be calculated by the same manner as we have done for ($X$ , $\mathcal{P}^{\tau}$). The Hamiltonian for both ($X$ , $\mathcal{P}^{\tau}$) and ($\tilde{X}$ ,$\tilde{\mathcal{P}^{\tau}}$ ) is same and written as

$$H = \frac{2}{\alpha'}(p^i p^i + p^2 + w^2) + N^{\perp} + \bar{N}^{\perp} - 2 \tag{5.36}$$

In the order to make a map between these two ($X$, $\mathcal{P}^{\tau}$) and ($\tilde{X}$, $\tilde{\mathcal{P}^{\tau}}$ ), the above (5.34) transform as

$$\begin{Bmatrix} x_0 \to \tilde{q}_0 \\ q_0 \to \tilde{x}_0 \end{Bmatrix} \begin{Bmatrix} p \to \tilde{w} \\ w \to \tilde{p} \end{Bmatrix} \begin{Bmatrix} \alpha_n \to -\tilde{\alpha}_n \\ \bar{\alpha}_n \to \ \tilde{\bar{\alpha}}_n \end{Bmatrix} \tag{5.37}$$

This map makes the physical equivalence of the two different theories and hence, T-duality is the symmetry, which exists between the different string theories. A complete summery is given in the following table (5.1)

| Theory with radius $R$ | Theory with radius $\tilde{R}$ |
|---|---|
| $H(R) = \frac{2}{\alpha'}(p^i p^i + p^2 + w^2) + N^{\perp} + \bar{N}^{\perp} - 2$ | $H(\tilde{R}) = \frac{2}{\alpha'}(p^i p^i + \tilde{p}^2 + \tilde{w}^2) + \tilde{N}^{\perp} + \tilde{\bar{N}}^{\perp} - 2$ |
| $x_0$ lives on a circle of radius R | $\tilde{x}_0$ lives on a circle of radius $\tilde{R}$ |
| $p$ has Eigen values $nR^{-1}$ | $\tilde{p}$ has Eigen values $mR\alpha'^{-1}$ |
| $q_0$ lives on a circle of radius $\alpha' R^{-1}$ | $q_0$ lives on a circle of radius $R$ |
| $w$ has Eigen values $mR\alpha'^{-1}$ | $w$ has Eigen values $nR^{-1}$ |
| $[x_0, p] = [q_0, w] = i$ | $[\tilde{x}_0, \tilde{p}] = [\tilde{q}_0, \tilde{w}] = i$ |
| $[\bar{\alpha}_m, \bar{\alpha}_n] = [\alpha_m, \alpha_n] = m\delta_{m+n,0}$<br>$[\alpha_m, \bar{\alpha}_n] = 0$ | $[\tilde{\bar{\alpha}}_m, \tilde{\bar{\alpha}}_n] = [\tilde{\alpha}_m, \tilde{\alpha}_n] = m\delta_{m+n,0}$<br>$[\tilde{\alpha}_m, \tilde{\bar{\alpha}}_n] = 0$ |

Table: (5.1) complete dual theory

### 5.2.1 Type II superstrings and T-duality

Let us consider $X^9$ coordinate is curled up into a circle of radius $R$ in type II superstring theories and the T-duality transformation carried out on this coordinate as

$$X_L^9 \to X_L^9 \text{ and } X_R^9 \to -X_R^9$$

This interchanged the momentum and the winding numbers similarly, the word sheet fermions will also transform under T-duality as

$$\psi_L^9 \to \psi_L^9 \text{ and } \psi_R^9 \to -\psi_R^9$$

This means that the T-duality reversed the chirality of the right moving ground states of Ramond sector. As the relative chirality of both left and right moving ground states is the thing which distinguished the type IIA and type IIB theories. If one is compactified on a circle of $R$, let say Type IIA, then the T-duality gives the type IIB which will be compactified on $\tilde{R}$ [3].

## 5.3 T-duality of open strings

### 5.3.1 T-duality and open strings

In the case of open strings, the situation is a little bit different when we compact a dimension because the open string cannot winds around the compact dimension. Let we have a D25-brane and a compactified $X^{25}$ circle having the radius $R$, the open string has quantize momentum $p^{25}$ but no winding but after the T-duality transformation, we will have a D24-brane and a compactified $X^{25}$ circle of radius $\tilde{R}$. The Dirichlet boundary conditions will impose a zero momentum constraint but the open string will have winding now. The open string mass squared of the two theories will coincide when $\tilde{R} = \alpha' R^{-1}$, because of the momentum states contribute to $M^2$ in the first theory in the same way as the open string winding states contribute to $M^2$ in the second theory. By permitting the duality conversion to change the D-brane, we can write it as

$$(D25;\ R)\ \to\ (D24;\ \tilde{R})$$

Now to find how the T-duality in the case of open strings, let us start from the assumption that we have space filling D25-brane i.e. the open strings have NN-type coordinates and let the compactified dimension $X^{25}(\tau,\sigma) \equiv X(\tau,\sigma)$, for this the mode expansion can be written as

$$X(\tau,\sigma) = x_0 + \sqrt{2\alpha'}\alpha_0\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n cos\sigma e^{-in\tau} \tag{5.38}$$

With

$$\alpha_0 = \sqrt{2\alpha'}p = \sqrt{2\alpha'}\frac{n}{R} \tag{5.39}$$

Now separating the above string coordinate (5.38) into left moving and right moving, as

$$X(\tau,\sigma) = X_L(\tau+\sigma) + X_R(\tau-\sigma) \tag{5.40}$$

Where

$$\begin{aligned} X_L &= \frac{1}{2}(x_0+q_0) + \sqrt{\frac{\alpha'}{2}}\alpha_0(\tau+\sigma) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in\tau}e^{-in\sigma} \\ X_R &= \frac{1}{2}(x_0-q_0) + \sqrt{\frac{\alpha'}{2}}\alpha_0(\tau-\sigma) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in\tau}e^{+in\sigma} \end{aligned} \tag{5.41}$$

Here $q_0$ is an arbitrary constant. For the T-duality transformation, let the dual coordinate is

$$\tilde{X}(\tau,\sigma) \equiv X_L(\tau+\sigma) - X_R(\tau-\sigma) \tag{5.42}$$

For this $\tilde{X}$ the mode expansions are

$$\tilde{X}(\tau,\sigma) = q_0 + \sqrt{2\alpha'}\alpha_0\sigma + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n \sin\sigma e^{-in\tau} \tag{5.43}$$

This equation is some thing like equation (4.16) in which a string is stretched from one D-brane to another D-brane, and hence, there is a correspondences between the above equation (5.43) and (4.16), and gives the idea that the coordinate $\tilde{X}$ is of DD type i.e. The end points are fixed as $\partial_\tau\tilde{X} = 0$ for the end points, $\sigma = 0$ and $\sigma = \pi$. The open string stretches when $\sigma$ goes from 0 to $\pi$ such as

$$\tilde{X}(\tau,\pi) - \tilde{X}(\tau,0) = \sqrt{2\alpha'}\alpha_0(\pi-0) = 2\pi\tilde{R}n \tag{5.44}$$

Since n will have any possible integer values, the physics behind this are that, infinite collections of D24-branes having uniform spacing $2\pi\tilde{R}$ along the compactified $X^{25}$ direction. These configurations will be physically equivalent to the single D24-brane which is at some fixed positions on the circle of radius $\tilde{R}$. We notice that the T-duality maps the Neumann boundary conditions into Dirichlet boundary conditions as

$$\partial_\sigma X = X'_L(\tau+\sigma) - X'_R(\tau-\sigma) = \partial_\tau \tilde{X}$$
$$\partial_\tau X = X'_L(\tau+\sigma) + X'_R(\tau-\sigma) = \partial_\sigma \tilde{X} \tag{5.45}$$

T-duality has transformed the open string with Neumann boundary conditions, on a circle of radius $R$ to an open string with Dirichlet boundary conditions, on a circle of radius $\tilde{R}$. Now let we have D25-brane in a word in which $k$ spatial dimensions are compactified into circles, Then the T-duality will transformation on each circle will give a physically equivalent world in which we will have a D(25 - $k$ )-brane and each circle will be replaced by a circle with dual radius. And generally, if we have a Dp-brane which is stretched around a compact dimension, T-duality along this direction will give a D($p$ - 1 )-brane at some fixed position on a circle of dual radius [7].

### 5.3.2 Open strings and Wilson lines

To study the effects and its dual picture, of open strings on D-branes having the gauge fields which are characterizing by holonomies. Let us start from a compact dimension on circle and on this, a flat potential[4] will have non trivial physical effects as like Aharanov-Bohm effect. As when the component of the gauge potential along the circle of compactified dimension takes none zero constant values, it gives the holonomy $W$ or Wilson line, which can be written as

$$W \equiv \exp(iw) = \exp\left(iq \oint dx\ A_x\right) \tag{5.46}$$

Here $A_x$ is the gauge potential along the compact direction $x$. As $w = q\oint dx\ A_x$ lives on a unit circle due to compact dimension, so it means that $w \in [0, 2\pi]$ and $w = q\oint dx\ A_x$

[4] The potential which gives the vanishing field strength

is some thing like the angle $\theta$ such that

$$\theta \equiv w = q \oint dx\ A_x \tag{5.47}$$

Since the Wilson line $W \equiv \exp(i\theta)$ is a gauge invariant and the $\theta$ gauge equivalents will gives same holonomy $W$. For a constant gauge potential $A_x$ the above equation becomes

$$q\ A_x = \frac{\theta}{2\pi R} \tag{5.48}$$

Now let us consider the open strings which have the opposite charges on the end points, so that the string will act as a neutral and the Wilson line will be no effects. If a D-brane wraps around the compact dimension then the mass-squared operators can be written as

$$M^2 = p^2 + \frac{1}{\alpha'}(N^{\perp} - 1)\ ,\ p = \frac{l}{R} \tag{5.49}$$

In case of open string which having opposite charges on their end points and which lies on the same Dp-brane then shifts in momentum [5] is as $p - qA_x + qA_x = p$. Now lets us consider the two Dp-branes and the string stretching between these two, as each Dp-branes has own Maxwell field. Let the negatively charged end points lies on first Dp-brane and the positively charged end points lies on the second Dp-branes then the momentum will be shifting from $p$ to $p - qA_1 + qA_2$ and can be written explicitly as

$$\frac{l}{R}\ \rightarrow\ \frac{l}{R} - \frac{\theta_2}{2\pi R} + \frac{\theta_1}{2\pi R} \tag{5.50}$$

The mass squared operator then becomes as

$$M^2 = \left(\frac{2\pi l - (\theta_2 - \theta_1)}{2\pi R}\right)^2 + \frac{1}{\alpha'}(N^{\perp} - 1)\ ,\ l \in Z \tag{5.51}$$

If $\theta_1 = \theta_2$ then as a result the effects of holonomies will be cancel out and the equation will be reduces to equation (5.49). The T-duality picture of the two Dp-branes having the different parameters of $\theta$ will be consisting the two D($p$ - 1 )-branes having the different (angular) positions due to the values of $\theta$ [3, 7].

[5]For point particle the addition of Wilson line make a shift in $p$ as $p$-$q$A.

## 5.4 Electromagnetic fields on D-branes and T-duality

### 5.4.1 Maxwell fields coupling to open strings

In the presence of background Maxwell fields, the string end points will couple to Maxwell potential $A_m$. As we have written this coupling terms in equation (4.59), adding this to the string action

$$S = \int d\tau d\sigma L(\dot{X}, X') + \int d\tau \ \mathrm{A}_m(X)\partial_\tau X^m|_{\sigma=\pi} - \int d\tau \ \mathrm{A}_m(X)\partial_\tau X^m|_{\sigma=0} \qquad (5.52)$$

Now let us consider those background fields which has constant electromagnetic field[6] strength $F_{mn}$, for this let define the gauge potential

$$\mathrm{A}_m(x) = \frac{1}{2}F_{mn}x^m \qquad (5.53)$$

Putting this gauge potential in the above action (5.52) and then by using the variation, we find the boundary conditions as

$$\mathcal{P}^\sigma_m + F_{mn}\partial_\tau X^n = 0, \ \sigma = 0, \pi \qquad (5.54)$$

We get this by taking the usual Dirichlet boundary condition $\delta X^a = 0$ for the coordinates normal to the brane. There are the background electromagnetic fields which changed the boundary conditions. Now using the explicit form of the $\mathcal{P}^\sigma_m$ and simplifying, we find

$$\partial_\sigma X_m - 2\pi\alpha' F_{mn}\partial_\tau X^n = 0, \ \sigma = 0, \pi \qquad (5.55)$$

This is the boundary condition for open string in the case of background field.

### 5.4.2 D-branes with Electric fields and T-dualities

Let us consider a D-brane which wraps around $x^{25}$ compact dimension and carrying constant electric fields along $x^{25}$ such that

$$F_{25,0} = E_{25} \equiv E \qquad (5.56)$$

[6]The background field will be purely electric if $F_{0i}(= -F_{i0})$ and purely magnetic if $F_{ij}$.

Since the boundary conditions (5.55) becomes as

$$\partial_\sigma X_0 - 2\pi\alpha' F_{25,0}\partial_\tau X^{25} = 0$$

$$\partial_\sigma X_2 5 - 2\pi\alpha' F_{25,0}\partial_\tau X^0 = 0 \tag{5.57}$$

As $X_0 = -X^0$ and let $X^{25} \equiv X$ then the above equations becomes

$$\partial_\sigma X^0 - \mathcal{E}\partial_\tau X = 0$$

$$\partial_\sigma X - \mathcal{E}\partial_\tau X^0 = 0 \tag{5.58}$$

Here $\mathcal{E}$ is the dimensionless electric field

$$\mathcal{E} \equiv 2\pi\alpha' E \tag{5.59}$$

Now writing the above equations (5.58) in a more beautiful way, for this let $\partial_\pm \equiv \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$ and by simplifying, we get

$$\partial_+ \begin{pmatrix} X^0 \\ X \end{pmatrix} = \begin{pmatrix} \frac{1+\mathcal{E}^2}{1-\mathcal{E}^2} & \frac{2\mathcal{E}}{1-\mathcal{E}^2} \\ \frac{2\mathcal{E}}{1-\mathcal{E}^2} & \frac{1+\mathcal{E}^2}{1-\mathcal{E}^2} \end{pmatrix} \partial_- \begin{pmatrix} X^0 \\ X \end{pmatrix} \tag{5.60}$$

This is the desired form of the boundary conditions (5.58) in the form of matrix. And the duality relations becomes

$$\partial_+ X = \partial_+ \tilde{X} \ , \ \partial_- X = -\partial_- \tilde{X} \tag{5.61}$$

In the order to find a dual description, let us consider a D($p$ - 1 )-brane which is moving with a constant velocity along the compact dimension. Let we have two frame of reference S and $S'$. The $S'$ is the rest frame of D($p$ - 1 )-brane while the relative parameter boost of these two are $\beta = v/c$, where is the speed of moving brane. Let the string coordinates in $S'$ are $X'^0$ and $\tilde{X}'$ then the Lorentz transformation can be written as

$$X'^0 = \gamma(X^0 - \beta\tilde{X})$$

$$\tilde{X}' = \gamma(-\beta X^0 + \tilde{X}) \tag{5.62}$$

Now using the duality relations and simplifying, we find

$$\partial_+ \begin{pmatrix} X^0 \\ X \end{pmatrix} = \begin{pmatrix} \frac{1+\beta^2}{1-\beta^2} & \frac{2\beta}{1-\beta^2} \\ \frac{2\beta}{1-\beta^2} & \frac{1+\beta^2}{1-\beta^2} \end{pmatrix} \partial_- \begin{pmatrix} X^0 \\ X \end{pmatrix} \tag{5.63}$$

These are the boundary conditions for the dual D($p$ - 1 )-brane. Comparing these to equation (5.60) then we get

$$\mathcal{E} = 2\pi\alpha' E = \beta \tag{5.64}$$

This shows that a moving D(p - 1 )-brane with the boost parameter $\beta = v/c$ is actually the T-dual to the Dp-brane which is wrapped around on the dual circle having the electric field $\mathcal{E} = \beta$ along in the direction of circle. Since the T-duality relates the Dp-brane with electric field is physically equal to the moving D($p$ - 1 )-brane with no electric field [7, 28].

### 5.4.3 D-branes with Magnetic fields and T-dualities

Now we will consider the magnetic fields in background. For this, let a Dp-brane having its two directions lies on $(x^2, \tilde{x}^3)$ plane, $\tilde{x}^3$ is compactified to a circle of radius $\tilde{R}_3$ such that both $x^2$ and $\tilde{x}^3$ give a cylinder of circumference of $2\pi\tilde{R}_3$. Let the open string coordinates are $X^2$ and $\tilde{X}^3$, which are Neumann. After T-duality on $\tilde{X}^3$, gives $X^3$ as a Dirichlet, since the dual picture is now a D($p$ - 1 )-brane which is stretch along $X^2$ at some fixed position $X^3 = 0$. Now the magnetic field on this Dp-brane is let $F_{23} = B$. For this the boundary conditions (5.55) becomes as

$$\begin{aligned} \partial_\sigma X^2 - \mathcal{B}\partial_\tau \tilde{X}^3 &= 0 \\ \partial_\sigma \tilde{X}^3 + \mathcal{B}\partial_\tau X^2 &= 0 \end{aligned} \tag{5.65}$$

Here $\mathcal{B}$ is the dimensionless electric field

$$\mathcal{B} \equiv 2\pi\alpha' B \tag{5.66}$$

And similarly we can write these boundary conditions as

$$\partial_+ \begin{pmatrix} X^2 \\ \tilde{X}^3 \end{pmatrix} = \begin{pmatrix} \frac{1-\mathcal{B}^2}{1+\mathcal{B}^2} & \frac{2\mathcal{B}}{1+\mathcal{B}^2} \\ -\frac{2\mathcal{B}}{1+\mathcal{B}^2} & \frac{1-\mathcal{B}^2}{1+\mathcal{B}^2} \end{pmatrix} \partial_- \begin{pmatrix} X^2 \\ \tilde{X}^3 \end{pmatrix} \tag{5.67}$$

These are the boundary condition in the presence of background magnetic fields. In the case of dual picture, let a D($p$-1)-brane which have tilted on the cylinder, then the boundary conditions can be written in the form of $X'^2$ and $X'^3$ that is rotated by an angle $\alpha$ . The and are Neumann and Dirichlet and the proper rotation can be written as

$$\begin{pmatrix} X'^2 \\ X'^3 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} X^2 \\ X^3 \end{pmatrix} \tag{5.68}$$

Using the duality relation and simplifying, we get

$$\partial_+ \begin{pmatrix} X^2 \\ \tilde{X}^3 \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix} \partial_- \begin{pmatrix} X^2 \\ \tilde{X}^3 \end{pmatrix} \tag{5.69}$$

Now comparing this result with the boundary conditions (5.67), first let the diagonals elements which gives

$$\frac{1-\mathcal{B}^2}{1+\mathcal{B}^2} = \cos 2\alpha \tag{5.70}$$

Simplifying this, we get $\mathcal{B} = \pm\tan\alpha$, we will take the negative sign such as

$$\mathcal{B} = 2\pi\alpha' B = -\tan\alpha \tag{5.71}$$

This equation gives that the zero magnetic field cannot rotate the D-brane and it will required an infinite magnetic field to rotate it. But putting the $\mathcal{B} = -\tan\alpha$, we can easily confirm the off diagonals. In equation (5.71) the negative sign is necessary for the confirmation of the off diagonals elements. Since the titled D-brane is the dual picture of the D-brane which have magnetic fields [7].

## 5.5 String coupling and the dilaton

The massless scalar field dilaton has an interesting property such as its expectation value can controls the coupling of string and this coupling is a dimensionless number which can set the strength of the interactions of the strings.

In string theory, the string coupling is not a constant and can be written in the form of dilaton field. The closed string coupling $g_s$ can be evaluated from the dilaton field $\phi(x)$, such as

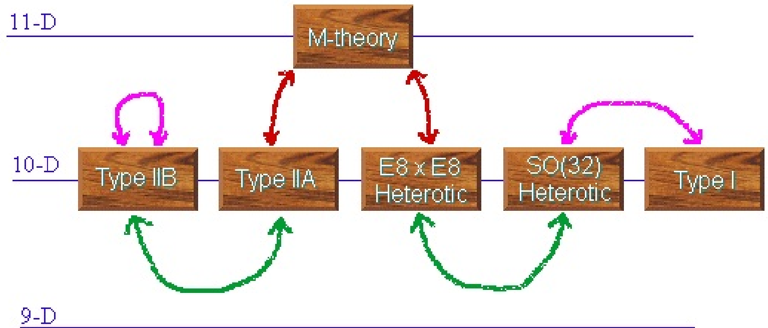
$$g_s \sim e^{\phi}$$

The string coupling $g_s$ is the not an adjustable parameter but a dynamical parameter in string theory. The dynamical nature of coupling is an ideal property for the unification of all the other interactions. Whenever the string coupling is small then the amplitudes for the string interactions can be easily calculable by using the Riemann surfaces. The Riemann surfaces can also allow us to understand the infinites which come in the amplitudes of general relativity [3, **?**].

## 5.6 S-duality

Another kind of duality which is called S-duality which relates the string coupling $g_s$ to $1/g_s$ as likes the T-duality which relates the radius $R$ to $\alpha' R^{-1}$. The S-duality relates the type I superstring theory to the SO(32) Heterotic superstring theory, and the type IIB superstring theory to itself. As when the coupling $g_s$ is small then we use the perturbation theory but whenever the coupling $g_s$ is large then we can use the S-duality i.e. the large coupling $g_s$ of the type I superstring theory is equivalent to the weak coupling $g_s$ of SO(32) Heterotic theory. S-duality tell us that how these three superstring theories behave at strong coupling but when the coupling is large, in Type IIA and $E_8 \times E_8$ Heterotic, then both of them developed an eleventh dimension of size the $l_s g_s$ . The 11th dimension is a circle in the type IIA and a line interval in the Heterotic[7].

[7]Under the T-duality, one can get easily $\tilde{g}_s = \frac{\sqrt{\alpha'}}{R} g_s$.




M-theory is a new type of quantum theory which lives in 11 dimensions of spacetime. At the low energies, it is approximately equivalent to supergravity, a classical field theory lives in 11-dimensions of space time, but the M-theory is much more than supergravity [3, 7, 30, 31, 32].

# Appendix A

# Appendix A

# The Massless States Of Closed String

As the massless level of closed string, the general state of fixed momentum. We write it as

$$\sum_{l,j} R_{lj} a_1^{l\dagger} \bar{a}_1^{j\dagger} \left| p^+, \vec{p}_T \right\rangle$$

Here $R_{lj}$ are the elements of an arbitrary square matrix of the size $(D-2)$. Any square matrix can be decomposed into its symmetric part and its antisymmetric part as

$$R_{lj} = \frac{1}{2}(R_{lj} + R_{jl}) + \frac{1}{2}(R_{lj} - R_{jl}) \equiv S_{lj} + A_{lj}$$

And the symmetric part $S_{lj}$ can be written as

$$S_{lj} = \left( S_{lj} - \frac{1}{D-2}\delta_{lj} S \right) + \frac{1}{D-2}\delta_{lj} S$$

with $S \equiv S^{ll} = \delta^{lj} S_{lj}$ The 1st term is traceless as

$$\delta^{lj} \left( S_{lj} - \frac{1}{D-2}\delta_{lj} S \right) = S - \frac{1}{D-2}\delta^{lj}\delta_{lj} S = 0$$

Hence $R_{lj}$ is decomposed into a traceless matrix plus a multiple of the unit matrix. Let say the traceless part of $R_{lj}$ is $\hat{S}_{lj}$ and $S' = \frac{S}{D-2}$ then we have

$$R_{lj} = \hat{S}_{lj} + A_{lj} + S'\delta_{lj}$$

Hence we decomposed the Matrix $R_{lj}$ into a symmetric-traceless part, an antisymmetric and a traceless part. Since we split the total massless states into three linear independent states as

$$\sum_{l,j} \hat{S}_{lj} a_1^{l\dagger} \bar{a}_1^{j\dagger} \left|p^+, \vec{p}_T\right\rangle \tag{A-1}$$

$$\sum_{l,j} A_{lj} a_1^{l\dagger} \bar{a}_1^{j\dagger} \left|p^+, \vec{p}_T\right\rangle \tag{A-2}$$

$$S' a_1^{l\dagger} \bar{a}_1^{j\dagger} \left|p^+, \vec{p}_T\right\rangle \tag{A-3}$$

The equation (A-1) is similar to the quantum theory of the free gravitational field states. Hence equation (A-1) represents Graviton states. The equation (A-2) corresponds to the one-particle states of the Kalb-Ramond field, an antisymmetric tensor field $B_{\mu\nu}$ with two indices. The equation (A-3) corresponds to a one-particle state of a massless scalar field. This field is called the dilaton [7].

# Appendix B

# Appendix B

# The Spinors Algebra In 2D

As the Clifford algebra satisfies

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}$$

In 2D it gives

$$\Gamma_1^2 = \eta_{11} \ , \ \Gamma_2^2 = \eta_{22} \ , \ \Gamma_1\Gamma_2 + \Gamma_2\Gamma_1 = 0.$$

Lorentizian:

In this case we have

$$\Gamma_1^2 = +1 \ , \ \Gamma_2^2 = -1 \ , \ \Gamma_1\Gamma_2 + \Gamma_2\Gamma_1 = 0$$

Now we can choose the representation of these operators as a matrices such that

$$\Gamma_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \Gamma_2 = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

And the generator of the $spin(1,1)$ is then given by

$$\Gamma_{12} = \frac{1}{4}\left[\Gamma_1, \Gamma_2\right] = \frac{1}{2}\Gamma_1\Gamma_2 = \frac{1}{2}\sigma_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix},$$

with

$$e^{\gamma\Gamma_{12}} = e^{\frac{\gamma 1}{2}\sigma_3} = \begin{pmatrix} e^{\frac{\gamma}{2}} & 0 \\ 0 & e^{-\frac{\gamma}{2}} \end{pmatrix}.$$

Given a vector $v = v_1\Gamma_1 + v_2\Gamma_2$ the Lorentz transformation is then given by

$$v \mapsto e^{\gamma\Gamma_{12}} v e^{-\gamma\Gamma_{12}},$$

By simplifying, we get

$$\mapsto \begin{pmatrix} 0 & e^{+\gamma(v_1+v_2)} \\ e^{-\gamma(v_1-v_2)} & 0 \end{pmatrix}.$$

And hence

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Thus the Clifford algebra generated by $\{1, \sigma_1, -i\sigma_2, \sigma_3\}$ is the same as a $2 \times 2$ real matrices and the even part of the Clifford algebra $Cl(1,1)$ is generated by 1 and $\sigma_3$ both of which are the diagonal matrices and therefore $Cl(1,1)_{even} \cong \Re \otimes \Re$. Because the $\Gamma_{12}$ is real and diagonal therefore the irreducible representations of $spin(1,1)$ are one dimensional (Weyl) and real (Majorana). They transform as

$$\psi^+ \mapsto e^{+\frac{\gamma}{2}}\psi \text{ (Positive chirality)}$$

$$\psi^- \mapsto e^{-\frac{\gamma}{2}}\psi \text{(Negative chirality)}$$

Euclidean:

In the case of Euclidean, we have

$$\Gamma_1^2 = +1 \ , \ \Gamma_2^2 = +1 \ , \ \Gamma_1\Gamma_2 + \Gamma_2\Gamma_1 = 0$$

Now by the similar way we can choose the representations of these operators as matrices

$$\Gamma_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \Gamma_2 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The generator of $spin(2)$ is then given by

$$\Gamma_{12} = \frac{1}{4}[\Gamma_1, \Gamma_2] = \frac{1}{2}\Gamma_1\Gamma_2 = \frac{-i}{2}\sigma_2 = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix},$$

With

$$e^{\gamma\Gamma_{12}} = e^{-i\frac{\gamma}{2}\frac{1}{2}\sigma_2} = \begin{pmatrix} \cos(\frac{\gamma}{2}) & -\sin(\frac{\gamma}{2}) \\ \sin(\frac{\gamma}{2}) & \cos(\frac{\gamma}{2}) \end{pmatrix}.$$

Given a vector $v = v_1\Gamma_1 + v_2\Gamma_2$ then the $spin(2)$

$$v \mapsto e^{\gamma\Gamma_{12}} v e^{-\gamma\Gamma_{12}},$$

By simplifying, we get

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Thus the Clifford algebra generated by $\{1, \sigma_1, -i\sigma_2, \sigma_3\}$ is the same as a $2 \times 2$ real matrices and the even part of the Clifford algebra $Cl(2)$ is generated by 1 and $\Gamma_1\Gamma_2$ since $(\Gamma_1\Gamma_2)^2 = -1$ therefore $Cl(1,1)_{even} \cong C$ and the spinors transforms as

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos(\frac{\gamma}{2}) & -\sin(\frac{\gamma}{2}) \\ \sin(\frac{\gamma}{2}) & \cos(\frac{\gamma}{2}) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

We can also form the irreducible representation by taking

$$\psi_+ = \psi_1 + i\psi_2,\ \psi_- = \psi_1 - i\psi_2.$$

$\psi_\pm$ transform under $Spin(2)$ as

$$\psi_+ \mapsto e^{+i\frac{\gamma}{2}}\psi_+ (\text{Positive chirality})$$

$$\psi_- \mapsto e^{-i\frac{\gamma}{2}}\psi_- (\text{Negative chirality})$$

The chirality is the eigenvalue under

$$\Gamma_3 = \Gamma_1\Gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto \Gamma_3 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} -\psi_2 \\ +\psi_1 \end{pmatrix}$$

and

$$\psi_+ \mapsto i\psi_+,\, \psi_- \mapsto -i\psi_-.$$

Thus in both Lorentzian and Euclidean signature, we have chiral (Weyl) spinors. The supersymmetry generators are spinors and in the two dimensions we can choose $p$ Weyl spinors of positive chirality and $q$ Weyl of negative chirality. This gives us $(p, q)$ supersymmetry in the two dimensions [33, 34, 35].

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