A prospect proof of the Goldbach's conjecture

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Abstract

Based on, the well-ordering $(\mathbb{N}, <)$ of the set of natural numbers $\mathbb{N}$, and some basic concepts of number theory, and using the proof by contradiction and the inductive proof on $\mathbb{N}$, we prove that the validity of the Goldbach's statement:

every even integer $2n \geq 4$, with $n \geq 2$, *is the sum of two primes.*

This result confirms the Goldbach conjecture, which allow to inserting it as theorem in number theory.

Key Words: Well-ordering $(\mathbb{N}, <)$, basic concepts and theorems on number theory, the indirect and inductive proofs on natural numbers.
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1 A brief history and some results on the conjecture

Historically, from the reference [6], the conjecture dating since 1742 in a letter addressed to Euler from Goldbach expresses the following fact:

Any natural number $n > 5$ *is the sum of three primes.*

The mathematician Euler replied that this fact is equivalent to the following statement:

Every even integer $2n \geq 4$ is the sum of two primes.

Since then, three major lines of attack to this famous conjecture emerged : "asymptotic study", "almost primes study" and finally "basis".

The first result, obtained in the asymptotic case is due to Hardy and Littlewood in 1923 under the consideration of Riemann hypothesis. In 1937, Vingradov showed the same result without using this assumption.

Theorem 1 *(asymptotic theorem). There exists a natural number n_0 such that every odd number $n \geq n_0$ is the sum of three primes.*

A natural number $n = \prod_{i=1}^{r} p_i^{e_i}$ (where each p_i is a prime) is called a k-almost prime when $\sum_{i=1}^{r} e_i = k$; the set of k-almost primes is denoted by P_k. The approach via almost-primes consists in showing that there exists $h, k \geq 1$ such that every sufficiently large even integer is in the set $P_h + P_k$ of sums of integers of P_h and P_k. The first result in this line of study was obtained by Brun in 1919 by showing that: every sufficiently large even number is in $P_9 + P_9$. In 1950, Selberg further improves the result by showing that: every sufficiently large even integers it is in $P_2 + P_3$. The best result in this direction is due to Chen(announcement of results in 1966, proofs in detail in 1973 and 1978) proving that:

Every sufficiently large even integer may be written as $2n = p + m$, where p is a prime and $m \in P_2$

1.1 The result of this paper

To proving the conjecture, we consider for any even natural number $2n > 4$, with $n > 2$, the finite sequence of natural numbers $S_m(n) = (s_i(n))_{i \in \{1,2,\ldots,m\}}$ defined by: $s_i(n) = 2n - p_i$ where p_i is the ith prime number in the finite

strictly ordered sequence of primes

$$P_m := p_1 = 2 < p_2 = 3 < p_3 = 5 < \ldots < p_m$$

where $m = \pi(2n)$ denotes the number of primes p such that $p < 2n$. Using two stages of proofs: the proof by contradiction (or reductio ad absurdum) including the inductive proof or mathematical induction, we proof that: for any natural number $n > 2$, there exists at least one prime number $s_r(n) = 2n - p_r$ belonging to the sequence $S_m(n)$, which confirms the result $2n = s_r(n) + p_r$ where p_r is the rth prime number of the sequence P_m of primes. We give also a second indirect proof, of the above result, without using the induction. These proofs confirms the validity of this conjecture which becomes then a theorem of number theory.

2 Preliminary and theoretical elements essential to the paper

The set of natural numbers $\mathbb{N} := 1, 2, ..., n, ...$, is well-ordered using the usual ordering relation denoted by $\leq$, where any subset of $\mathbb{N}$ contains a least element (this fact is an axiom called the least integer principal). Another way to see the well-ordering of $\mathbb{N}$, is that any natural number n can be reached in finite counting steps by ascent (adding 1) or descent (subtracting 1) from any other natural number m, there isn't an infinite descent on natural numbers. This significant characteristic property of the set of naturals numbers $\mathbb{N}$, is the key of almost results of properties of natural numbers.The concept of well-ordering is of fundamental importance in view of the mathematical induction to proving, in two steps only, the validity of a property $H(n)$ depending on natural number n. For a natural numbers a, b, we say a divides b, if there is a natural number q such that $b = aq$. In this case, we also say that b is divisible by a, or that a is divisor of b, or that a is a factor of b, or that b is a multiple of a. If a is not a divisor of b, then we write $a \nmid b$. A natural number $p > 1$ is called prime, if it is not divisible by any natural number other than 1 and p. Another way of saying this is that, an natural number $p > 1$ is a prime if it cannot be written as the product $p = t_1 t_2$ of two smaller natural numbers t_1, t_2 not equal to 1. A natural number $b > 1$ that is not a prime is called composite. The number 1 is considered neither prime nor composite because the factors of 1 are redundant $1 = 1 \times 1 = 1 \times 1 \times ..... \times 1$. We shall write

$$p_1 = 2 < p_2 = 3 < p_3 = 5 < p_4 = 7 < ... < p_i < ...$$

the infinite increasing sequence of primes, where p_i is the ith prime in this sequence. The Euclid's theorem ensures that there are infinitely many primes, without knowing their pattern and indication of how to determine the ith prime number. There is not a regularity in the distribution of these primes on the chain $(\mathbb{N}, \leq)$(in certain situation they are twins, i.e., there exists a positif integer k such $p_{k+1} = p_k + 2$, like $p_2 = 3$ and $p_3 = 5$, $p_5 = 11$ and $p_6 = 13$ (it is not known today whether they are an infinitely many twin primes), in the same time, for any integer $k \geq 2$, the sequence of the successive $k-1$ naturals numbers $k! + 2, k! + 3, k! + 4, ..., k! + k$, are all composite, for the simple reason that, any term $k! + t$, for $2 \leq t \leq k$, is divisible by t.

The fundamental theorem of arithmetic shows that any natural number $n > 1$ can be written as the product of primes and this factorization is unique up to the order of the prime factors. The primes are regarded as "atoms" where every natural number is built , in unique way, out of prime numbers. For a natural number $n \geq 2$, we denote by $\pi(n)$ the number of primes $p \leq n$, ($\pi(n)$ is called also the prime counting function, for example $\pi(4) = 2$, $\pi(5) = 3$,...and so on). The fundamental theorem of primes (Tcheybecheff an empiric estimation around 1850, Hadamard and de Vallée-Poussin theoritical proof at the end of 19*th century*) shows that, for any large natural number n, we have $\pi(n) \sim \frac{n}{\ln n}$ and then $p_n \sim n \ln n$ where ln denotes the natural logarithm of base $e = 2,71....$ Finally, the Bertrand's postulate(1845) and the Tchebycheff theorem provides that between any natural number $n \geq 2$ and its double $2n$ there exists at least one prime. Equivalently, this may be stated as $\pi(2n) - \pi(n) \geq 1$, for $n \geq 2$, or also in compact form $p_{n+1} < p_n$ for $n \geq 1$.

Finally, the proof by contradiction and the inductive proof can be stated as follows. Proving by so called proof by contradiction or reduction to absurdum, the validity of the property H, consists to assume that the hypothesis H is false, which is then logically equivalent to (*non* H) is true and derived from this, by rules of logic, a false statement or contradiction c of the form $c = (nonR) \wedge R$, this result confirms that the hypothesis H is not false, i.e., *non*(*non* H) is true, we deduce then that H must be true(the absurdity or non sense or contradiction follows by the assumption that H is false).

The mathematical induction is just pattern of the direct proof based on the well-ordering of the set of natural numbers $\mathbb{N}$. Proving the statement $H(n)$ depending on the natural number n, consists to verify in the first step, the validity of the statement H for certain element $n_0 \in \mathbb{N}$, this step is called the base case of induction. And in the second step, assuming the validity of the statement $H(n)$ for $n \in \mathbb{N}$, (called the inductive hypothesis), then prove directly the truth of $H(n+1)$ (this is the inductive case), we can conclude then, based on the well-ordering of $\mathbb{N}$, the truth of the statement $H(n)$ for all $n \geq n_0$.

3 The construction and analysis of the sequence $S_m(n)$

Prior to the construction and analysis of the sequence $S_m(n)$, we begin by these simple lemmas in view of their utilities for the rest of the paper.

Lemma 2 *If the odd integer $t > 1$ is not prime, then it can be factored only on the form $t = t_1 t_2$ where t_1, t_2 are proper factors $\neq 1$, and each factor t_1 or t_2 it is also an odd natural number greater or equal to the number 3.*

Proof. By definition of prime, if the integer $t > 1$ is not prime then it is composite. Let $t = t_1 t_2$ be any possible factorization of t with t_1, t_2 are the proper factors $\neq 1$. If one of these factors (or both) is an even integer then the product $t_1 t_2 = t$ will be also an even integer, but the number t is odd. Then each of the factors $t_1$and t_2 must odd and then greater or equal to the number 3. ∎

Lemma 3 *Any natural number $b \neq 1$ admits a prime divisor. If b is not prime, then there is a prime p divisor of b such that $p^2 \leq b$.*

Proof. By the definition of prime number, if the natural number $b \neq 1$ admits only the number b as proper divisor then b is a prime number. If b is not prime, then it can be factored as $b = pq$ such that: $1 < p < b$ and

$1 < q < b$ with p is the smallest, under the usual ordering $\leq$, proper factor of the number b. Since p is the smallest proper factor of the number b then p must be a prime otherwise, it is not then the least factor of b. As p is the least factor of b then $p \leq q$. Multiplying both sides by p, we obtain: $pp = p^2 \leq pq = b$. $\blacksquare$

Let $m \geq 1$ be natural number. We denote by $I_m = \{1, 2, ..., m\}$ the finite sequence of consecutive naturals numbers from 1 to m. Let $n \geq 2$ be a natural number, we consider the finite strictly increasing sequence of prime numbers

$$p_1 = 2 < p_2 = 3 < p_3 = 5 < ... < p_i ... < p_m$$

where $m = \pi(2n)$ denotes the number of primes $p < 2n$. Let $P_m = (p_i)_{i \in I_m}$ denote this finite successive primes less strictly than $2n$. The Tschebycheff's theorem asserts that at least the prime p_m is between n and its double $2n$. For any natural number $n > 2$, we consider the finite sequence $S_m(n) = (s_i(n))_{i \in I_m}$ of natural numbers defined by: $s_i(n) = 2n - p_i$ where p_i is the ith prime of P_m. Then we have:

$s_1(n) = 2n - 2,$
$s_2(n) = 2n - 3,$
$s_3(n) = 2n - 5,$
.
.
.
$s_i(n) = 2n - p_i,$
.
.
.
$s_m(n) = 2n - p_m.$

Example 4 *For $n = 10$, the finite sequence of primes less than 20 is then: $p_1 = 2 < p_2 = 3 < p_3 = 5 < p_4 = 7 < p_5 = 11 < p_6 = 13 < p_7 = 17 < p_8 = 19$. Consequently $\pi(20) = 8$ and the terms of the sequence $S_8(n) = S_8(10) = (s_i(10))_{i \in \{1,2,...,8\}}$ are:*

$s_1(10) = 20 - 2 = 18,$ $s_2(10) = 20 - 3 = 17,$ $s_3(10) = 20 - 5 = 15,$

$s_4(10) = 20 - 7 = 13,$ $s_5(10) = 20 - 11 = 9,$ $s_6(10) = 20 - 13 = 7,$

$s_7(10) = 20 - 17 = 3,\ s_8(10) = 20 - 19 = 1.$

Lemma 5 *For the natural number $n > 2$ with $m = \pi(2n)$, the finite sequence of natural numbers $S_m(n) = (s_i(n))_{i \in I_m}$ defined by $s_i(n) = 2n - p_i$, with $1 \leq i \leq m$, is strictly decreasing from $s_1(n) = 2n - 2 = Max(S_m(n))$ to $s_m(n) = 2n - p_m = \min(S_m(n)) \geq 1$, and each element $s_i(n)$ of this sequence is an odd natural number except the first term $s_1(n) = 2n - 2$ that is evidently an even number. The last term $s_m(n)$ is equal to 1 only in the case when $p_m = 2n - 1$.*

Proof. Let $n > 2$ be a natural number with $m = \pi(2n)$. Since the finite sequence of primes $p_1 = 2 < p_2 = 3 < p_3 = 5 < ... < p_i < ...p_m$ is strictly increasing, and each term $s_i(n)$ is defined by $2n - p_i$ then the sequence $S_m(n)$ is strictly decreasing from $s_1(n)$ to $s_m(n)$. In fact, we have $p_{i+1} > p_i$ for $1 \leq i \leq m - 1$ and then $s_i(n) = 2n - p_i > s_{i+1}(n) = 2n - p_{i+1}$, this shows that we have:

$s_1(n) = 2n - p_1 = 2n - 2 > s_2(n) = 2n - p_2 = 2n - 3 > s_3(n) = 2n - p_3 = 2n - 5 > ...s_i(n) = 2n - p_i > s_{i+1}(n) = 2n - p_{i+1} > ..... > s_m(n) = 2n - p_m \geq 1$. Since for all i, with $2 \leq i \leq m$, the prime p_i is odd then also the term $s_i(n) = 2n - p_i$ is odd. The first term $s_1(n) = 2n - 2$ is the unique even number in the sequence $S_m(n)$. The last term $s_m(n) = 2n - p_m = \min(S_m(n))$ can be equal to the number 1 if and only if $p_m = 2n - 1$. In fact, if $p_m = 2n - 1$ then $s_m(n) = 2n - p_m = 2n - (2n - 1) = 1$. In the reverse case, we have $\forall p \in P_m$, $p < 2n$ and then $2n - p > 0 \Longleftrightarrow 2n - p \geq 1$ and we have $2n - p = 1$ only in the case when $p = 2n - 1 = p_m$. In example 4, we have this situation, as $p_8 = 19$ then $s_8(10) = 20 - 19 = 1$. ■

4 Existence of prime in the sequence $S_m(n) = (s_i(n))_{i \in I_m}$, for all natural number $n > 2$ with $m = \pi(2n)$

Theorem 6 *For any natural number $n > 2$ with $m = \pi(2n)$, the finite sequence of natural numbers $S_m(n) = (s_i(n))_{i \in I_m}$ defined by $s_i(n) = 2n - p_i$,* *with* $1 \leq i \leq m$*, contains at least one prime* $s_r \in P_m \cap S_m(n)$*.*

Proof. For any natural number $n > 2$*, let* $S_m(n) = (s_i(n))_{i \in I_m}$ be the finite sequence of natural numbers as defined in the section 3 above. The proof is by contradiction, and so begin by assuming that the following hypothesis $H(n)$ is true for some natural number $n > 2$.

The hypothesis $H(n)$:
"there exists a natural number $n > 2$, such that: each term $s_i(n) \in S_m(n)$, for any $i \in I_m$, is not a prime number".

This is equivalent to:

"there exists a natural number $n > 2$, such that: each term $s_i(n) \in S_m(n)$, for all $i \in I_m$, is a composite number or equal to the natural number 1".

Symbolicaly the hypothsis $H(n)$ can be written:
"$\exists\ (n > 2) \in \mathbb{N},\ \forall\ i \in \{1, 2, 3, ...m\}$: the term $s_i(n)$ is not a prime number"
But, the unique term $s_i(n)$ of $S_m(n)$, which can be equal to the number 1 is, the last term $s_m(n) = 2n - p_m$ in the case when $p_m = 2n - 1$ (see the lemma 5). The last term $s_m(n) = 2n - p_m$ is the unique term of the sequence $S_m(n)$, which it is neither prime nor a composite number in the case when $p_m = 2n - 1$, i.e., in the case when $s_m(n) = 1$.

To contradict or denying the hypothesis $H(n)$ for all $n > 2$, (in symbolic terms this contradiction is written: $\forall (n > 2) \in \mathbb{N}, \exists i \in \{1, 2, 3, ...m\}$ such that $s_i(n)$ is a prime number), we compute the sum of the terms of the sequence $S_m(n)$ in two different ways: In the first way, we compute the sum $\sum_{i=1}^{m} s_i(n)$ without any hypothesis, which represents the sum of the real terms. In the second way, we compute the sum $\sum_{i=1}^{m} s_i(n)$, where each term $s_i(n) \neq 1$ of $S_m(n)$ it is supposed to be a composite natural number, under the hypothesis $H(n)$, for all $n > 2$.

In the first way:

$$Sumrel\,(n) = \sum_{i=1}^{m} s_i\,(n) = \sum_{i=1}^{m} (2n - p_i)$$

Where, $Sumrel\,(n)$ represents the sum of the terms $s_i\,(n)$ without the hypothesis $H\,(n)$, which it is the sum of real terms for $n > 2$.
In the second way:

$$Sumhyp\,(n) = \sum_{i=1}^{m} s_i\,(n)$$

Where, $Sumhyp\,(n)$ represents the sum of the terms under the hypothesis $H\,(n)$, for $n > 2$, with each term $s_i\,(n) > 1$ it is to be assumed a composite number for all $i \in I_m$ or $i \in I_{m-1}$ in the case when $s_m\,(n) = 2n - p_m = 1$.

In the first way, we have:

$$\begin{aligned} Sumrel\,(n) &= \sum_{i=1}^{m} s_i\,(n) = \sum_{i=1}^{m} (2n - p_i) = \\ &= (2n-2) + (2n-3) + ... + (2n - p_i) + ... + (2n - p_m) \\ &= (2n + 2n + ... + 2n) - (2 + 3 + ... + p_i + ... + p_m) \\ &= \sum_{i=1}^{m} 2n - \sum_{i=1}^{m} p_i = 2nm - \sum_{i=1}^{m} p_i. \end{aligned}$$

This positive integer value represents, for $n > 2$, the real sum of all the terms of the sequence $S_m\,(n)$ with $m = \pi\,(2n)$. Evidently m and then $Sumrel\,(n)$ are depending on the natural number $n > 2$, when n describes $\mathbb{N}$. In the second way, from the lemma 5, all the terms of sequence $S_m\,(n)$ are odd numbers except the first $s_1\,(n) = (2n-2)$. Since, under the hypothesis $H\,(n)$, each term $s_i\,(n) \neq 1$ it is supposed to be a composite number, we consider then the possible factorization of each term $s_i\,(n)$ as the following form:

$$s_i\,(n) \underset{(\text{under } H(n))}{=} p_i^{'}\,(n)\, q_i\,(n)$$

such that $p_i^{'}\,(n)$ is a prime number, the existence of this prime factor it is assured by the fundamental theorem of arithmetic or it suffices to see the lemma 3, and $q_i\,(n)$ is the other propre factor. According to the lemmas 2 and 5, the factors $p_i^{'}\,(n)$, $q_i\,(n)$ are odd ≥ 3, for all $i \in \{2, ..., m\}$ in the case when $p_m \neq 2n-1$, and for all $i \in \{2, ..., m-1\}$ in the case when $p_m = 2n-1$ (because in this case, we have $s_m = 2n - p_m = 2n - (2n-1) = 1$). The term $s_1(n) = 2n-2 = 2(n-1)$ is the only natural even number of the sequence $S_m(n)$, which it is evidently a composite number.

Two cases are to consider for $Sumhyp\,(n) = \sum_{i=1}^{m} s_i(n)$,

depending on whether $p_m \neq 2n-1$ (in this case $s_m(n) = 2n - p_m \geq 3$ it is also a composite number) or, $p_m = 2n-1$ (in this case $s_m(n) = 2n - p_m = 1$ it is neither prime nor a composite number).

$\mathbf{1}^{rt}case$: if $p_m \neq 2n-1$, then all the terms $s_i(n)$ are to be a composite number, in view of $H(n)$, and we have then:

$$Sumhyp\,(n) = \sum_{i=1}^{m} s_i(n) = s_1(n) + \sum_{i=2}^{m} s_i(n) = (2n-2) + \sum_{i=2}^{m} p_i^{'}(n)\, q_i(n).$$

(from the hyp. $H(n)$ and the lemmas 2 and 5 : $p_i^{'}(n)$ is a prime ≥ 3, $q_i(n)$ is a proper factor ≥ 3. The first term $s_1(n) = 2n-2$ is written separately in the summation).

$\mathbf{2}^{nd}case$: if $p_m = 2n-1$, in this case we have $s_m(n) = 2n - p_m = 2n - (2n-1) = 1$, thus we have:

$$Sumhyp\,(n) = \sum_{i=1}^{m} s_i(n) = s_1(n) + \sum_{i=2}^{m-1} s_i(n) + s_m(n) =$$
$$(2n-2) + \sum_{i=2}^{m-1} p_i^{'}(n)\, q_i(n) + 1.$$

(from the hyp. $H(n)$ and the lemmas $2, 3$ and 5, $p_i^{'}(n)$ is a prime ≥ 3, $q_i(n)$ is a proper factor ≥ 3. The first term $s_1(n) = 2n-2$, and the last $s_m(n) = 1$ are written separately in the summation).

Since all the terms $s_i(n)$, in the step n, are between the numbers 1 and $2n-2$, we have then:

$$T_{prim}(n) = \left\{p_2^{'}(n), p_3^{'}(n), ...p_m^{'}(n)\right\} \subseteq P_m$$

In the same case for the proper factors, we have:

$$T_{fact}(n) = \{q_2(n), q_3(n) ..., q_m(n)\}$$

with $3 \le q_i(n) < 2n-2$, $\forall i \in \{2,3,...,m\}$(according to the hyp. $H(n)$ and lemmas 2 and 5). Our objective at this stage is to prove that, under the hypothesis $H(n)$, we will have for any natural number $n > 2$:

$$Sumhyp(n) > Sumrel(n)$$

In the cases 1 and 2 cited above.
The inequality $Sumhyp(n) > Sumrel(n)$ can be written Symbolicaly in the cases 1 and 2 as the following forms:

$\mathbf{1}^{rt} case$:

$$(2n-2) + \sum_{i=2}^{m} p_i^{'}(n)\, q_i(n) > 2nm - \sum_{i=1}^{m} p_i$$

$\mathbf{2}^{nd} case$:

$$(2n-2) + \sum_{i=2}^{m-1} p_i^{'}(n)\, q_i(n) + 1 > 2nm - \sum_{i=1}^{m} p_i$$

Note that, the first even term $s_1(n) = 2n - p_1 = 2n-2 = 2(n-1)$ is written separately in the sum of $Sumhyp(n)$, because the objective of this method is to show that there is at least one odd prime number in the sequence $S_m(n)$ for all $m = \pi(2n)$ with $n > 2$.
Note that, this study relates to the finite case and not to the asymptotic. That is, for each natural number $n > 2$, there is a finite number $m = \pi(2n)$ of prime numbers and consequently the finite sequences $P_m(n)$ and $S_m(n)$ that are being considered, for each n, when the variable n runs over $\mathbb{N}$. The functions or expressions $Sumrel(n)$, $Sumhyp(n)$ are depending on the natural variable $n > 2$. To this end, we apply the mathematical induction over $\mathbb{N}$ to ensure that if the condition $Sumhyp(n) > Sumrel(n)$ is valid or not for any natural number $n > 2$. It is noted here that the use of induction on the integer variable n does not affects the prime factors of $s_i(n) = 2n - p_i = p_i^{'}(n)\, q_i(n)$ at the step n, and those of the term $s_i(n+1) = 2(n+1) - p_i = p_i'(n+1)\, q_i(n+1)$ at the step $n+1$, which are assumed to be given according to the hypothesis $H(n)$. We simply want to ensure that if it will be or not $Sumhyp(n+1) > Sumrel(n+1)$ at the step $n+1$ given that we have $Sumhyp(n) > Sumrel(n)$ valid at the step n.

Verification over the set $\mathbb{N}-\{1,2\}$ the veracity of condition "$\forall n>2, Sumhyp\,(n)>Sumrel\,(n)$"

1.Verification if $n=2$ can be taken as the basis for the recurrence.
For $n=2$ we have $2n=4$ and then:
$m=\pi\,(2n)=\pi\,(4)=2=|\{p_1,p_2\}|=|\{2,3\}|$,
and consequently
$Sumrel\,(2)=2nm-(p_1+p_2)=2\times 2\times 2-(2+3)=8-5=3$.
In the other hand, since $p_2=3=2\times 2-1$, therefore the $2^{nd}case$ which will be used to calculte $Sumhyp\,(2)$. We have:
$s_1\,(2)=2\times 2-2=2\times 1$ (the even term),
$s_2\,(2)=2\times 2-3=1$ (the last term).
And then, $T_{prim}\,(2)=T_{fact}\,(2)=\emptyset$ (because there isn't any odd terms ≥ 3 assumed to be a composite number according to the hypothesis $H\,(2)$). Then,
$Sumhyp\,(2)=s_1\,(2)+s_2\,(2)=2\times 1+1\times 1=3$,
therefore $Sumrel\,(2)=Sumhyp\,(2)=3$ (this situation happened only for the case $n=2$, that is due to non-existence of the presumed factors $s_i\,(2)\geq 3$ by the hypothesis $H\,(2)$). But, evidently we have $4=2+2$
Verification for the integer $n=3$.
On the one hand, we have $2n=6$ and then:
$m=\pi\,(2n)=\pi\,(6)=|\{p_1,p_2,p_3\}|=|\{2,3,5\}|=3$ with $(2+3+5)=10$ and consequently, $Sumrel\,(3)=2nm-10=2\times 3\times 3-10=8$.
On the other hand, since $p_3=5=2n-1=2\times 3-1$, then the $2^{nd}case$ which it will be used to calculate $Sumhyp\,(3)$. We have:
$s_1\,(3)=6-2=4$ (the even term),
$s_2\,(3)\underset{(\text{under } H(3))}{=}p_2'\,(3)\,q_2\,(3)=3\times q_2\,(3)$, with the presumed proper factor $q_2\,(3)$ is between $4>q_2\,(3)\geq 3$ under the hypothesis $H\,(3)$.
$s_3\,(3)=6-5=1$.
We have $T_{prim}\,(3)=\{3\}$ and $T_{fact}\,(3)=\{q_2\,(3)\,/\quad 4>q_2\,(3)\geq 3\}$.
For $n=3$, we have then:
$Sumhyp\,(3)=(2\times 3-2)+3\times q_2\,(3)+1=4+3\times q_2\,(3)+1=5+3\times q_2\,(3)$, with $4>q_2\,(3)\geq 3$. And consequently,
$Sumhyp\,(3)=5+3\times q_2\,(3)\geq 14>8=Sumrel\,(3)$.
Therefore $Sumhyp\,(3)>Sumrel\,(3)$ and then the base case for the induction is, the natural number 3.

Verification again for the integer $n = 4$.
On the one hand, we have $2n = 8$ and then:
$m = \pi(2n) = \pi(8) = |\{p_1, p_2, p_3, p_4\}| = |\{2, 3, 5, 7\}| = 4$ with
$(2 + 3 + 5 + 7) = 17$ and consequently,
$Sumrel(4) = 2nm - 17 = 2 \times 4 \times 4 - 17 = 15$.
On the other hand, since $p_4 = 2 \times n - 1 = 2 \times 4 - 1 = 7$, then the $2^{nd} case$ which will be used to calculate $Sumhyp(4)$.We have:
$s_1(4) = 8 - 2 = 6$ (the even term),
$s_2(4) \underset{(\text{under } H(4))}{=} p'_2(4)\, q_2(4) = 5 \times q_2(4)$,
with the presumed factor $q_2(4)$ is between $6 > q_2(4) \geq 3$ under the hypothesis $H(4)$)
$s_3(4) \underset{(\text{under } H(4))}{=} p'_3(4)\, q_3(4) = 3 \times q_3(4)$,
with the presumed factor $q_3(4)$ is between $6 > q_3(4) \geq 3$ under the hypothesis $H(4)$)
$s_3(4) = 8 - 7 = 1$. We have then:
$T_{prim}(4) = \{5, 3\}$ and $T_{fact}(4) = \{q_2(4), q_3(4) / \quad 6 > q_2(4), q_3(4) \geq 3\}$.
Then for $n = 4$,
$Sumhyp(4) = (2 \times 4 - 2) + 5 \times q_2(4) + 3 \times q_3(4) + 1$
$= 6 + 5 \times q_2(4) + 3 \times q_3(4) + 1 = 7 + 5 \times q_2(4) + 3 \times q_3(4)$
with $6 > q_2(4), q_3(4) \geq 3$. And consequently,
$Sumhyp(4) = 7 + 5 \times q_2(4) + 3 \times q_3(4) \geq 7 + 5 \times 3 + 3 \times 3 = 7 + 15 + 9 = 31 > 15 = Sumrel(4)$,
this shows that we have also the inequality $Sumhyp(4) > Sumrel(4)$ it is satisfied.

2. verification if we will have $Sumhyp(n+1) > Sumrel(n+1)$ at the step $n + 1$, given that we have $Sumhyp(n) > Sumrel(n)$ satisfied at the step n.
Recall that we have two cases to consider at the step $n + 1$.
- If the prime $p_{m+1} \neq 2n + 1$, in this case $p_{m+1} > 2n + 2$ and consequently we have $\pi(2n) = \pi(2n + 2) = m$, this shows that we are in the $\mathbf{1}^{er} case$ with $s_m(n + 1) = 2(n + 1) - p_m \geq 3$ because $p_m < 2n$, and consequently the sequence S at the step $n + 1$ will have also m terms, with each term $s_i(n + 1)$, according to $H(n + 1)$ and lemmas 2 and 5, is a composite odd natural number for each $i \in \{2, ..., m\}$, with the even term $s_1(n + 1) = 2(n + 1) - 2$ it is given a composite number from the start.
- If the prime $p_{m+1} = 2n + 1$, in this case $\pi(2n + 2) = m + 1$, this shows that

there is $m+1$ terms with $s_{m+1}(n+1)=1$(the factor $s_{m+1}(n+1)=1$ is non composite and not prime and all the others terms $s_i(n+1)$ are composite according to the hypothesis $H(n+1)$, for all $i\in\{2,...,m\}$, with the even term $s_1(n+1)=2(n+1)-2$ is given a composite number from the start).
Let be the following factorization of the sequence $S_m(n)=(s_i(n))_{i\in\{1,...,m\}}$ with $Sumhyp(n)>Sumrel(n)$ it is given true at the step n.
$s_1(n)=2n-p_1=2n-2,$
$s_{2(n)}=2n-p_2=2n-3\underset{(\text{under } H(n))}{=}p_2'(n)\,q_2(n),$
$s_3(n)=2n-p_3=2n-5\underset{(\text{under } H(n))}{=}p_3'(n)\,q_3(n),$
.
$s_i(n)=2n-p_i\underset{(\text{under } H(n))}{=}p_i'(n)\,q_i(n),$
.
.
$s_m(n)=2n-p_m\underset{(\text{under } H(n))}{=}p_m'(n)\,q_m(n).$
At the step $n+1$, in the $\mathbf{1}^{er}case$ when $p_{m+1}\neq 2n+1$, we have:
On the one hand,

$$Sumrel(n+1)=\sum_{i=1}^{m}(2(n+1)-p_i)=\sum_{i=1}^{m}((2n-p_i)+2)$$
$$=\sum_{i=1}^{m}(2n-p_i)+2m=Sumrel(n)+2m.$$

On the other hand,
$s_1(n+1)=2(n+1)-p_1=2(n+1)-2,$
$s_2(n+1)=2(n+1)-p_2=2(n+1)-3\underset{(\text{under } H(n+1))}{=}p_2'(n+1)\,q_2(n+1),$
$s_3(n+1)=2(n+1)-p_3=2(n+1)-5\underset{(\text{under } H(n+1))}{=}p_3'(n+1)\,q_3(n+1),$
.
$s_i(n+1)=2(n+1)-p_i\underset{(\text{under } H(n+1))}{=}p_i'(n+1)\,q_i(n+1),$
.
.
$s_m(n+1)=2(n+1)-p_m\underset{(\text{under } H(n+1))}{=}p_m'(n+1)\,q_m(n+1).$
With for all $i\in\{2,3,...,m\}$, $2(n+1)-2>p_i'(n+1)\,,\ q_i(n+1)\geq 3.$
But, we have also for all $i\in\{2,3,...,m\}$,
$s_i(n+1)=p_i'(n+1)\,q_i(n+1)=2(n+1)-p_i=$
$=2n+2-p_i=(2n-p_i)+2=s_i(n)+2=p_i'(n)\,q_i(n)+2,$
and then:

$$Sumhyp\,(n+1) = \sum_{i=1}^{m} s_i\,(n+1) = s_1\,(n+1) + \sum_{i=2}^{m} s_i\,(n+1)$$
$$= 2\,(n+1) - 2 + \sum_{i=2}^{m} (s_i\,(n) + 2) = (2n-2) + 2 + \sum_{i=2}^{m} s_i\,(n) + 2\,(m-1)$$
$$= \left((2n-2) + \sum_{i=2}^{m} s_i\,(n)\right) + 2m = Sumhyp\,(n) + 2m.$$
Since the condition $Sumhyp\,(n) > Sumrel\,(n)$ it is given valid at the step n, we obtain then

$Sumhyp\,(n+1) = Sumhyp\,(n) + 2m > Sumrel\,(n) + 2m = Sumrel\,(n+1)$
and consequently $Sumhyp\,(n+1) > Sumrel\,(n+1)$.
We conclude then the condition it is true in the $\mathbf{1}^{rt} case$:

$$\forall n > 2, Sumhyp\,(n) > Sumrel\,(n)$$

$\mathbf{2}^{nd} case$ if $p_{m+1} = 2n+1$:
On the one hand we have
$$Sumrel\,(n+1) = \sum_{i=1}^{m+1} s_i\,(n+1) = \sum_{i=1}^{m} (2\,(n+1) - p_i) + s_{m+1}\,(n+1) =$$
$$\sum_{i=1}^{m} ((2n - p_i) + 2) + 1 = \sum_{i=1}^{m} (2n - p_i) + 2m + 1 = Sumrel\,(n) + 2m + 1.$$
On the other hand,
$$Sumhyp\,(n+1) = \sum_{i=1}^{m+1} s_i\,(n+1) = s_1\,(n+1) + \sum_{i=2}^{m} s_i\,(n+1) + s_{m+1}\,(n+1)$$
$$= 2\,(n+1) - 2 + \sum_{i=2}^{m} (s_i\,(n) + 2) + 1 = (2n-2) + 2 + \sum_{i=2}^{m} s_i\,(n) + 2\,(m-1) + 1$$
$$= (2n-2) + \sum_{i=2}^{m} s_i\,(n) + 2\,(m-1) + 3 = \left((2n-2) + \sum_{i=2}^{m} s_i\,(n)\right) + 2m + 1 =$$
$$Sumhyp\,(n) + 2m + 1.$$
Since we have $Sumhyp\,(n) > Sumrel\,(n)$ at the step n, then:
$Sumhyp\,(n+1) = Sumhyp\,(n) + 2m + 1 > Sumrel\,(n) + 2m + 1 = Sumrel\,(n+1)$,
which shows that we have also the condition verified at the step $n+1$,

$$\forall n > 2, Sumhyp\,(n+1) > Sumrel\,(n+1)$$

From the cases **1** and **2**, we conclude then:

$$\forall n > 2, Sumhyp\,(n) > Sumrel\,(n)$$

In conclusion, we obtain then according the hypothesis $H(n)$,
$Sumhyp(n) > Sumrel(n)$ for all $n > 2$.
This results shows that there isn't any natural number $n > 2$ such that we will have $Sumhyp(n) = Sumrel(n)$ under the hypothesis $H(n)$. In other words, the hypothesis conduct or forces the inequality between the quantities $Sumrel(n)$ and $Sumhyp(n)$ which represents normally the same value, this is a mathematically absurd. The sum $\sum_{i=1}^{m} s_i(n)$ is equal to one integer value depending on the natural number $n > 2$. The consideration of this assumption, i.e., the hypothesis $H(n)$, for $n > 2$, conduct to the contradiction and the impossibility (the same integer expression $\sum_{i=1}^{m} s_i(n)$ with two different values, absurdity), and therefore the categorical refutation of the hypothesis $H(n)$, for all $n > 2$. Consequently for each $n > 2$, there is at least one odd prime number $s_r(n) \in S_m(n)$. Note that, the inequality between $Sumrel(n)$ and $Sumhyp(n)$ is due to the fact that the hypothesis $H(n)$ forces only every proper factor $q_i(n)$ of each term $s_i(n) = p_i^{'}(n)\, q_i(n)$ to be greater or equal to 3, for $i \in \{2, 3, ...m\}$ if $p_m \neq 2n-1$ otherwise $i \in \{2, 3, ...m-1\}$ if $p_m = 2n-1$. The existence of the prime $p_i^{'}(n)$ as factor ≥ 3 is provided by the lemma 2 with or without the hypothesis $H(n)$. Since we have isolated, from our comparaison, the first term $s_1(n)$ and the last term $s_m(n)$ in the case when $s_m(n) = 1$, and to have this necessary equality between $Sumhyp(n)$ and $Sumrel(n)$, for all $n > 2$, it is necessary then that some factor, at least one, $q_i(n)$ must be strictly less than the natural number 3. And since this factor $q_i(n)$ cannot be even equal to 2 (otherwise, the term $s_i(n)$ would be also even, but we have all the terms are odd numbers except the first term), and therefore, this factor must be equal to the natural number 1(i.e., $q_i(n) = 1$). In conclusion, it is necessary then the existence of at least one term $s_r(n) = p_r^{'}(n)\, q_r(n) \in S_m(n)$ with $q_r(n) = 1$ and consequently, we have $s_r(n) = p_r^{'}(n)\, q_r(n) = p_r^{'}(n) \times 1 = p_r^{'}(n) \in T_{prim}(n) = \left\{p_2^{'}(n), p_3^{'}(n), ...p_m^{'}(n)\right\} \subseteq P_m$ and then $s_r(n)$ is a prime number. Thus, we obtain for each $n > 2$, an odd prime number $s_r(n)$ claimed by the theorem.

■

Note that, in the first proof of theorem 6, the hypothesis which it is taken into contradiction using the induction on the natural number $n \in \mathbb{N}$

is $H(n) :=$ "$\exists$ $n\in\mathbb{N}$, $\forall$ $i\in\{1,2,3,...m\}$: $s_i(n)$ is not a prime number" the negation of $H(n)$ is "$\forall$ $n\in\mathbb{N}$, $\exists$ $i\in\{1,2,3,...m\}$ such that the term $s_i(n)$ of the sequence $S_m(n)$ is a prime number", is proved using the induction on the natural $n\in\mathbb{N}$. In this subsection, we give another indirect proof without the induction of the hypothesis $H_2(n)$: " For any naturel number $n>2$, $\exists i\in\{1,2,3,...m\}$ such that the term $s_i(n)$ of the sequence $S_m(n)$ is a prime number".

4.1 Second Proof of existence of prime in the sequence $S_m(n) = (s_i(n))_{i\in I_m}$, for all natural number $n>2$ with $m=\pi(2n)$ (Indirect proof without the recurrence).

Proof. For the natural number $n>2$, we consider the finite sequence of primes $P_m := (p_1 = 2 < p_2 = 3 < .... < p_m)$ with $m=\pi(2n)$ and let $S_m(n) = (s_i(n))_{i\in I_m}$ be the finite sequence of natural numbers defined by $s_i(n) = 2n-p_i$, where p_i is the ith prime of P_m. Let $H_2(n)$ be the hypothesis defined by:

$H_2(n)$: " For any naturel number $n>2$, $\exists i\in\{1,2,3,...m\}$ such that: the term $s_i(n)$ of the sequence $S_m(n)$ is a prime number".

Suppose that the hypothesis $H_2(n)$ is false, there exists then a natural $n_0>2$, such that: $\forall i\in\{1,2,3,...m\}$ the natural number $s_i(n_0)$ is not a prime number with $m=\pi(2n_0)$. This negation of this hypothesis is equivalent to :

$\exists$ $n_0>2$ such that: $\forall$ $i\in\{1,2,3,...m\}$, with $m=\pi(2n_0)$, the natural number $s_i(n_0)$ is a composite number.

We show in this, we will have $Sumhyp(n_0) > Sumrel(n_0)$ for $n_0>6$, and we obtain then a contradiction. This result asserts then that, the hypothesis $H_2(n)$ it is never false.

On the one hand we have:

$Sumrel(n_0) = \displaystyle\sum_{i=1}^{m} s_i(n_0) = \sum_{i=1}^{m}(2n_0-p_i) = 2n_0m - \sum_{i=1}^{m} p_i$. Since we have $p_1 = 2 < p_2 = 3 < p_3 = 5 < ... < p_i... < p_m$ then $p_1 = 2 = \min P_m = \min(p_i)_{i\in I_m}$. This shows that we have:

$Sumrel(n_0) = 2n_0m - \displaystyle\sum_{i=1}^{m} p_i \ll 2n_0m - 2m = 2m(n_0-1)$.

On the other hand, we have to consider for $Sumhyp\,(n)$ two cases: the case if $p_m \neq 2n-1$ and the case if $p_m = 2n-1$.
If $p_m \neq 2n-1$ then:
$Sumhyp\,(n_0) = (2n_0 - 2) + \sum_{i=2}^{m} p_i^{'}\,(n_0)\,q_i\,(n_0)$
with $3 \leq p_i'\,(n_0)\,, q_i\,(n_0) < 2n_0 - 2$, $\forall i \in \{2, 3, ..., m\}$.
This shows that we have $Sumhyp\,(n_0) \gg (2n_0 - 2) + \sum_{i=2}^{m} 9 = (2n_0 - 2) + 9\,(m-1)$. Then we have:
$Sumhyp\,(n_0) \gg (2n_0 - 2) + \sum_{i=2}^{m} 9 =$
$= (2n_0 - 2) + 9\,(m-1) > 2m\,(n_0 - 1) \gg Sumrel\,(n_0)$ for $n_0 > 6$. In fact,
$(2n_0 - 2) - 2m\,(n_0 - 1) > -9\,(m-1) \Longleftrightarrow$
$2\,(n_0 - 1) - 2m\,(n_0 - 1) > -9\,(m-1) \Longleftrightarrow$
$2\,(n_0 - 1)\,(1-m) > -9\,(m-1) \Longleftrightarrow$
$(n_0 - 1) > \frac{-9(m-1)}{2(1-m)} = 9/2 \Longrightarrow n_0 > [9/2 + 1] = 6$.
In the same manner, if $p_m = 2n-1$ then $Sumhyp\,(n_0) = (2n_0 - 2) + \sum_{i=2}^{m-1} p_i^{'}\,(n_0)\,q_i\,(n_0) + 1 \gg (2n_0 - 2) + 9\,(m-2) + 1 > 2m\,(n_0 - 1) \gg Sumrel\,(n_0)$ for $n_0 > 5$. The above results shows that, with large negligence of real terms $\sum_{i=2}^{m} p_i^{'}\,(n_0)\,q_i\,(n_0)$ normally added to $Sumhyp\,(n_0)$ and $\sum_{i=1}^{m} p_i$ subtracted to $Sumrel\,(n_0)$, the non-existence of such natural number n_0 and consequently there isn't any natural number $n_0 > 2$ such that: the hypothesis $H_2\,(n_0)$ it is false. This result asserts that the hypothesis $H_2\,(n)$ is true for all natural number $n > 2$. Finally, we obtain then the existence of a prime number in the sequence $S_m\,(n)$ for all $n > 2$. $\blacksquare$

Theorem 7 *Every even integer* $2n \geq 4$*, with* $n \geq 2$*, is the sum of two primes.*

Proof. If $n = 2$ then $4 = 2 + 2$. If the natural number $n > 2$, consider then the finite sequence of primes $P_m := (p_1 = 2 < p_2 = 3 < .... < p_m)$ with $m = \pi\,(2n)$ and let $S_m\,(n) = (s_i\,(n))_{i \in I_m}$ be the finite sequence of natural numbers defined by $s_i\,(n) = 2n - p_i$, where p_i is i^{th} prime of P_m. From the theorem 6 there exists at least one prime number $s_r\,(n) \in S_m\,(n)$. As we have

$s_r(n) = 2n - p_r$ with p_r is the rth *prime number of sequence* P_m. Therefore we have $s = s_r(n) = 2n - p_r$ and consequently $2n = p_r + s$. It follows that the Goldbach's conjecture is effectively a theorem of number theory. ■

As consequence of this results, given an even natural number $2n \geq 4$ with $n \geq 2$ to find the pair of prime numbers (p, s) such that $2n = p + s$, it suffices that the algorithm run through the finite sequence $S_m(n) = (s_i(n))_{i \in I_m}$ which contains, at least one, solution claimed.

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