A Geometric Proof that e is Irrational and a New Measure of its Irrationality

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1. INTRODUCTION. While there exist geometric proofs of irrationality for $\sqrt{2}$ [**2**], [**27**], no such proof for e, π, or $\ln 2$ seems to be known. In section 2 we use a geometric construction to prove that e is irrational. (For other proofs, see [**1**, pp. 27-28], [**3**, p. 352], [**6**], [**10**, pp. 78-79], [**15**, p. 301], [**16**], [**17**, p. 11], [**19**], [**20**], and [**21**, p. 302].) The proof leads in section 3 to a new measure of irrationality for e, that is, a lower bound on the distance from e to a given rational number, as a function of its denominator. A connection with the greatest prime factor of a number is discussed in section 4. In section 5 we compare the new irrationality measure for e with a known one, and state a number-theoretic conjecture that implies the known measure is almost always stronger. The new measure is applied in section 6 to prove a special case of a result from [**24**], leading to another conjecture. Finally, in section 7 we recall a theorem of G. Cantor that can be proved by a similar construction.

2. PROOF. The irrationality of e is a consequence of the following construction of a nested sequence of closed intervals I_n. Let $I_1 = [2,3]$. Proceeding inductively, divide the interval I_{n-1} into n (≥ 2) equal subintervals, and let the second one be I_n (see Figure 1). For example, $I_2 = \left[\frac{5}{2!},\frac{6}{2!}\right]$, $I_3 = \left[\frac{16}{3!},\frac{17}{3!}\right]$, and $I_4 = \left[\frac{65}{4!},\frac{66}{4!}\right]$.

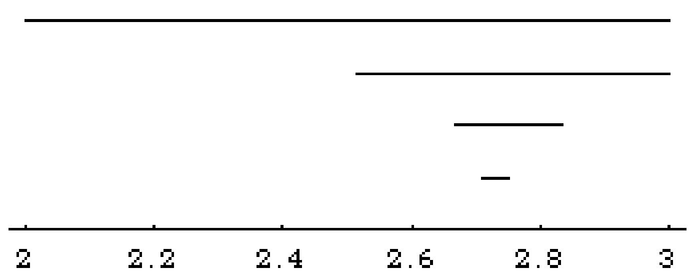

Figure 1. The intervals I_1, I_2, I_3, I_4.

The intersection

$$\bigcap_{n=1}^{\infty} I_n = \{e\} \tag{1}$$

is then the geometric equivalent of the summation (see the Addendum)

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e\,. \tag{2}$$

When $n > 1$ the interval I_{n+1} lies strictly between the endpoints of I_n, which are $\frac{a}{n!}$ and $\frac{a+1}{n!}$ for some integer $a = a(n)$. It follows that the point of intersection (1) is not a fraction with denominator $n!$ for any $n \geq 1$. Since a rational number p/q with $q > 0$ can be written

$$\frac{p}{q} = \frac{p \cdot (q-1)!}{q!}, \tag{3}$$

we conclude that e is irrational. •

Question. The nested intervals I_n intersect in a number—let's call it b. It is seen by the Taylor series (2) for e that $b = e$. Using only standard facts about the natural logarithm (including its definition as an integral), but *not* using any series representation for log, can one see directly from the given construction that $\log b = 1$?

3. A NEW IRRATIONALITY MEASURE FOR *e*. As a bonus, the proof leads to the following measure of irrationality for e.

Theorem 1. *For all integers p and q with* $q > 1$

$$\left| e - \frac{p}{q} \right| > \frac{1}{(S(q)+1)!}, \tag{4}$$

where $S(q)$ *is the smallest positive integer such that* $S(q)!$ *is a multiple of* q.

For instance, $S(q) = q$ if $1 \leq q \leq 5$, while $S(6) = 3$. In 1918 A. J. Kempner [**13**] used the prime factorization of q to give the first algorithm for computing

$$S(q) = \min\{k > 0 : q \mid k!\} \tag{5}$$

(the so-called Smarandache function [**28**]). We do not use the algorithm in this note.

Proof of Theorem 1. For $n > 1$ the left endpoint of I_n is the closest fraction to e with denominator not exceeding $n!$. Since e lies in the interior of the second subinterval of I_n,

$$\left| e - \frac{m}{n!} \right| > \frac{1}{(n+1)!} \tag{6}$$

for any integer m. Now given integers p and q with $q > 1$, let $m = p \cdot S(q)!/q$ and $n = S(q)$. In view of (5), m and n are integers. Moreover,

$$\frac{p}{q} = \frac{p \cdot S(q)!/q}{S(q)!} = \frac{m}{n!}. \tag{7}$$

Therefore, (6) implies (4). •

As an example, take q to be a prime. Clearly, $S(q) = q$. In this case, (4) is the (very weak) inequality

$$\left| e - \frac{p}{q} \right| > \frac{1}{(q+1)!}. \tag{8}$$

In fact, (4) implies that (8) holds for *any* integer q larger than 1, because $S(q) \le q$ always holds. But (4) is an improvement of (8), just as (7) is a refinement of (3).

Theorem 1 would be false if we replaced the denominator on the right side of (4) with a smaller factorial. To see this, let p/q be an endpoint of I_n, which has length $\frac{1}{n!}$. If we take $q = n!$, then since evidently

$$S(n!) = n \tag{9}$$

and e lies in the interior of I_n,

$$\left| e - \frac{p}{q} \right| < \frac{1}{S(q)!}. \tag{10}$$

(If $q < n!$, then (10) still holds, since $n > 2$, so p/q is not an endpoint of I_{n-1}, hence $S(q) = n$.)

4. THE LARGEST PRIME FACTOR OF *q*. For $q \ge 2$ let $P(q)$ denote the largest prime factor of q. Note that $S(q) \ge P(q)$. Also, $S(q) = P(q)$ if and only if $S(q)$ is prime. (If $S(q)$ were prime but greater than $P(q)$, then since q divides $S(q)!$, it would also divide $(S(q)-1)!$, contradicting the minimality of $S(q)$.)

P. Erdős and I. Kastanas [**9**] observed that

$$S(q) = P(q) \qquad \text{(almost all } q\text{)}. \tag{11}$$

(Recall that a claim C_q is true *for almost all* q if the counting function $N(x) = \#\{q \le x : C_q \text{ is false}\}$ satisfies the asymptotic condition $N(x)/x \to 0$ as $x \to \infty$.) It follows that Theorem 1 implies an irrationality measure for e involving the simpler function $P(q)$.

Corollary 1. *For almost all q, the following inequality holds with any integer p*:

$$\left| e - \frac{p}{q} \right| > \frac{1}{(P(q)+1)!}. \tag{12}$$

When q is a factorial, the statement is more definite.

Corollary 2. *Fix* $q = n! > 1$. *Then* (12) *holds for all p if and only if n is prime.*

Proof. If n is prime, then $P(q) = n$, so (4) and (9) imply (12) for all p. Conversely, if n is composite, then $P(q) < n$, and (10) shows that (12) fails for certain p. •

Thus when $q > 1$ is a factorial, (12) is true for all p if and only if $S(q) = P(q)$. To illustrate this, take $\frac{p}{q} = \frac{65}{4!}$ to be the left endpoint of I_4. Then $P(q) = 3 < 4 = S(q)$, and (12) does not hold, although of course (4) does:

$$0.00833\ldots = \frac{1}{5!} < \left| e - \frac{65}{24} \right| = 0.00994\ldots < \frac{1}{4!} = 0.04166\ldots .$$

5. A KNOWN IRRATIONALITY MEASURE FOR *e*. The following measure of irrationality for e is well known: *given any* $\varepsilon > 0$ *there exists a positive constant* $q(\varepsilon)$ *such that*

$$\left| e - \frac{p}{q} \right| > \frac{1}{q^{2+\varepsilon}} \tag{13}$$

for all p and q with $q \geq q(\varepsilon)$. This follows easily from the continued fraction expansion of e. (See, for example, [**23**]. For sharper inequalities than (13), see [**3**, Corollary 11.1], [**4**], [**7**], [**10**, pp. 112-113], and especially the elegant [**26**].)

Presumably, (13) is usually stronger than (4). We state this more precisely, and in a number-theoretic way that does not involve e.

Conjecture 1. *The inequality* $q^2 < S(q)!$ *holds for almost all* q. *Equivalently,* $q^2 < P(q)!$ *for almost all* q.

(The equivalence follows from (11).) This is no doubt true; the only thing lacking is a proof. (Compare [**12**], where A. Ivić proves an asymptotic formula for the counting function $N(x) = \#\{q \leq x : P(q) < S(q)\}$ and surveys earlier work, including [**9**].)

Conjecture 1 implies that (13) is almost always a better measure of irrationality for e than those in Theorem 1 and Corollary 1. On the other hand, Theorem 1 applies to all $q > 1$. Moreover, (4) is stronger than (13) for certain q. For example, let $q = n!$ once more. Then (4) and (9) give (6), which is stronger than (13) if $n > 2$, since

$$(n+1)! < (n!)^2 \qquad (n \geq 3). \tag{14}$$

6. PARTIAL SUMS VS. CONVERGENTS. Theorem 1 yields other results on rational approximations to e [**24**]. One is that *for almost all n, the n-th partial sum* s_n *of series* (2) *for e is* not *a convergent to the simple continued fraction for e*. Here $s_0 = 1$ and s_n is the left endpoint of I_n for $n \geq 1$. (In 1840 J. Liouville [**14**] used the partial sums of the Taylor series for e^2 and e^{-2} to prove that the equation $ae^2 + be^{-2} = c$ is impossible if a, b, and c are integers with $a \neq 0$. In particular, e^4 is irrational.)

Let q_n be the denominator of s_n in lowest terms. When $q_n = n!$ (see [**22**, sequence A102470]), the result is more definite, and the proof is easy.

Corollary 3. *If* $q_n = n!$ *with* $n \geq 3$, *then* s_n *cannot be a convergent to* e.

Proof. Use (4), (9), (14), and the fact that every convergent satisfies the reverse of inequality (13) with $\varepsilon = 0$ [**10**, p. 24], [**17**, p. 61]. •

When $q_n < n!$ (for example, $q_{19} = 19!/4000$—see [**22**, sequence A093101]), another argument is required, and we can only prove the assertion for almost all n. However, numerical evidence suggests that much more is true.

Conjecture 2. *Only two partial sums of series* (2) *for e are convergents to e, namely,* $s_1 = 2$ *and* $s_3 = 8/3$.

7. CANTOR'S THEOREM. A generalization of the construction in section 2 can be used to prove the following result of Cantor [**5**].

Theorem 2. *Let* $a_0, a_1, \ldots$ *and* $b_1, b_2, \ldots$ *be integers satisfying the inequalities* $b_n \geq 2$ *and* $0 \leq a_n \leq b_n - 1$ *for all* $n \geq 1$. *Assume that each prime divides infinitely many of the* b_n. *Then the sum of the convergent series*

$$a_0 + \frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \frac{a_3}{b_1 b_2 b_3} + \cdots$$

is irrational if and only if both $a_n > 0$ *and* $a_n < b_n - 1$ *hold infinitely often*.

For example, series (2) for e and all subseries (such as $\Sigma_{n \geq 0} \frac{1}{(2n)!} = \cosh 1$ and $\Sigma_{n \geq 0} \frac{1}{(2n+1)!} = \sinh 1$) are irrational, but the sum $\Sigma_{n \geq 1} \frac{n-1}{n!} = 1$ is rational.

An exposition of the "if" part of Cantor's theorem is given in [**17**, pp. 7-11]. For extensions of the theorem, see [**8**], [**11**], [**18**], and [**25**].

ADDENDUM. Here are some details on why the nested closed intervals I_n constructed in section 2 have intersection e. Recall that $I_1 = [2, 3]$, and that for $n \geq 2$ we get I_n from I_{n-1} by cutting it into n equal subintervals and taking the second one. The left-hand endpoints of $I_1, I_2, I_3, \ldots$ are $2, 2 + \frac{1}{2!}, 2 + \frac{1}{2!} + \frac{1}{3!}, \ldots$, which are also partial sums of the series (2) for e. Since the endpoints approach the intersection of the intervals, whose lengths tend to zero, the intersection is the single point e.

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