Notes on Uniqueness Solutions of Navier-Stokes Equations

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Abstract: § 1: *remembering the need of imposed the boundary condition $u(x,t)=0$ at infinity to ensure uniqueness solutions to the Navier-Stokes equations. This section is historical only.* § 2: *verifying that for potential and incompressible flows there is no uniqueness solutions when the velocity is equal to zero at infinity. More than this, when the velocity is equal to zero at infinity for all $t \geq 0$ there is no uniqueness solutions, in general case. Exceptions when $u^0=0$.* § 3: *non-uniqueness in time for incompressible and potential flows, if $u \neq 0$.* § 4: *a more general solution of Euler and Navier-Stokes equations for incompressible and irrotational (potential) flows, given the initial velocity.* § 5: *Solution for Euler and Navier-Stokes equations using Taylor's series of powers of t around $t=0$.*

§ 1

Recently I wrote a paper named "A Naive Solution for Navier-Stokes Equations"[1] where I concluded that it is possible does not exist the uniqueness of solutions in these equations for $n=3$, even with all terms and for any $t>0$.

This conclusion inhibited me to publish officially my other article "Three Examples of Unbounded Energy for $t>0$"[2], also a very important paper.

This distressful and no way out situation disappears when we impose the boundary condition $\lim_{|x|\to\infty} u(x,t)=0$, which guarantees the desired uniqueness of solutions at least in a finite and not null time interval $[0,T]$. Possibly others boundary conditions also arrive at the uniqueness, but null velocity at infinite may imply a minimum volume of $|u|^2$ and the respective total kinetic energy.

Thus is necessary do some changes in the expressions of external forces, pressures and velocities used in [2] to establish again the breakdown solution in [3], due to occurrence of unbounded energy $\int_{\mathbb{R}^3}|u|^2dx \to \infty$ in $t>0$. In special, a general example, for $1 \leq i \leq 3$ and $\nabla\cdot u=\nabla\cdot u^0=\nabla\cdot v=0$, is

$$u_i(x,t)=u_i^0(x)e^{-t}+v_i(x)e^{-t}(1-e^{-t}),\ u,u^0,v,x\in\mathbb{R}^3,$$

$$u_i^0(x)\in S(\mathbb{R}^3),\ v_i(x)\in C^\infty(\mathbb{R}^3),\ v\notin L^2(\mathbb{R}^3),\ \lim_{|x|\to\infty} v(x)=0,$$

$$p\in C^\infty\big(\mathbb{R}^3\times[0,\infty)\big),$$

$$f_i = \left(\frac{\partial p}{\partial x_i} + \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} - \nu \nabla^2 u_i\right) \in S\big(\mathbb{R}^3 \times [0,\infty)\big).$$

The conditions (4) for initial velocity and (5) for external force, conforming description given in [3],

(4) $|\partial_x^\alpha u^0(x)| \le C_{\alpha K}(1+|x|)^{-K}: \mathbb{R}^3,\ \forall \alpha, K$

(5) $|\partial_x^\alpha \partial_t^m f(x,t)| \le C_{\alpha m K}(1+|x|+t)^{-K}: \mathbb{R}^3 \times [0,\infty)\ ,\forall \alpha, m, K$

is a kind of *straitjacket,* and for me do not seem good conditions to make possible physically reasonable solutions, rather only restricts the solutions to a very limited and very artificial set of possibilities. If it were possible to the external force be in the set $C^\infty\big(\mathbb{R}^3 \times [0,\infty)\big)$, such as the velocity and pressure in $t > 0$, even being only limited functions and equals zero as $|x| \to \infty$, instead Schwartz Space, the possible solutions will be much more interesting and realistic.

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§ 2

As we know, when $\nabla \times u = 0$ exist a potential function ϕ such that $u = \nabla\phi$. When $\nabla \times u = 0$ and $\nabla \cdot u = 0$ then $\nabla^2\phi = 0$ and $\nabla^2 u = 0$, therefore the Navier-Stokes equations are reduced to Euler's equations and the solutions for velocity are given by Laplace's equation, they are harmonic functions, i.e.,

$$\nabla^2 u = \nabla(\nabla \cdot u) - \nabla \times (\nabla \times u) = (\nabla^2 u_1, \nabla^2 u_2, \nabla^2 u_3) = 0$$

and

$$u = \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right), \nabla \cdot u = 0 \Longrightarrow \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) = 0.$$

It is clear that there is no uniqueness solutions in all cases, in special when the velocity is both irrotational and incompressible, even if the velocity vanishes at infinity. Defining $\phi(x,t) = \phi^0(x)T(t)$, $T(0) = 1$, then we have $u = \nabla\phi = T(t)\nabla\phi^0 = T(t)u^0(x)$ and so there are endless possibilities for constructing u given u^0, because there are endless possibilities for constructing $T(t)$ with $T(0) = 1$, even if $\lim_{|x|\to\infty} u = T(t) \lim_{|x|\to\infty} u^0 = 0$.

According proof in my other paper [4], if $u(x,y,z,0) = u^0(x,y,z)$ is the initial velocity of the system, valid solution in $t = 0$, then $u(x,y,z,t) = u^0(x+t, y+t, z+t)$ is a solution for velocity in $t \ge 0$. Similarly, $p(x,y,z,t) = p^0(x+t, y+t, z+t)$ is the correspondent solution for pressure in $t \ge 0$, being $p^0(x,y,z)$ the initial condition for pressure. More than this, the velocities $u^0(x+t,y,z)$, $u^0(x,y+t,z)$ and $u^0(x,y,z+t)$ are also solutions, and respectively also the pressures $p^0(x+t,y,z)$, $p^0(x,y+t,z)$ and $p^0(x,y,z+t)$. That is, when the velocity is equal to zero at infinity for all $t \geq 0$ there is no uniqueness solutions, in general case. Apparently, an additional complication if the uniqueness condition is required.

Exception to the two previous paragraphs when $u^0 = 0$.

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§ 3

In line with previous date, if $\nabla \cdot u = 0$ and $\nabla \times u = 0$ then $\nabla^2 u = 0$. For $u = (u_1, u_2, u_3)$ and $w = (w_1, w_2, w_3)$, defining $w_i = A(t)u_i + B_i(t), 1 \leq i \leq 3$, we will have $\nabla \cdot w = 0$, $\nabla \times w = 0$ and $\nabla^2 w = 0$.

If $u = \nabla\phi$ solves the Navier-Stokes equations then

$$\nabla p + \frac{\partial u}{\partial t} + (u \cdot \nabla)u = \nu \nabla^2 u$$

$$\nabla p + \nabla\left(\frac{\partial \phi}{\partial t}\right) + (\nabla \times u) \times u + \frac{1}{2}\nabla|u|^2 =$$
$$= \nu\big(\nabla(\nabla \cdot u) - \nabla \times (\nabla \times u)\big)$$

$$\nabla p + \nabla\left(\frac{\partial \phi}{\partial t}\right) + \nabla\left(\frac{1}{2}|u|^2\right) = 0$$

$$\nabla\left(p + \frac{\partial \phi}{\partial t} + \frac{1}{2}|u|^2\right) = 0$$

$$p + \frac{\partial \phi}{\partial t} + \frac{1}{2}|u|^2 = \theta(t),$$

which is the Bernouilli's law without external force.

With a gradient external force $f = \nabla U$ we will have

$$p + \frac{\partial \phi}{\partial t} + \frac{1}{2}|u|^2 = U + \theta(t).$$

For w defined as above, substituting $u \mapsto w$ in the Navier-Stokes equations comes

$$p + \frac{\partial \varphi}{\partial t} + \frac{1}{2}|w|^2 = U + \theta(t),$$

where $\varphi = A(t)\phi + B_1 x + B_2 y + B_3 z$, and p is the new pressure for the velocity $w = A(t)u + B(t), B = (B_1, B_2, B_3)$.

If $A(0)=1$ and $B_i(0)=0,\ 1\le i\le 3$, then u and w obey the same initial condition and both solve the Navier-Stokes (and Euler) equations and they are incompressible and potential flows. In this case, there is no uniqueness solution, for $A(t)\not\equiv 1$ or $B(t)\not\equiv 0$, i.e., $u\not\equiv w$.

Imposing the boundary condition at infinity $u|_{r\to\infty}=0, r=\sqrt{x^2+y^2+z^2}$, the velocity $w=A(t)u$ obey the same boundary condition, for $A(0)=1,\ A(t)\not\equiv 1$ finite for all $t\ge 0$, i.e. $w(x,y,z,t)=A(t)u(x,y,z,t)$ and $u(x,y,z,t)$ obey the same initial and boundary conditions, so there is no uniqueness solutions for Navier-Stokes (and Euler) equations in this case of incompressible and potential flows with velocity zero at infinity, if $u\ne 0$.

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§ 4

Other class of solutions for velocity is built through of the transformations $x_i\mapsto\alpha(t)x_i+ct,\ 1\le i\le 3,\ \alpha(t),c\in\mathbb{R},\ \alpha(t)\ne 0,\ \alpha(0)=1$, in the parameters of the initial velocity, i.e.,

(4.1) $$\boldsymbol{u}(x,y,z,t)=A(t)\boldsymbol{u}^{\boldsymbol{0}}(\alpha x+ct,\alpha y+ct,\alpha z+ct)+\boldsymbol{B}(t),$$

because if

(4.2.1) $$\nabla^2\boldsymbol{u}^{\boldsymbol{0}}(x,y,z)=\nabla^2[A(t)\boldsymbol{u}^{\boldsymbol{0}}(x,y,z)+\boldsymbol{B}(t)]=\boldsymbol{0}$$

(4.2.2) $$\nabla\cdot\boldsymbol{u}^{\boldsymbol{0}}(x,y,z)=\nabla\cdot[A(t)\boldsymbol{u}^{\boldsymbol{0}}(x,y,z)+\boldsymbol{B}(t)]=0$$

(4.2.3) $$\nabla\times\boldsymbol{u}^{\boldsymbol{0}}(x,y,z)=\nabla\times[A(t)\boldsymbol{u}^{\boldsymbol{0}}(x,y,z)+\boldsymbol{B}(t)]=\boldsymbol{0}$$

then also

(4.3.1) $$\nabla^2[A(t)\boldsymbol{u}^{\boldsymbol{0}}(\alpha x+ct,\alpha y+ct,\alpha z+ct)+\boldsymbol{B}(t)]=\boldsymbol{0}$$

(4.3.2) $$\nabla\cdot[A(t)\boldsymbol{u}^{\boldsymbol{0}}(\alpha x+ct,\alpha y+ct,\alpha z+ct)+\boldsymbol{B}(t)]=0$$

(4.3.3) $$\nabla\times[A(t)\boldsymbol{u}^{\boldsymbol{0}}(\alpha x+ct,\alpha y+ct,\alpha z+ct)+\boldsymbol{B}(t)]=\boldsymbol{0}$$

α a function of time, that is, the velocity (4.1) with $A(0)=1,\ \boldsymbol{B}(0)=\boldsymbol{0},\ \alpha(0)=1$, $\alpha(t)\ne 0$, is a solution for Euler (and Navier-Stokes) equations with initial velocity $\boldsymbol{u}^{\boldsymbol{0}}(x,y,z)$, a general solution for incompressible and irrotational (potential) flows, in the case of conservative external forces. The respective pressure is again given by the Bernouilli's law, (4.4) $$p+\frac{\partial\phi}{\partial t}+\frac{1}{2}|\boldsymbol{u}|^2=U+\theta(t),$$

$\boldsymbol{u}=\nabla\phi,\ \boldsymbol{f}=\nabla U,$ also without uniqueness solution due to $\theta(t)$ and $\boldsymbol{u}$.

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§ 5

In [5] we got to a great result, the complete solution for Euler and Navier-Stokes equations for incompressible fluids, using the expansion of the velocity u in a Taylor's series of powers of time around $t=0$. This work was a natural evolution of [6], which in turn is a consequence of [4]. In [4] the external force is equal to zero, and in [6] the external force need be conservative, derived from a potential. In [5] a very general condition is accepted, since that the initial velocity, external force and pressure belong to C^∞, and of general way they can be expressed in Taylor's series of time t.

This expansion of the velocity in a Taylor´s series in relation to time around $t=0$, considering x,y,z as constant, for $1\le i\le 3$, is

(5.1) $$u_i=u_i|_{t=0}+\frac{\partial u_i}{\partial t}|_{t=0}\ t+\frac{\partial^2 u_i}{\partial t^2}|_{t=0}\ \frac{t^2}{2}+\frac{\partial^3 u_i}{\partial t^3}|_{t=0}\ \frac{t^3}{6}+\cdots$$
$$+\frac{\partial^k u_i}{\partial t^k}|_{t=0}\ \frac{t^k}{k!}+\cdots$$

or

(5.2) $$u_i=u_i^0+\sum_{k=1}^{\infty}\frac{\partial^k u_i}{\partial t^k}|_{t=0}\ \frac{t^k}{k!}.$$

For the calculation of $\frac{\partial u_i}{\partial t},\frac{\partial^2 u_i}{\partial t^2},\frac{\partial^3 u_i}{\partial t^3},\ldots$ we use the values that are obtained directly from the Navier-Stokes equations and its derivatives in relation to time, i.e.,

(5.3) $$\frac{\partial u_i}{\partial t}=-\frac{\partial p}{\partial x_i}-\sum_{j=1}^{3}u_j\frac{\partial u_i}{\partial x_j}+\nu\,\nabla^2 u_i+f_i,$$

(5.4) $$\frac{\partial^2 u_i}{\partial t^2}=-\frac{\partial^2 p}{\partial t\,\partial x_i}-\sum_{j=1}^{3}\left(\frac{\partial u_j}{\partial t}\frac{\partial u_i}{\partial x_j}+u_j\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\right)+\nu\,\nabla^2\frac{\partial u_i}{\partial t}+\frac{\partial f_i}{\partial t},$$

and using induction we come to

(5.5) $$\frac{\partial^k u_i}{\partial t^k}=-\frac{\partial^k p}{\partial t^{k-1}\,\partial x_i}-\sum_{j=1}^{3}N_j^{k-1}+\ \nu\,\nabla^2\frac{\partial^{k-1}u_i}{\partial t^{k-1}}+\frac{\partial^{k-1}f_i}{\partial t^{k-1}},$$
$$N_j^{k-1}=\frac{\partial}{\partial t}N_j^{k-2}=\sum_{l=0}^{k-1}\binom{k-1}{l}\partial_t^{k-1-l}u_j\ \frac{\partial}{\partial x_j}\partial_t^l u_i,$$

$$\partial_t^0 u_n = u_n,\ \partial_t^m u_n = \frac{\partial^m u_n}{\partial t^m},\ \binom{k-1}{l} = \frac{(k-1)!}{(k-1-l)!\, l!}.$$

In (5.1) and (5.2) it is necessary to know the values of the derivatives $\frac{\partial u_i}{\partial t}, \frac{\partial^2 u_i}{\partial t^2}, \ldots, \frac{\partial^k u_i}{\partial t^k}$ in $t = 0$ then we must to calculate, from (5.3) to (5.5),

(5.6) $$\frac{\partial u_i}{\partial t}|_{t=0} = -\frac{\partial p^0}{\partial x_i} - \sum_{j=1}^3 u_j^0 \frac{\partial u_i^0}{\partial x_j} + \nu\, \nabla^2 u_i^0 + f_i^0,$$

the superior index 0 meaning the value of the respective function at $t = 0$,

(5.7) $$\frac{\partial^2 u_i}{\partial t^2}|_{t=0} = -\frac{\partial^2 p}{\partial t\, \partial x_i}|_{t=0} - \sum_{j=1}^3 N_j^1\, |_{t=0} + \\ + \nu\, \nabla^2 \frac{\partial u_i}{\partial t}|_{t=0} + \frac{\partial f_i}{\partial t}|_{t=0},$$

$$N_j^1|_{t=0} = \sum_{j=1}^3 \left(\frac{\partial u_j}{\partial t}|_{t=0} \frac{\partial u_i^0}{\partial x_j} + u_j^0 \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t}|_{t=0} \right),$$

and of generic form,

(5.8) $$\frac{\partial^k u_i}{\partial t^k}|_{t=0} = -\frac{\partial^k p}{\partial t^{k-1}\, \partial x_i}|_{t=0} - \sum_{j=1}^3 N_j^{k-1}\, |_{t=0} + \\ + \nu\, \nabla^2 \frac{\partial^{k-1} u_i}{\partial t^{k-1}}|_{t=0} + \frac{\partial^{k-1} f_i}{\partial t^{k-1}}|_{t=0},$$

$$N_j^{k-1}|_{t=0} = \sum_{l=0}^{k-1} \binom{k-1}{l} \partial_t^{k-1-l} u_j|_{t=0} \frac{\partial}{\partial x_j} \partial_t^l u_i|_{t=0},$$

$$\partial_t^0 u_n|_{t=0} = u_n^0,\ \partial_t^m u_n|_{t=0} = \frac{\partial^m u_n}{\partial t^m}|_{t=0}.$$

This solution is an explicit representation of the non-uniqueness solution of the Euler's ($\nu = 0$) and Navier-Stokes equations because it is possible choose any smooth pressure to be a solution for the problem, in special in the cases without boundary conditions, where the domain of the position (x, y, z) is the whole space $\mathbb{R}^3$ and be of the class C^∞ is a necessary condition for the velocity, in each point of space and time.

The presented method can be implemented in other equations, of course, for example, heat equation, Schrödinger equation, wave equation and many others. In linear equations the facility of operations seems to be great.

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References

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