Factored closed-form expressions for the sums of cubes of Fibonacci and Lucas numbers*

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Abstract

We obtain *explicit* factored closed-form expressions for Fibonacci and Lucas sums of the form $\sum_{k=1}^{n} F_{2rk}^3$ and $\sum_{k=1}^{n} L_{2rk}^3$, where r and n are integers.

1 Introduction

The Fibonacci numbers, F_n, and Lucas numbers, L_n, are defined, for $n \in \mathbb{Z}$, as usual, through the recurrence relations $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$, with $F_{-n} = (-1)^{n-1} F_n$ and $L_{-n} = (-1)^n L_n$.

Clary and Hemenway [1] derived the remarkable formulas

$$4\sum_{k=1}^{n} F_{2k}^3 = \begin{cases} F_n^2 L_{n+1}^2 F_{n-1} L_{n+2} & \text{if } n \text{ is even,} \\ L_n^2 F_{n+1}^2 L_{n-1} F_{n+2} & \text{if } n \text{ is odd}\,, \end{cases} \tag{1.1}$$

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and

$$8\sum_{k=1}^{n} F_{4k}^3 = F_{2n}^2 F_{2n+2}^2 (L_{4n+2} + 6)\,. \tag{1.2}$$

In this present paper we will derive the following corresponding Lucas counterparts of (1.1) and (1.2):

$$4\sum_{k=1}^{n} L_{2k}^3 = \begin{cases} 5F_nF_{n+1}(L_nL_{n+1}L_{2n+1} + 16) & \text{if } n \text{ is even,}\\ L_nL_{n+1}(5F_nF_{n+1}L_{2n+1} + 16) & \text{if } n \text{ is odd.}\end{cases} \tag{1.3}$$

and

$$8\sum_{k=1}^{n} L_{4k}^3 = F_{2n}L_{2n+2}(5L_{2n}F_{2n+2}F_{4n+2} + 32)\,. \tag{1.4}$$

In fact we will derive the following more general results:

- If r is odd, then

$$L_{3r}\sum_{k=1}^{n} F_{2rk}^3 = \begin{cases} F_{rn}^2L_{rn+r}^2(L_{rn}F_{rn+r} + F_r) & \text{if } n \text{ is even,}\\ L_{rn}^2F_{rn+r}^2(F_{rn}L_{rn+r} + F_r) & \text{if } n \text{ is odd}\,,\end{cases} \tag{1.5}$$

$$L_{3r}\sum_{k=1}^{n} L_{2rk}^3 = \begin{cases} 5F_{rn}F_{rn+r}(L_{rn}L_{rn+r}L_{2rn+r} + 4(L_{2r}+1)) & \text{if } n \text{ is even,}\\ L_{rn}L_{rn+r}(5F_{rn}F_{rn+r}L_{2rn+r} + 4(L_{2r}+1)) & \text{if } n \text{ is odd}\,.\end{cases} \tag{1.6}$$

- If r is even, then

$$F_{3r}\sum_{k=1}^{n} F_{2rk}^3 = F_{rn}^2F_{rn+r}^2(L_{rn}L_{rn+r} + L_r)\,, \tag{1.7}$$

$$F_{3r}\sum_{k=1}^{n} L_{2rk}^3 = F_{rn}L_{rn+r}(5L_{rn}F_{rn+r}F_{2rn+r} + 4(L_{2r}+1))\,. \tag{1.8}$$

As variations on identities (1.5) and (1.7) we will prove

- If r is odd, then

$$L_{3r}\sum_{k=1}^{n} F_{2rk}^3 = \begin{cases} F_{rn}L_{rn+r}(L_{rn}F_{rn+r}F_{2rn+r} - 2F_r^2) & \text{if } n \text{ is even,}\\ L_{rn}F_{rn+r}(F_{rn}L_{rn+r}F_{2rn+r} - 2F_r^2) & \text{if } n \text{ is odd}\,.\end{cases}$$

- If r is even, then

$$5F_{3r}\sum_{k=1}^{n} F_{2rk}^3 = F_{rn}F_{rn+r}(L_{rn}L_{rn+r}L_{2rn+r} - 2L_r^2)\,.$$

2 Required identities and preliminary results

2.1 Telescoping summation identity

The following telescoping summation identity is a special case of more general identities proved in [3].

Lemma 2.1. *If $f(k)$ is a real sequence and m, q and n are positive integers, then*

$$\sum_{k=1}^{n}\left[f(mk+mq)-f(mk)\right]=\sum_{k=1}^{q}f(mk+mn)-\sum_{k=1}^{q}f(mk)\,.$$

2.2 First-power Fibonacci summation identities

Lemma 2.2. *If r and n are integers, then*

(i) If r is even, then

$$F_r\sum_{k=1}^{n}F_{2rk}=F_{rn}F_{rn+r}\,,$$

(ii) if r is odd, then

$$L_r\sum_{k=1}^{n}F_{2rk}=\begin{cases}F_{rn}L_{rn+r} & \text{if } n \text{ is even,}\\ L_{rn}F_{rn+r} & \text{if } n \text{ is odd.}\end{cases}$$

Proof. Setting $v=2r$ and $u=2rk$ in the identity

$$L_{u+v}-(-1)^{v}L_{u-v}=5F_uF_v \tag{2.1}$$

gives

$$L_{2rk+2r}-L_{2rk-2r}=5F_{2r}F_{2rk}\,. \tag{2.2}$$

Taking $f(k)=L_{k-2r}$, $q=2$ and $m=2r$ in Lemma 2.1 and employing identity (2.2) we have

$$\begin{aligned}5F_{2r}\sum_{k=1}^{n}F_{2rk}&=\sum_{k=1}^{2}L_{2rk+2rn-2r}-\sum_{k=1}^{2}L_{2rk-2r}\\&=L_{2rn+2r}+L_{2rn}-L_{2r}-2\,.\end{aligned} \tag{2.3}$$

If r is even, then on account of the identity

$$L_{u+v} + (-1)^v L_{u-v} = L_u L_v, \tag{2.4}$$

we have

$$L_{2rn+2r} + L_{2rn} = L_r L_{2rn+r}, \quad L_{2r} + 2 = L_r^2\,,$$

and since

$$F_{2u} = F_u L_u\,, \tag{2.5}$$

identity (2.3) now becomes

$$\begin{aligned} 5F_r \sum_{k=1}^{n} F_{2rk} &= L_{2rn+r} - L_r \\ &= 5F_{rn}F_{rn+r}\,, \quad \text{by (2.1)}\,, \end{aligned} \tag{2.6}$$

that is,

$$F_r \sum_{k=1}^{n} F_{2rk} = F_{rn}F_{rn+r}, \quad r \text{ even}\,,$$

and the first part of Lemma 2.2 is proved.

If r is odd, then on account of the identities (2.1) and (2.4), we have

$$L_{2rn+2r} + L_{2rn} = 5F_r F_{2rn+r}, \quad L_{2r} + 2 = 5F_r^2\,,$$

and identity (2.3) reduces to

$$\begin{aligned} L_r \sum_{k=1}^{n} F_{2rk} &= F_{2rn+r} - F_r \\ &= \begin{cases} F_{rn}L_{rn+r} & \text{if } n \text{ is even,} \\ L_{rn}F_{rn+r} & \text{if } n \text{ is odd,} \end{cases} \end{aligned}$$

and the second part of Lemma 2.2 is proved. In the last stage of the above derivation we made use of the identities

$$F_{u+v} - (-1)^v F_{u-v} = F_v L_u \tag{2.7}$$

and

$$F_{u+v} + (-1)^v F_{u-v} = L_v F_u\,. \tag{2.8}$$

□

2.3 First-power Lucas summation identities

Lemma 2.3. *If r and n are integers, then*

(i) If r is even, then

$$F_r \sum_{k=1}^{n} L_{2rk} = F_{rn} L_{rn+r}\,,$$

(ii) if r is odd, then

$$L_r \sum_{k=1}^{n} L_{2rk} = \begin{cases} 5F_{rn}F_{rn+r} & \text{if } n \text{ is even,}\\ L_{rn}L_{rn+r} & \text{if } n \text{ is odd.}\end{cases}$$

Proof. Setting $v = 2r$ and $u = 2rk$ in the identity (2.7) gives

$$F_{2rk+2r} - F_{2rk-2r} = F_{2r} L_{2rk}\,. \tag{2.9}$$

Taking $f(k) = F_{k-2r}$, $q = 2$ and $m = 2r$ in Lemma 2.1 and employing identity (2.9) we have

$$\begin{aligned} F_{2r} \sum_{k=1}^{n} L_{2rk} &= \sum_{k=1}^{2} F_{2rk+2rn-2r} - \sum_{k=1}^{2} F_{2rk-2r} \\ &= F_{2rn+2r} + F_{2rn} - F_{2r}\,. \end{aligned} \tag{2.10}$$

If r is even, then choosing $v = r$ and $u = 2rn + r$ in identity (2.8) gives

$$F_{2rn+2r} + F_{2rn} = L_r F_{2rn+r} \tag{2.11}$$

and, on account of identity (2.5), the identity (2.10) reduces to

$$\begin{aligned} F_r \sum_{k=1}^{n} L_{2rk} &= F_{2rn+r} - F_r \\ &= F_{rn+r+rn} - F_{rn+r-rn} \\ &= F_{rn} L_{rn+r}\,, \quad \text{by identity (2.7),} \end{aligned}$$

and the first part of Lemma 2.3 is proved.

If r is odd, then choosing $v = r$ and $u = 2rn + r$ in identity (2.7) gives

$$F_{2rn+2r} + F_{2rn} = F_r L_{2rn+r} \tag{2.12}$$

and, again on account of identity (2.5), the identity (2.10) now reduces to

$$\begin{aligned}
L_r \sum_{k=1}^{n} L_{2rk} &= L_{2rn+r} - L_r \\
&= L_{rn+r+rn} - L_{rn+r-rn} \\
&= \begin{cases} 5F_{rn}F_{rn+r} & \text{if } n \text{ is even,} \\ L_{rn}L_{rn+r} & \text{if } n \text{ is odd,} \end{cases}
\end{aligned}$$

where in the last step we used the identities (2.1) and (2.4).

□

2.4 Other identities

Lemma 2.4. *If r and n are integers, then*

$$\frac{F_{3rn}F_{3rn+3r}}{F_{rn}F_{rn+r}} = L_{rn}L_{rn+r}L_{2rn+r} + L_{2r} + (-1)^{r-1}.$$

Proof. Using the identity (equation (36) of [1], also (3.3) of [4])

$$F_{3u} = 5F_u^3 + 3(-1)^u F_u\,, \tag{2.13}$$

we have

$$\begin{aligned}
\frac{F_{3rn}F_{3rn+3r}}{F_{rn}F_{rn+r}} &= (5F_{rn}^2 + 3(-1)^{rn})(5F_{rn+r}^2 + 3(-1)^{rn+r}) \\
&= (L_{rn}^2 - (-1)^{rn})(L_{rn+r}^2 - (-1)^{rn+r}) \\
&= L_{rn}^2 L_{rn+r}^2 - (-1)^{rn+r}L_{rn}^2 - (-1)^{rn}L_{rn+r}^2 + (-1)^r\,,
\end{aligned} \tag{2.14}$$

where we have also made use of the identity

$$5F_u^2 - L_u^2 = (-1)^{u-1}4\,. \tag{2.15}$$

Now,

$$\begin{aligned}
L_{rn}^2 L_{rn+r}^2 &= L_{rn}L_{rn+r}(L_{rn}L_{rn+r}) \\
&= L_{rn}L_{rn+r}(L_{2rn+r} + (-1)^{rn}L_r) \quad \text{by (2.4)} \\
&= L_{rn}L_{rn+r}L_{2rn+r} + (-1)^{rn}L_{rn}L_{rn+r}L_r\,.
\end{aligned}$$

Therefore

$$\begin{aligned}\frac{F_{3rn}F_{3rn+3r}}{F_{rn}F_{rn+r}} &= L_{rn}L_{rn+r}L_{2rn+r}\\ &\quad + (-1)^{rn}L_{rn+r}(L_{rn}L_r - L_{rn+r})\\ &\quad\quad - (-1)^{rn+r}L_{rn}^2 + (-1)^r\,.\end{aligned}$$

But

$$\begin{aligned}&(-1)^{rn}L_{rn+r}(L_{rn}L_r - L_{rn+r})\\ &= (-1)^{rn}L_{rn+r}(L_{rn+r} + (-1)^r L_{rn-r} - L_{rn+r}), \quad \text{by (2.4)}\\ &= (-1)^{rn+r}L_{rn+r}L_{rn-r}\\ &= (-1)^{rn+r}(L_{2rn} + (-1)^{rn-r}L_{2r}), \quad \text{again by (2.4)}\\ &= (-1)^{rn+r}L_{2rn} + L_{2r}\,.\end{aligned}$$

Thus

$$\begin{aligned}\frac{F_{3rn}F_{3rn+3r}}{F_{rn}F_{rn+r}} &= L_{rn}L_{rn+r}L_{2rn+r}\\ &\quad + (-1)^{rn+r}L_{2rn} + L_{2r}\\ &\quad\quad - (-1)^{rn+r}L_{rn}^2 + (-1)^r\\ &= L_{rn}L_{rn+r}L_{2rn+r}\\ &\quad + (-1)^{rn+r}(L_{2rn} - L_{rn}^2)\\ &\quad\quad + L_{2r} + (-1)^r\,.\end{aligned}$$

Finally, using the identity

$$L_{2u} = L_u^2 + (-1)^{u-1}2\,, \tag{2.16}$$

obtained by setting $v = u$ in identity (2.4), we have the statement of the Lemma. □

Lemma 2.5. *If r and n are integers, then*

$$\frac{L_{3rn}L_{3rn+3r}}{L_{rn}L_{rn+r}} = 5F_{rn}F_{rn+r}L_{2rn+r} + L_{2r} + (-1)^{r-1}.$$

Proof. Using the following identity, (equation (1.6) of [4])

$$L_{3u} = L_u^3 - 3(-1)^u L_u\,, \tag{2.17}$$

we have

$$\begin{aligned}\frac{L_{3rn}L_{3rn+3r}}{L_{rn}L_{rn+r}} &= (L_{rn}^2 - 3(-1)^{rn})(L_{rn+r}^2 - 3(-1)^{rn+r})\\ &= (5F_{rn}^2 + (-1)^{rn})(5F_{rn+r}^2 + (-1)^{rn+r}), \quad \text{by (2.15)}\\ &= 25F_{rn}^2F_{rn+r}^2 + (-1)^{rn+r}5F_{rn}^2 + (-1)^{rn}5F_{rn+r}^2 + (-1)^r\,,\end{aligned}$$

and the rest of the calculation then proceeds as in the proof of Lemma 2.4, the basic required identities now being (2.1), (2.8) and the identity

$$L_{2u} = 5F_u^2 + (-1)^u 2, \tag{2.18}$$

obtained by setting $v = u$ in identity (2.1). □

Lemma 2.6. *If r and n are integers, then*

$$\frac{L_{3rn}F_{3rn+3r}}{L_{rn}F_{rn+r}} = 5F_{rn}L_{rn+r}F_{2rn+r} + L_{2r} + (-1)^r.$$

Lemma 2.7. *If r and n are integers, then*

$$\frac{F_{3rn}L_{3rn+3r}}{F_{rn}L_{rn+r}} = 5L_{rn}F_{rn+r}F_{2rn+r} + L_{2r} + (-1)^r.$$

Different but equivalent versions of Lemmata 2.4—2.7 are given below:

Lemma 2.8. *If r and n are integers, then*

$$\frac{F_{3rn}F_{3rn+3r}}{F_{rn}F_{rn+r}} = L_{2rn+r}^2 + (-1)^{nr}L_{rn+r}^2 + (-1)^{(n-1)r}L_{rn}^2 + L_r^2 + (-1)^{r-1}7\,.$$

Proof. The proof is similar to that of Lemma 2.4, but here we use

$$\begin{aligned}L_{rn}^2L_{rn+r}^2 &= (L_{2rn+r} + (-1)^{rn}L_r)^2\\ &= L_{2rn+r}^2 + L_r^2 + 2(-1)^{rn}\{L_rL_{2rn+r}\}\\ &= L_{2rn+r}^2 + L_r^2 + 2(-1)^{rn}\{L_{2rn+2r} + (-1)^rL_{2rn}\}\\ &= L_{2rn+r}^2 + L_r^2 + 2(-1)^{rn}\{L_{rn+r}^2 + (-1)^{rn+r-1}2\\ &\qquad + (-1)^r[L_{rn}^2 + (-1)^{rn-1}2]\}\end{aligned}$$

and substitute in (2.14). □

Lemma 2.9. *If r and n are integers, then*

$$\frac{L_{3rn}L_{3rn+3r}}{L_{rn}L_{rn+r}} = L_{2rn+r}^2 + (-1)^{nr-1}L_{rn+r}^2 - (-1)^{(n-1)r}L_{rn}^2 + L_r^2 + (-1)^r\,.$$

Lemma 2.10. *If r and n are integers, then*

$$\frac{L_{3rn}F_{3rn+3r}}{L_{rn}F_{rn+r}} = 5F_{2rn+r}^2 + (-1)^{nr-1}5F_{rn+r}^2 + (-1)^{(n-1)r}5F_{rn}^2 + 5F_r^2 + (-1)^r 3\,.$$

Lemma 2.11. *If r and n are integers, then*

$$\frac{F_{3rn}L_{3rn+3r}}{F_{rn}L_{rn+r}} = 5F_{2rn+r}^2 + (-1)^{nr}5F_{rn+r}^2 - (-1)^{(n-1)r}5F_{rn}^2 + 5F_r^2 + (-1)^r 3\,.$$

3 Main results

3.1 Sums of cubes of Fibonacci numbers

Theorem 3.1. *If r and n are integers such that r is odd, then*

$$L_{3r}\sum_{k=1}^n F_{2rk}^3 = \begin{cases} F_{rn}L_{rn+r}(L_{rn}F_{rn+r}F_{2rn+r} - 2F_r^2) & \text{if } n \text{ is even,}\\ L_{rn}F_{rn+r}(F_{rn}L_{rn+r}F_{2rn+r} - 2F_r^2) & \text{if } n \text{ is odd}\,. \end{cases}$$

Proof. Setting $u = 2rk$ in identity (2.13) and summing, we have

$$5\sum_{k=1}^n F_{2rk}^3 = \sum_{k=1}^n F_{6rk} - 3\sum_{k=1}^n F_{2rk}\,,$$

so that,

$$\begin{aligned} 5L_{3r}\sum_{k=1}^n F_{2rk}^3 &= L_{3r}\sum_{k=1}^n F_{6rk} - 3\frac{L_{3r}}{L_r}L_r\sum_{k=1}^n F_{2rk}\\ &= L_{3r}\sum_{k=1}^n F_{6rk} - 3(L_r^2+3)L_r\sum_{k=1}^n F_{2rk}\,. \end{aligned} \tag{3.1}$$

- If n is even, then, by Lemma 2.2, identity (3.1) can be written as

$$5L_{3r}\sum_{k=1}^n F_{2rk}^3 = F_{3rn}L_{3rn+3r} - 3(L_r^2+3)F_{rn}L_{rn+r}\,,$$

so that

$$\begin{aligned}\frac{5L_{3r}\sum_{k=1}^{n}F_{2rk}^{3}}{F_{rn}L_{rn+r}} &= \frac{F_{3rn}L_{3rn+3r}}{F_{rn}L_{rn+r}} - 3(L_r^2+3)\\ &= 5L_{rn}F_{rn+r}F_{2rn+r} + L_{2r} - 1 - 3L_r^2 - 9, \quad \text{by Lemma 2.7}\\ &= 5L_{rn}F_{rn+r}F_{2rn+r} - 10F_r^2, \quad \text{by (2.15) and (2.16)}\,.\end{aligned}$$

- If n is odd, then, by Lemma 2.2, we have

$$5L_{3r}\sum_{k=1}^{n}F_{2rk}^{3} = L_{3rn}F_{3rn+3r} - 3(L_r^2+3)L_{rn}F_{rn+r}\,,$$

so that

$$\begin{aligned}\frac{5L_{3r}\sum_{k=1}^{n}F_{2rk}^{3}}{L_{rn}F_{rn+r}} &= \frac{L_{3rn}F_{3rn+3r}}{L_{rn}F_{rn+r}} - 3(L_r^2+3)\\ &= 5F_{rn}L_{rn+r}F_{2rn+r} + L_{2r} - 1 - 3L_r^2 - 9, \quad \text{by Lemma 2.7}\\ &= 5F_{rn}L_{rn+r}F_{2rn+r} - 10F_r^2, \quad \text{by (2.15) and (2.16)}\,.\end{aligned}$$

□

Theorem 3.2. *If r and n are integers such that r is even, then*

$$5F_{3r}\sum_{k=1}^{n}F_{2rk}^{3} = F_{rn}F_{rn+r}(L_{rn}L_{rn+r}L_{2rn+r} - 2L_r^2)\,.$$

Proof.

$$\begin{aligned}5F_{3r}\sum_{k=1}^{n}F_{2rk}^{3} &= F_{3r}\sum_{k=1}^{n}F_{6rk} - 3\frac{F_{3r}}{F_r}F_r\sum_{k=1}^{n}F_{2rk}\\ &= F_{3rn}F_{3rn+3r} - 3(5F_r^2+3)F_{rn}F_{rn+r},\\ &\quad\text{by Lemma 2.2 and identity (2.13)}\,,\end{aligned}$$

so that

$$\begin{aligned}\frac{5F_{3r}\sum_{k=1}^{n}F_{2rk}^{3}}{F_{rn}F_{rn+r}} &= \frac{F_{3rn}F_{3rn+3r}}{F_{rn}F_{rn+r}} - 3(5F_r^2+3)\\ &= L_{rn}L_{rn+r}L_{2rn+r} + L_{2r} - 1 - 15F_r^2 - 9\\ &\quad\text{(by Lemma 2.4 and identity (2.13))}\,,\\ &= L_{rn}L_{rn+r}L_{2rn+r} - 2L_r^2, \quad \text{by (2.15), (2.16) and (2.18)}\,.\end{aligned}$$

□

Theorem 3.3. *If r and n are integers such that r is odd, then*

$$L_{3r}\sum_{k=1}^{n}F_{2rk}^{3}=\begin{cases}F_{rn}^{2}L_{rn+r}^{2}(L_{rn}F_{rn+r}+F_r) & \text{if } n \text{ is even,}\\ L_{rn}^{2}F_{rn+r}^{2}(F_{rn}L_{rn+r}+F_r) & \text{if } n \text{ is odd,}\end{cases}$$

Proof. • if n is even, then from Lemma 2.2 and identity (3.1) we have

$$\begin{aligned}\frac{5L_{3r}\sum_{k=1}^{n}F_{2rk}^{3}}{F_{rn}L_{rn+r}}&=\frac{F_{3rn}L_{3rn+3r}}{F_{rn}L_{rn+r}}-3(L_r^2+3)\\&=5F_{2rn+r}^{2}+5F_{rn+r}^{2}+5F_{rn}^{2}+5F_r^2-3-3L_r^2-9,\ \text{by Lemma 2.11}\\&=5F_{2rn+r}^{2}+5F_{rn+r}^{2}+5F_{rn}^{2}-10F_r^2\quad\text{by identity (2.15)},\end{aligned}$$

so that

$$\begin{aligned}\frac{L_{3r}\sum_{k=1}^{n}F_{2rk}^{3}}{F_{rn}L_{rn+r}}&=F_{2rn+r}^{2}+F_{rn+r}^{2}+F_{rn}^{2}-2F_r^2\\&=(F_{2rn+r}^{2}-F_r^2)+(F_{rn+r}^{2}+F_{rn}^{2})-F_r^2\,.\end{aligned}$$

Using the following identity, derived in [5],

$$F_u^2+(-1)^{u+v-1}F_v^2=F_{u-v}F_{u+v}\,, \tag{3.2}$$

we have

$$\begin{aligned}\frac{L_{3r}\sum_{k=1}^{n}F_{2rk}^{3}}{F_{rn}L_{rn+r}}&=F_{2rn}F_{2rn+2r}+F_rF_{2rn+r}-F_r^2\\&=F_{2rn}F_{2rn+2r}+F_r(F_{2rn+r}-F_r)\\&=F_{2rn}F_{2rn+2r}+F_rF_{rn}L_{rn+r},\ \text{by identity (2.7)}\\&=F_{rn}L_{rn+r}L_{rn}F_{rn+r}+F_rF_{rn}L_{rn+r}\\&=F_{rn}L_{rn+r}(L_{rn}F_{rn+r}+F_r)\,.\end{aligned}$$

• if n is odd, then from Lemma 2.2 and identity (3.1) we have

$$\begin{aligned}\frac{5L_{3r}\sum_{k=1}^{n}F_{2rk}^{3}}{L_{rn}F_{rn+r}}&=\frac{L_{3rn}F_{3rn+3r}}{L_{rn}F_{rn+r}}-3(L_r^2+3)\\&=5F_{2rn+r}^{2}+5F_{rn+r}^{2}+5F_{rn}^{2}+5F_r^2-3-3L_r^2-9,\ \text{by Lemma 2.10}\\&=5F_{2rn+r}^{2}+5F_{rn+r}^{2}+5F_{rn}^{2}-10F_r^2\quad\text{by identity (2.15)},\end{aligned}$$

so that

$$\frac{L_{3r}\sum_{k=1}^n F_{2rk}^3}{L_{rn}F_{rn+r}} = F_{2rn+r}^2 + F_{rn+r}^2 + F_{rn}^2 - 2F_r^2$$
$$= (F_{2rn+r}^2 - F_r^2) + (F_{rn+r}^2 + F_{rn}^2) - F_r^2\,.$$

Using identity (3.2), we have

$$\frac{L_{3r}\sum_{k=1}^n F_{2rk}^3}{L_{rn}F_{rn+r}} = F_{2rn}F_{2rn+2r} + F_rF_{2rn+r} - F_r^2$$
$$= F_{2rn}F_{2rn+2r} + F_r(F_{2rn+r} - F_r)$$
$$= F_{2rn}F_{2rn+2r} + F_rL_{rn}F_{rn+r},\text{ by identity (2.8)}$$
$$= F_{rn}L_{rn+r}L_{rn}F_{rn+r} + F_rL_{rn}F_{rn+r}$$
$$= L_{rn}F_{rn+r}(F_{rn}L_{rn+r} + F_r)\,.$$

□

Theorem 3.4. *If r and n are integers such that r is even, then*

$$F_{3r}\sum_{k=1}^n F_{2rk}^3 = F_{rn}^2F_{rn+r}^2(L_{rn}L_{rn+r} + L_r)\,. \tag{3.3}$$

Proof.

$$5F_{3r}\sum_{k=1}^n F_{2rk}^3 = F_{3r}\sum_{k=1}^n F_{6rk} - 3\frac{F_{3r}}{F_r}F_r\sum_{k=1}^n F_{2rk}$$
$$= F_{3rn}F_{3rn+3r} - 3(5F_r^2 + 3)F_{rn}F_{rn+r}\,,$$

so that

$$\frac{5F_{3r}\sum_{k=1}^n F_{2rk}^3}{F_{rn}F_{rn+r}} = \frac{F_{3rn}F_{3rn+3r}}{F_{rn}F_{rn+r}} - 3(5F_r^2 + 3)$$
$$= L_{2rn+r}^2 + L_{rn+r}^2 + L_{rn}^2 + L_r^2 - 7 - 15F_r^2 - 9,\quad\text{by Lemma 2.8}$$
$$= L_{2rn+r}^2 + L_{rn+r}^2 - 2L_r^2 + 5F_{rn}^2,\quad\text{by (2.15)}$$
$$= (L_{2rn+r}^2 - L_r^2) + (L_{rn+r}^2 - L_r^2) + 5F_{rn}^2\,.$$

Using the identity (derived in [5])

$$L_u^2 + (-1)^{u+v-1}L_v^2 = 5F_{u-v}F_{u+v}\,, \tag{3.4}$$

we see that

$$L_{2rn+r}^2 - L_r^2 = 5F_{2rn}F_{2rn+2r} = 5F_{rn}F_{rn+r}L_{rn}L_{rn+r} \tag{3.5}$$

and

$$L_{rn+r}^2 - L_r^2 = 5F_{rn}F_{rn+2r}\,. \tag{3.6}$$

Thus,

$$\begin{aligned}
\frac{F_{3r}\sum_{k=1}^{n} F_{2rk}^3}{F_{rn}F_{rn+r}} &= F_{rn}F_{rn+r}L_{rn}L_{rn+r} + F_{rn}F_{rn+2r} + F_{rn}^2\\
&= F_{rn}F_{rn+r}L_{rn}L_{rn+r} + F_{rn}(F_{rn} + F_{rn+2r})\\
&= F_{rn}F_{rn+r}L_{rn}L_{rn+r} + F_{rn}F_{rn+r}L_r, \quad \text{by identity (2.8)}\\
&= F_{rn}F_{rn+r}(L_{rn}L_{rn+r} + L_r)\,.
\end{aligned}$$

□

3.2 Sums of cubes of Lucas numbers

Theorem 3.5. *If r and n are integers such that r is odd, then*

$$L_{3r}\sum_{k=1}^{n} L_{2rk}^3 = \begin{cases} 5F_{rn}F_{rn+r}(L_{rn}L_{rn+r}L_{2rn+r} + 4(L_{2r}+1)) & \text{if } n \text{ is even,}\\ L_{rn}L_{rn+r}(5F_{rn}F_{rn+r}L_{2rn+r} + 4(L_{2r}+1)) & \text{if } n \text{ is odd}\,. \end{cases}$$

Proof. Using identity (2.17) with $u = 2rk$, we have

$$\sum_{k=1}^{n} L_{2rk}^3 = \sum_{k=1}^{n} L_{6rk} + 3\sum_{k=1}^{n} L_{2rk}\,,$$

so that

$$\begin{aligned}
L_{3r}\sum_{k=1}^{n} L_{2rk}^3 &= L_{3r}\sum_{k=1}^{n} L_{6rk} + 3\frac{L_{3r}}{L_r}L_r\sum_{k=1}^{n} L_{2rk}\\
&= L_{3r}\sum_{k=1}^{n} L_{6rk} + 3(L_r^2+3)L_r\sum_{k=1}^{n} L_{2rk}, \quad \text{by (2.17)}\,.
\end{aligned}$$

- If n is even, then by Lemma 2.3 we have

$$L_{3r}\sum_{k=1}^{n} L_{2rk}^3 = 5F_{3rn}F_{3rn+3r} + 3(L_r^2+3)5F_{rn}F_{rn+r}\,, \tag{3.7}$$

so that

$$\begin{aligned}\frac{L_{3r}\sum_{k=1}^{n}L_{2rk}^3}{5F_{rn}F_{rn+r}} &= \frac{F_{3rn}F_{3rn+3r}}{F_{rn}F_{rn+r}} + 3(L_r^2+3)\\ &= L_{rn}L_{rn+r}L_{2rn+r} + L_{2r} + 1 + 3L_r^2 + 9, \quad \text{by Lemma 2.4}\\ &= L_{rn}L_{rn+r}L_{2rn+r} + 4(L_{2r}+1), \quad \text{by (2.16)}\,.\end{aligned}$$

- If n is odd, then by Lemma 2.3 we have

$$L_{3r}\sum_{k=1}^{n}L_{2rk}^3 = L_{3rn}L_{3rn+3r} + 3(L_r^2+3)L_{rn}L_{rn+r}\,, \tag{3.8}$$

so that

$$\begin{aligned}\frac{L_{3r}\sum_{k=1}^{n}L_{2rk}^3}{L_{rn}L_{rn+r}} &= \frac{L_{3rn}L_{3rn+3r}}{L_{rn}L_{rn+r}} + 3(L_r^2+3)\\ &= 5F_{rn}F_{rn+r}L_{2rn+r} + L_{2r} + 1 + 3L_r^2 + 9, \quad \text{by Lemma 2.5}\\ &= 5F_{rn}F_{rn+r}L_{2rn+r} + 4(L_{2r}+1), \quad \text{by (2.16)}\,.\end{aligned}$$

□

Theorem 3.6. *If r and n are integers such that r is even, then*

$$F_{3r}\sum_{k=1}^{n}L_{2rk}^3 = F_{rn}L_{rn+r}(5L_{rn}F_{rn+r}F_{2rn+r} + 4(L_{2r}+1))\,.$$

Proof.

$$\begin{aligned}F_{3r}\sum_{k=1}^{n}L_{2rk}^3 &= F_{3r}\sum_{k=1}^{n}L_{6rk} + 3\frac{F_{3r}}{F_r}F_r\sum_{k=1}^{n}L_{2rk}\\ &= F_{3r}\sum_{k=1}^{n}L_{6rk} + 3(5F_r^2+3)F_r\sum_{k=1}^{n}L_{2rk}, \quad \text{by identity (2.13)}\\ &= F_{3rn}L_{3rn+3r} + 3(5F_r^2+3)F_{rn}L_{rn+r}, \quad \text{by Lemma 2.3}\,.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{F_{3r}\sum_{k=1}^{n}L_{2rk}^3}{F_{rn}L_{rn+r}} &= \frac{F_{3rn}L_{3rn+3r}}{F_{rn}L_{rn+r}} + 3(5F_r^2+3)\\ &= 5L_{rn}F_{rn+r}F_{2rn+r} + L_{2r} + 1 + 15F_r^2 + 9, \quad \text{by Lemma 2.7}\\ &= 5L_{rn}F_{rn+r}F_{2rn+r} + 4(L_{2r}+1), \quad \text{by (2.16) and (2.18)}\,.\end{aligned}$$

□

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