

# On Integral Representations for Harmonic Number and Digamma Function

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“Then Simon Peter answered him, Lord, to whom shall we go? thou hast the words of eternal life.” - John 6:68.

ABSTRACT. In this paper, we demonstrate some integral representations for harmonic number and digamma function.

## 1. INTRODUCTION

Leonhard Euler established the following integral representation [1] for harmonic number

$$H_n = \int_0^1 \frac{1-x^n}{1-x} dx,$$

and, accordingly, we can deduce the integral representation for the digamma function [2]

$$\psi(s+1) = -\gamma + \int_0^1 \frac{1-x^s}{1-x} dx.$$

In this article, we prove that

$$H_n = 2 \int_0^1 \frac{(1-x^n)(1-x^{n+1})}{1-x^2} dx,$$

$$H_n = 2m \int_0^1 \frac{x^{m-1}(1-x^{nm})(1-x^{(n+1)m})}{1-x^{2m}} dx$$

and

$$\psi(s+1) = -\gamma - 2 \int_0^1 \frac{(1-x)^s - (1-x^2)^s}{x} dx.$$

## 2. INTEGRAL REPRESENTATIONS FOR HARMONIC NUMBER

**Lemma 1.** *If  $k = 1, 2, 3, \dots$ , then*

$$\frac{1}{k} = 2 \int_0^1 x^{k-1}(1-x^k) dx. \quad (1)$$

**Proof.** We know the elementary identity

$$\frac{1}{k} = \frac{a}{ak-1} - \frac{1}{k(ak-1)} \quad (2)$$

and the integral representation

$$\frac{1}{k} = \int_0^1 x^{k-1} dx, \quad (3)$$

for  $\text{Re}(k) > 0$ .

From (2) and (3), it follows that

$$\begin{aligned} \frac{1}{k} &= \int_0^1 x^{\frac{ak-1}{a}-1} dx - \int_0^1 x^{k(ak-1)-1} dx \\ &= \int_0^1 \left[ x^{\frac{ak-1}{a}-1} - x^{k(ak-1)-1} \right] dx. \end{aligned} \quad (4)$$

We take  $a = \frac{2}{k}$  in Eq. (4)

$$\frac{1}{k} = \int_0^1 \frac{x^{\frac{k}{2}} - x^k}{x} dx. \quad (5)$$

Let  $k \rightarrow 2k$  in Eq. (5)

$$\begin{aligned} \frac{1}{k} &= 2 \int_0^1 \frac{x^k - x^{2k}}{x} dx \\ &= 2 \int_0^1 x^{k-1} (1 - x^k) dx, \end{aligned}$$

which is the desired result.  $\square$

**Theorem 2.** If  $n = 1, 2, 3, \dots$ , then

$$H_n = 2 \int_0^1 \frac{(1 - x^n)(1 - x^{n+1})}{1 - x^2} dx,$$

where  $H_n$  denotes the harmonic number.

**Proof.** We knew the definition [1]

$$H_n \equiv \sum_{k=1}^n \frac{1}{k}. \quad (6)$$

From (1) and (6), we conclude that

$$\begin{aligned} H_n &= \sum_{k=1}^n 2 \int_0^1 x^{k-1} (1 - x^k) dx \\ &= 2 \int_0^1 \sum_{k=1}^n x^{k-1} (1 - x^k) dx \\ &= 2 \int_0^1 \frac{(1 - x^n)(1 - x^{n+1})}{1 - x^2} dx, \end{aligned}$$

which is the desired result.  $\square$

**Theorem 3.** If  $n = 1, 2, 3, \dots$  and  $m = 1, 2, 3, \dots$ , then

$$H_n = 2m \int_0^1 \frac{x^{m-1} (1 - x^{nm})(1 - x^{(n+1)m})}{1 - x^{2m}} dx,$$

where  $H_n$  denotes the harmonic number.

**Proof.** We use the induction. Let  $m = 1$ , thereafter,

$$\begin{aligned} H_n &= 2 \cdot 1 \cdot \int_0^1 \frac{1}{x} \sum_{k=1}^n (x^k - x^{2k}) dx \\ &= 2 \int_0^1 \frac{1}{x} \cdot \frac{x(1 - x^n)(1 - x^{n+1})}{1 - x^2} dx \\ &= 2 \int_0^1 \frac{(1 - x^n)(1 - x^{n+1})}{1 - x^2} dx, \end{aligned}$$

as shown in previous theorem. Assume that the formula is valid for  $m = j$ . Therefore, for  $m = j + 1$ , we have

$$\begin{aligned} H_n &= 2(j+1) \int_0^1 \frac{1}{x} \sum_{k=1}^n (x^{k(j+1)} - x^{2k(j+1)}) dx \\ &= 2(j+1) \int_0^1 \frac{1}{x} \cdot \frac{x^{j+1} (1 - x^{n(j+1)}) (1 - x^{(n+1)(j+1)})}{1 - x^{2(j+1)}} dx \\ &= 2(j+1) \int_0^1 \frac{x^j (1 - x^{n(j+1)}) (1 - x^{(n+1)(j+1)})}{1 - x^{2(j+1)}} dx, \end{aligned}$$

consequently, setting  $j + 1 \rightarrow m$ , we obtain

$$H_n = 2m \int_0^1 \frac{x^{m-1}(1-x^{nm})(1-x^{(n+1)m})}{1-x^{2m}} dx,$$

which is the desired result.  $\square$

### 3. INTEGRAL REPRESENTATION FOR DIGAMMA FUNCTION

**Theorem 4.** *If  $\operatorname{Re}(s) > -1$ , then*

$$\psi(s+1) = -\gamma - 2 \int_0^1 \frac{(1-x)^s - (1-x^2)^s}{x} dx, \quad (7)$$

where  $\psi(s)$  denotes the digamma function and  $\gamma$  denotes the Euler-Mascheroni constant.

**Proof.** We consider the Newton series [2] for the digamma function

$$\psi(s+1) = -\gamma - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \binom{s}{k}. \quad (8)$$

Substitute the right hand side of (1) in the right hand side of (8)

$$\begin{aligned} \psi(s+1) &= -\gamma - \sum_{k=1}^{\infty} (-1)^k 2 \int_0^1 x^{k-1} (1-x^k) dx \binom{s}{k} \\ &= -\gamma - 2 \int_0^1 \sum_{k=1}^{\infty} (-1)^k x^{k-1} (1-x^k) \binom{s}{k} dx \\ &= -\gamma - 2 \int_0^1 \frac{(1-x)^s - (1-x^2)^s}{x} dx, \end{aligned}$$

which is the desired result.  $\square$

**Remark 5.** With the aid of Theorem 4, we evaluate some special values for digamma functions:

$$\begin{aligned} \psi\left(\frac{5}{2}\right) &= -\gamma + \frac{8}{3} - 2 \ln 2, \\ \psi\left(\frac{7}{3}\right) &= -\gamma + \frac{15}{4} - \frac{\pi\sqrt{3}}{6} - \frac{3 \ln 3}{2}, \\ \psi\left(\frac{9}{4}\right) &= -\gamma + \frac{24}{5} - \frac{\pi}{2} - 3 \ln 2, \\ \psi\left(\frac{11}{5}\right) &= -\gamma + \frac{35}{6} - \frac{\pi}{2} \sqrt{1 + \frac{2}{\sqrt{5}}} - \frac{\sqrt{5}}{4} \ln\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right) - \frac{15 \ln 5}{12}, \\ \psi\left(\frac{13}{6}\right) &= -\gamma + \frac{48}{7} - \frac{\pi\sqrt{3}}{2} - 2 \ln 2 - \frac{3 \ln 3}{2}. \end{aligned}$$

### REFERENCES

- [1] [en.wikipedia.org/wiki/Harmonic\\_number](http://en.wikipedia.org/wiki/Harmonic_number), available in March 6, 2015.
- [2] [en.wikipedia.org/wiki/Digamma\\_function](http://en.wikipedia.org/wiki/Digamma_function), available in March 6, 2015.