Convergence sums and the derivative of a sequence at infinity

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Abstract

For convergence sums, by threading a continuous curve through a monotonic sequence, a series difference can be made a derivative. Series problems with differences can be transformed and solved in the continuous domain. At infinity, a bridge between the discrete and continuous domains is made. Stolz theorem at infinity is proved. Alternating convergence theorem for convergence sums is proved.

1 Introduction

There have always been relationships between series with discrete change and integrals with continuous change. In solving both problems and proofs we observe similarities and differences.

Series have no chain rule. However, for monotonic sequences satisfying the convergence sums criteria we can construct a continuous function at infinity where the chain rule can be applied. This can be combined with convergence sums integral test.

In topology, a coffee cup can be transformed by stretching into a donut. Similarly, we can consider a monotonic sequence which by stretching deforms into a strictly monotonic sequence.

Consider a positive monotonic continuous function and its integral at infinity. Provided that the function's plateaus do not sum to infinity, the integral has the same convergence or divergence as the strictly deformed function's integral.

Since convergence sums are monotonic, and can be deformed to be strictly monotonic, the correlation between the series and integrals can be coupled in a way that results in a non-zero derivative. The derivative of a sequence follows.

We believe the derivative of a sequence significantly changes convergence testing by allowing an interchange between sums and integrals with the integral theorem via sequences and functions in a fluid way.

At infinity with infinireals we provide a classical explanation of a geometric construction of a curve threaded through a sequence of points (see Figure 1).

This simplicity explains what can be highly technical arguments on integer sums and theorems, which are not transferable between sums and integrals. The mirrored discrete formula may use integer arguments in the proof specific to number theory whereas the continuous formula may be proved again by altogether different means. Never shall they meet.

We again find that the acceptance of infinity, be it initially disturbing compared with classical arguments, ends up augmenting, upgrading or replacing them.

The derivative of a sequence is a bridge between the continuous and discrete convergence sums at infinity.

2 Derivative at infinity

When solving problems with sequences, there is no chain rule for sequences, as there is for the continuous variable. However, forward and backward differences are used in numerical analysis to calculate derivatives in the continuous domain.

In the discrete domain of integers, sequences, by contrast may use an equivalent theorem such as Stolz theorem or Cauchy's condensation test, as an effective chain rule.

If we consider a calculus of sequences, the change is an integer change, hence the goal is to construct a derivative that has meaning there.

Consider the following example which motivates the possibility of having a derivative at infinity, by constructing a derivative with powers at infinity.

Since a function can be represented by a power series, we now can convert between a difference and a derivative at infinity. This uses non-reversible arithmetic.

Example 2.1. *Let* $f(x) = x^2$. $f(x+1) - f(x)|_{x=\infty} = (x+1)^2 - x^2|_{x=\infty} = x^2 + 2x + 1 - x^2|_{x=\infty}$ $= 2x + 1|_{x=\infty} = 2x|_{x=\infty} = f'(x)$, *as* $2x \succ 1|_{x=\infty}$.

Lemma 2.1. *Generalizing the derivative of a power at infinity. If* $f(x) = x^p|_{x=\infty}$ *then* $\frac{df}{dx} = f(x+1) - f(x)|_{x=\infty}$

Proof. $f(x+1) - f(x)|_{x=\infty} = (x+1)^p - x^p|_{x=\infty} = (x^p + \binom{p}{1}x^{p-1} + \binom{p}{2}x^{p-2} + \ldots) - x^p|_{x=\infty}$ $= px^{p-1}|_{x=\infty}$, as $x^{k+1} \succ x^k|_{x=\infty}$. $\square$

Example 2.2. *Find the derivative of* $\sin x$. *Since* $\sin x$ *behaves the same as it does for finite values as it does at infinity, take the difference at infinity. Let* $f(x) = \sin x$. $f(x+1) - f(x)|_{x=\infty} = \sin(x+1) - \sin x|_{x=\infty} = ((x+1) - \frac{1}{3!}(x+1)^3 + \frac{1}{5!}(x+1)^5 - \ldots)\ -(x -$

$\frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots)|_{x=\infty} = 1 + (-\frac{1}{3!}(x+1)^3 + \frac{1}{5!}(x+1)^5 - \ldots) + (\frac{1}{3!}x^3 - \frac{1}{5!}x^5 + \ldots)|_{x=\infty}$ $= 1 + \sum_{k=1}^{\infty}(-1)^k(\frac{1}{(2k+1)!}(x+1)^{2k+1} - \frac{1}{(2k+1)!}x^{2k+1})|_{x=\infty}$

Consider $\frac{1}{(2k+1)!}(x+1)^{2k+1}|_{x=\infty}$*, taking the two most significant terms,* $\frac{1}{(2k+1)!}(x+1)^{2k+1}|_{x=\infty}$ $= \frac{1}{(2k+1)!}x^{2k+1} + \frac{1}{(2k+1)!}\binom{2k+1}{1}x^{2k}|_{x=\infty} = \frac{1}{(2k+1)!}x^{2k+1} + \frac{1}{(2k)!}x^{2k}|_{x=\infty}$

Substituting the expression into the previous sum, $f(x+1)-f(x)|_{x=\infty} = 1+\sum_{k=1}^{\infty}(-1)^k(\frac{1}{(2k+1)!}x^{2k+1} + \frac{1}{(2k)!}x^{2k} - \frac{1}{(2k+1)!}x^{2k+1})|_{x=\infty} = 1 + \sum_{k=1}^{\infty}(-1)^k\frac{1}{(2k)!}x^{2k}|_{x=\infty} = \cos x|_{x=\infty}$*, since a power series* $f'(x) = \cos x$*.*

Given a function $f(x)$, we can determine its derivative at infinity by converting $f(x)$ to a power series, taking the difference, and converting from the power series back into a function.

Theorem 2.1. *When* $f(x) = \sum_{k=0}^{\infty} c_i x^i$*,*

$$\frac{df(x)}{dx} = f(x+1) - f(x)|_{x=\infty}$$

Proof. Given $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $\frac{df(x)}{dx} = \sum_{k=0}^{\infty}\frac{d}{dx}a_k x^k = \sum_{k=0}^{\infty} k a_k x^{k-1}$. Consider the difference, $f(x+1) - f(x) = \sum_{k=0}^{\infty}(a_k(x+1)^k - a_k x^k) = \sum_{k=0}^{\infty} k a_k x^{k-1} = \frac{df(x)}{dx}$, by Lemma 2.1. □

An application of the derivative at infinity is, with the comparison logic, where rather than either assume that infinitesimally close expressions are equal or using orders of higher magnitude to simplify under addition by forming a difference we can obtain the derivative. Since the derivative is a function, we have an asymptotic result.

Example 2.3. *While solving for relation z:* $f + \ln(n+1) \;\; z \;\; g + \ln n|_{n=\infty}$*,* $f + \ln(n+1) - \ln n \;\; z \;\; g|_{n=\infty}$*,* $f + \frac{d}{dn}\ln n \;\; z \;\; g|_{n=\infty}$*,* $f + \frac{1}{n} \;\; z \;\; g|_{n=\infty}$

Without the derivative at infinity, with an assumed $f \simeq g|_{n=\infty}$ logical errors in the calculation are more easily made. This can be addressed by solving using magnitude arguments and non-reversible arithmetic; however, this does not yield an asymptotic error estimate.

With the use of the sequence derivative, an asymptotic expression of the difference is formed.

The sequence derivative can be an alternative to the use of the binomial theorem. (However, if there is any doubt other well known methods such as the binomial theorem are available.)

Example 2.4. *Using the binomial theorem,* $(2n+1)^{\frac{1}{2}} - (2n)^{\frac{1}{2}}|_{n=\infty} = (2n)^{\frac{1}{2}}(1 + \frac{1}{2n})^{\frac{1}{2}} - (2n)^{\frac{1}{2}}|_{n=\infty} = (2n)^{\frac{1}{2}}(1 + \frac{1}{2}\frac{1}{2n} + \ldots - 1)|_{n=\infty} = (2n)^{\frac{1}{2}}\frac{1}{4n} = \frac{1}{2n^{\frac{1}{2}}}|_{n=\infty} = 0$

The same calculation with the derivative at infinity and non-reversible arithmetic.

$(2n+1)^{\frac{1}{2}} - (2n)^{\frac{1}{2}}|_{n=\infty} = (2n+2)^{\frac{1}{2}} - (2n)^{\frac{1}{2}}|_{n=\infty} = (2(n+1))^{\frac{1}{2}} - (2n)^{\frac{1}{2}}|_{n=\infty} = \frac{d}{dn}(2n)^{\frac{1}{2}}|_{n=\infty}$ $= \frac{1}{2}(2n)^{-\frac{1}{2}}2|_{n=\infty} = \frac{1}{2n^{\frac{1}{2}}}|_{n=\infty} = 0.$ $2n+1 = 2n+2|_{n=\infty}$

The following definitions and results are given, as logarithms are extensively used with sequences and convergence tests.

Definition 2.1. *Let* $\ln_k$ *be* k *nested log functions, by default having variable* n*.* $\ln_k = \ln(ln_{k-1})$, $\ln_0 = n$.

Definition 2.2. *Let* $L_w = \prod_{k=0}^{w} \ln_k$.

Lemma 2.2. $\frac{d}{dn}\ln_w = \frac{1}{L_{w-1}}|_{n=\infty}$

Example 2.5. *In the following comparison,* $(\ln_3(x+1) - \ln_3 x)|_{x=\infty} = \frac{d}{dx}\ln_3 x|_{x=\infty} = \frac{1}{L_2(x)}|_{x=\infty} = 0$*. In a sense this is the error term. [1, Example 2.19].*

$$
\begin{aligned}
x^{\frac{p}{p+1}} \; z \; x^{\ln_2(x)/\ln_2(x+1)}|_{x=\infty} & \qquad \text{(Solve for relation } z\text{)} \\
\ln(x^{\frac{p}{p+1}}) \; (\ln z) \; \ln(x^{\ln_2(x)/\ln_2(x+1)})|_{x=\infty} & \\
\frac{p}{p+1}\ln x \; (\ln z) \; \frac{\ln_2(x)}{\ln_2(x+1)}\ln x|_{x=\infty} & \\
p\ln_2(x+1) \; (\ln z) \; (p+1)\ln_2(x)|_{x=\infty} & \\
\ln p + \ln_3(x+1) \; (\ln_2 z) \; \ln(p+1) + \ln_3(x)|_{x=\infty} & \\
\ln p + (\ln_3(x+1) - \ln_3 x) \; (\ln_2 z) \; \ln(p+1)|_{x=\infty} & \qquad \text{(Apply derivative)} \\
\ln p \; (\ln_2 z) \; \ln(p+1)|_{x=\infty} & \\
\ln p < \ln(p+1)|_{x=\infty} & \\
\ln_2 z = <, \; z = e^{e^{<}} = < &
\end{aligned}
$$

In working with integers, it is sometimes convenient to solve the problem for real numbers, then translate back into the integer domain.

The development of a way to convert between the integer domain or the domain of sequences, and the continuous domain, is similarly beneficial. For example, converting between sums and integrals.

By threading a continuous function through a monotonic sequence, we can construct a continuous function with the monotonic properties.

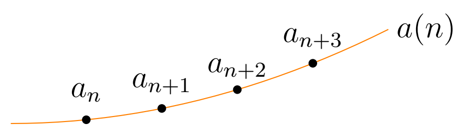

Figure 1: Monotonic function and sequence through points

Further, since a monotonic series or integral can be deformed to a strictly monotonic series or integral ([5, Theorem 3.1]), we need only consider the strictly monotonic case. By definition of the convergence Criterion E3 [3], cases where this cannot be done are said to be undefined.

Consider a positive sequence (a_n) in $*G$. Without loss of generality, let $(a_n)_{n=\infty}$ be strictly monotonic, either increasing or decreasing.

Fit a curve, with conditions: $a(n) = a_n$. Let $a(x) = \sum_{k=0}^{n} c_k x^k$, pass through $n+1$ points. Solve for (c_k).

Fitting a power series curve through a strictly monotonic sequence; the curve fitted is also strictly monotonic (within the index interval).

Since we determine convergence at infinity, we fit the curve for the sequence at infinity. Then $a(x)$ is strictly monotonic, and an analytic function. By converting a sequence difference, for example $a_{n+1} - a_n$ to the continuous power series representation, Theorem 2.1 can be applied and a derivative formed.

In solving for one domain and transferring to the other, we can bridge between sequences and continuous functions.

Definition 2.3. *Let $(a_n)|_{n=\infty}$ be a sequence at infinity and $a(n)$ a continuous function through the sequence.*

$$a_n = a(n)|_{n=\infty};\ n \in \mathbb{J}_\infty$$

Definition 2.4. *Let the derivative of a sequence at infinity be the difference of consecutive terms.*

$$a_{n+1} - a_n = \frac{da_n}{dn}|_{n=\infty} \text{ or } a_{2n+1} - a_{2n} = \frac{da_n}{dn} \text{ where } a_n|_{n=\infty} \neq \alpha \text{ a constant.}$$

How the derivative of a sequence is defined is problem dependent. It is up to the user. In a similar way we may start counting from 0 or 1. By the contiguous rearrangement theorem [5, Theorem 2.1], we need only determine one contiguous rearrangement to determine convergence or divergence.

Consider the technique of adequality [8, p.5] more generally to that of a principle of variation.

$$d(f(A)) = f(A+E) - f(A)$$

As a change in consecutive integers is 1,

$$dn = (n+1) - n$$

we can see a correspondence between a sequence derivative, and the continuous derivative.

$$d(a_n) = a_{n+1} - a_n = \frac{a_{n+1} - a_n}{1} = \frac{da_n}{dn}$$

To aid calculation, a convention of left to right equals symbol ordering is used to indicate which direction a conversion is taking place. Further, by redefining a variable from an integer to the continuous variable, will enable the transformation to be more natural and effortless.

Theorem 2.2. *By threading a continuous function $a(n)$ through sequence a_n and preserving monoticity.*

$$\frac{da_n}{dn} = \frac{da(n)}{dn}|_{n=\infty}$$

Proof. Let $a(n)$ be represented by a power series. $\frac{da_n}{dn} = a_{n+1} - a_n = a(n+1) - a(n) = \frac{da(n)}{dn}|_{n=\infty}$ by Theorem 2.1 □

Remark: 2.1. *The usefulness of the change of integers can be seen when considering the equality of the Riemann sum to the integral [6, Remark 2.1], hence discrete change has generality.*

On the assumption that $\frac{d^k a_n}{dn^k}$ can be similarly defined.

Definition 2.5. *Converting between the discrete sequence and continuous curve through the sequence, with left to right direction.*

$$f_n(a_n, \frac{da_n}{dn}, \ldots) = f(a(n), \frac{da(n)}{dn}, \ldots)|_{n=\infty} \textit{ sequence to function}$$

$$f(a(n), \frac{da(n)}{dn}, \ldots) = f_n(a_n, \frac{da_n}{dn}, \ldots)|_{n=\infty} \textit{ function to sequence}$$

Theorem 2.3. *For a strictly monotonic sequence, we can construct an associated strictly monotonic function that is continually differentiable.*

Proof. For a strictly monotonic sequence, the sequence derivative is never 0, a power series at infinity, say for N infinite number of points, solving N equations, the resulting curve is continually differentiable. □

With the interchangeability of the derivative between sequences and continuous functions, equations involving sequences can be solved as differential equations, and the result transformed back into the domain with sequences. Bridging the continuous and discrete domains at infinity.

Proposition 2.1. *If $a_{n+1} - a_n|_{n=\infty} = \alpha$, then $\frac{a_n}{n}|_{n=\infty} = \alpha$. [7, 2.3.14]*

Proof. As an alternative to the use of Stolz theorem, $a_{n+1} - a_n|_{n=\infty} = \frac{a_{n+1}-a_n}{dn}|_{n=\infty} = \alpha$, $\frac{da(n)}{dn} = \alpha$, separate the variables, $d(a(n)) = \int \alpha dn$, $a(n) = \alpha n|_{n=\infty}$, $a_n = \alpha n|_{n=\infty}$, $\frac{a_n}{n}|_{n=\infty} = \alpha$. □

Theorem 2.4. *Stolz theorem. Given sequence* $(y_n)|_{n=\infty}$ *is monotonically increasing and diverges,* $y_n|_{n=\infty} = \infty$, *and* $\frac{x_n - x_{n-1}}{y_n - y_{n-1}}|_{n=\infty} = g$, *then* $\frac{x_n}{y_n}|_{n=\infty} = g$

Proof. $\frac{x_n - x_{n-1}}{y_n - y_{n-1}}|_{n=\infty} = \frac{x_n - x_{n-1}}{dn}\frac{dn}{y_n - y_{n-1}}|_{n=\infty} = \frac{dx_n}{dn}\frac{dn}{dy_n}|_{n=\infty} = \frac{dx(n)}{dn}\frac{dn}{dy(n)}|_{n=\infty} = \frac{dx(n)}{dy(n)}|_{n=\infty} = g$, recognizing a separation of variables problem, separate and integrate the variables. $\int dx = g\int dy|_{n=\infty}$, $x(n) = gy(n)|_{n=\infty}$, $x_n = gy_n|_{n=\infty}$, $\frac{x_n}{y_n}|_{n=\infty} = g$. □

In applications with series expansions that include differences, when it is possible to arbitrarily truncate the series, apply the transforms for the new system.

Example 2.6. *Using the sequence derivative with a* $\sin$ *expansion.* $0 < a_1 < 1$, $a_{n+1} = \sin a_n$, *Show* $n^{\frac{1}{2}} a_n = 3^{\frac{1}{2}}|_{n=\infty}$.

Within the interval, $0 \leq \sin x < x$, *then* $a_{n+1} \leq a_n$. *Applying this to infinity,* $a_n|_{n=\infty} = 0$ *Using a Taylor series expansion, a one term expansion fails, giving a derivative of* 0. *However a two term expansion succeeds.*

$\sin x = x - \frac{x^3}{3!} + \ldots$, $a_{n+1} = \sin a_n|_{n=\infty} = a_n - \frac{a_n^3}{3!}|_{n=\infty}$, $a_{n+1} - a_n = -\frac{a_n^3}{6}|_{n=\infty}$, $\frac{da(n)}{dn} = -\frac{a^3}{6}|_{n=\infty}$, $\frac{da}{a^3} = -\frac{dn}{6}|_{n=\infty}$, $-\frac{1}{2a^2} = -\frac{n}{6}|_{n=\infty}$, $\frac{1}{a^2} = \frac{n}{3}|_{n=\infty}$, $3 = na^2$, $3^{\frac{1}{2}} = n^{\frac{1}{2}} a_n|_{n=\infty}$.

While it is standard practice of including the integral symbol when integrating, the integral itself may be subject to algebraic simplification, on occasions, it can be better to leave off the integral symbol.

Definition 2.6. *For a continuous variable, integration can be expressed without the integral symbol.* $(a\, dn)$ *means* $\int a\, dn$.

When considering a change of variable, as in the chain rule, a variable is used to express the change. However this is not necessarily required, By the $d()$ operator, integration and differentiation are possible. This can be more direct.

Example 2.7. $\int \frac{2u}{u^2+1} du$. *Let* $v = u^2 + 1$, $\frac{dv}{du} = 2u$. $\int \frac{2u}{u^2+1} du = \int \frac{dv}{du}\frac{1}{v} du = \int dv \frac{1}{v} = \ln v$

Alternatively without the variable, $\int \frac{2u}{u^2+1} du = \int \frac{d(u^2+1)}{du}\frac{1}{u^2+1} du = \int d(u^2+1)\frac{1}{u^2+1} = \ln(u^2+1)$

Formally the integral symbol $\int$ and the change in variable dx integrate the expression between them $\int y(x)dx$. However, when working with the algebra and cancelling, integration and differentiation become factors. The integral symbol is not always necessary, and the order of cancellation does not necessarily put the variable at the right end.

From the point of view of solving, the integral symbol $\int$ may be omitted, where trying different combinations of change may be beneficial.

Providing the context is clear, you can remove the integral symbols, but include the symbols at the end when communicating.

The generalised p-series test [4].

$$\sum \frac{1}{\prod_{k=0}^{w-1} \ln_k \cdot \ln_w^p}|_{n=\infty} = \begin{cases} 0 & \text{converges when } p > 1 \\ \infty & \text{diverges when } p \leq 1 \end{cases}$$

Example 2.8. *[7, p.89 3.3.6]. Given $s_n = \sum_{k=1}^n a_k$, $s_n|_{n=\infty} = \infty$.*
3.3.6.a Show $\sum \frac{a_{n+1}}{s_n \ln s_n}|_{n=\infty} = \infty$ diverges.

Transform the problem into the continuous domain. $\sum \frac{a_{n+1}}{s_n \ln s_n}|_{n=\infty} = \sum \frac{a_{n+1}}{s_n \ln s_n} dn|_{n=\infty} = \int \frac{a}{s \ln s} dn|_{n=\infty}$ where n has been redefined. Let $a(n)$ and $s(n)$ be continuous functions to replace a_n and s_n respectively. $s = a\, dn$, $s|_{n=\infty} = \infty$.

Observing $\frac{ds}{dn} = \frac{d(a\, dn)}{dn} = a$ then $\int \frac{a}{s \ln s} dn|_{n=\infty} = \int \frac{1}{s \ln s} \frac{ds}{dn} dn|_{n=\infty} = \int \frac{1}{s \ln s}\, ds|_{n=\infty} = \infty$ diverges.

Alternatively applying the chain rule. $\int \frac{a}{s \ln s} dn|_{n=\infty} = \int \frac{a}{(a\, dn) \ln (a\, dn)} dn|_{n=\infty} = \int \frac{a}{(a\, dn) \ln (a\, dn)} \frac{dn}{d(a\, dn)} d(a\, dn)|_{n=\infty}$ $= \int \frac{a}{(a\, dn) \ln (a\, dn)} \frac{1}{a} d(a\, dn)|_{n=\infty} = \int \frac{1}{(a\, dn) \ln (a\, dn)} d(a\, dn)|_{n=\infty} = \int \frac{1}{s \ln s} ds|_{s=\infty} = \infty$ as on the boundary.

The derivative of a sequence(Definition 2.4) leads to a chain rule with sequences.

Example 2.9. *Example 2.8, solved with the derivative, noticing that $a_{n+1} = s_{n+1} - s_n$ and constructing a derivative $\frac{ds_n}{dn}$.*

$\sum \frac{a_{n+1}}{s_n \ln s_n} dn|_{n=\infty} = \sum \frac{s_{n+1} - s_n}{s_n \ln s_n} dn|_{n=\infty} = \sum \frac{ds_n}{dn} \frac{1}{s_n \ln s_n} dn|_{n=\infty} = \sum \frac{1}{s_n \ln s_n} ds_n|_{S_n=\infty} = \infty$ diverges.

Example 2.10. *[7, p.89 3.3.6.b]. Continued from Example 2.8. Show $\sum \frac{a_n}{s_n (\ln s_n)^2}|_{n=\infty} = 0$ converges.*

$\sum \frac{a_n}{s_n (\ln s_n)^2}|_{n=\infty} = \sum \frac{s_n - s_{n-1}}{s_n (\ln s_n)^2} dn|_{n=\infty} = \sum \frac{ds_n}{dn} \frac{1}{s_n (\ln s_n)^2} dn|_{n=\infty} = \sum \frac{1}{s_n (\ln s_n)^2} ds_n|_{s_n=\infty} = \int \frac{1}{s (\ln s)^2} ds|_{s=\infty}$ $= 0$ converges (Generalised p-series, $p = 2 > 1$).

The exception to the derivative forming a difference is when $a_n|_{n=\infty} = \alpha$ is a constant, see Definition 2.4. The sum of the power series, instead of being an infinite sum, reduces to a single term, or an infinity of terms with a non-monotonic function. At infinity, the power series could not be monotonic, or have a strict relation.

Example 2.11. *To demonstrate the case, applying the derivative to the following problem.*

Let (a_n) be a sequence with $a_n|_{n=\infty} = \alpha \neq 0$, $a_n > 0$. Prove that the series $\sum_{k=1}^{\infty} (a_{n+1} - a_n)$ and $\sum_{k=1}^{\infty} (\frac{1}{a_{n+1}} - \frac{1}{a_n})$ both absolutely converge or both absolutely diverge. [7, 3.4.17]

Reorganising the problem, show $\sum(a_{n+1} - a_n)|_{n=\infty}$ *and* $\sum(\frac{1}{a_{n+1}} - \frac{1}{a_n})|_{n=\infty}$ *both absolutely converge or both absolutely diverge.*

Following the approach given in this paper. $\sum(a_{n+1}-a_n)|_{n=\infty} = \sum \frac{da_n}{dn}dn|_{n=\infty} = \int \frac{da}{dn}dn|_{n=\infty}$ $= \int da|_{n=\infty} = a|_{n=\infty} = \alpha$
$\sum \frac{a_n-a_{n+1}}{a_n a_{n+1}}dn|_{n=\infty} = \int -\frac{da}{dn}\frac{1}{a^2}dn|_{n=\infty} = \int -\frac{1}{a^2}da|_{n=\infty} = \frac{1}{a}|_{n=\infty} = \frac{1}{\alpha}$

Both the sums fail the convergence criterion E3 where we expect the sums at infinity to be either 0 *or* ∞*.*

This is suggesting that for a constant we need to treat the theory separately. Here the problem is reconsidered with the reasoning that a_n *is a constant, and* $a_{n+1} - a_n$ *is an infinitesimal,*

Proof. $\sum(\frac{1}{a_{n+1}} - \frac{1}{a_n})|_{n=\infty} = \sum \frac{a_n-a_{n+1}}{a_{n+1}a_n}|_{n=\infty} = \sum -\frac{1}{a_{n+1}a_n}(a_{n+1} - a_n)|_{n=\infty} = \sum -\frac{1}{\alpha^2}(a_{n+1} - a_n)|_{n=\infty} = \sum(a_{n+1} - a_n)|_{n=\infty}$. Since at infinity the sums are equal, so is their absolute value sum. □

When approximating numerically, solving for a variable by variation, it is common to incrementally approach the solution with numerical schemes.

$$\text{If } \delta_n \to 0 \text{ then } x_{n+1} - x_n = \delta_n, \;\; x_{n+1} - x_n = \frac{dx_n}{dn} = \frac{dx(n)}{dn}|_{n=\infty} = 0$$

The iterative scheme has a solution when its derivative is zero, corresponding to the solution of the problem.

Example 2.12. *[2, Example 2.4] We can show the derivative of* x_n*, successive approximations, as decreasing in the following algorithm.* $x \in *G$*;* $\delta \in \Phi$*;* $(x + \delta)^2 = 2$*. Develop an iterative scheme,* $x^2 + 2x\delta + \delta^2 = 2$*;* $x^2 + 2x\delta = 2$ *as* $2x\delta \succ \delta^2$*,* $x_n^2 + 2x_n\delta_n|_{n=\infty} = 2$*,* $\delta_n = \frac{1}{x_n} - \frac{x_n}{2}|_{n=\infty}$*. Couple by solving for* $x_{n+1} = x_n + \delta_n$*.*

In the ideal case, $(x_n + \delta_n)^2|_{n=\infty} \simeq 2$ *Provided* $\delta_n \to 0$*,* $(x_n)|_{n=\infty}$ *is a series of progressions towards the solution. This can be expressed as a derivative.* $x_{n+1} = x_n + \delta_n$*,* $x_{n+1} - x_n = \delta_n$*,* $\frac{dx_n}{dn} = \delta_n$*.*

Transferring the algorithm $*G \to \mathbb{R}$*, provided we observe the same decrease in* δ_n*, the algorithm finds the solution.*

Let $x_1 = 1.5$*,* $\delta_n : (-8.3\times10^{-2}, -2.45\times10^{-3}, -2.12\times10^{-6}, -1.59\times10^{-12}, \ldots)$ *As the gradient is negative and decreasing,* n *vs* x_n *is monotonically decreasing and asymptotic to the solution* $x_n|_{n=\infty} = \sqrt{2}$*.*

3 Convergence tests

Theorem 3.1. *The Alternating convergence theorem(ACT). If* $(a_n)|_{n=\infty}$ *is a monotonic decreasing sequence and* $a_n|_{n=\infty} = 0$ *then* $\sum(-1)^n a_n|_{n=\infty} = 0$ *is convergent.*

Proof. Compare against the boundary [4] between convergence and divergence.

$$\begin{aligned}
\sum(-1)^n a_n \; z \; \sum \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty} && \text{(Rearrangent, see [5])} \\
\sum a_{2n} - a_{2n-1} \; z \; \sum \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty} && \text{(A sequence derivative)} \\
\sum \frac{da_n}{dn} \; z \; \sum \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty} && \text{(Discrete to continuous } n\text{)} \\
\frac{da(n)}{dn} \; z \; \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty} && \text{(Separation of variables)} \\
da(n) \; z \; \int \frac{1}{\prod_{k=0}^{w} \ln_k} dn|_{n=\infty} && \\
a(n) \; z \; \ln_{w+1}|_{n=\infty} && \text{(substituting conditions, } a(n)|_{n=\infty} = 0\text{)} \\
0 \; z \; \infty, \quad z = < &&
\end{aligned}$$

□

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