# New sufficient conditions of signal recovery with tight frames via $l_1$-analysis[*]

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**Abstract.**   The paper discusses the recovery of signals in the case that signals are nearly sparse with respect to a tight frame $D$ by means of the $l_1$-analysis approach. We establish several new sufficient conditions regarding the $D$-restricted isometry property to ensure stable reconstruction of signals that are approximately sparse with respect to $D$. It is shown that if the measurement matrix $\Phi$ fulfils the condition $\delta_{ts} < t/(4-t)$ for $0 < t < 4/3$, then signals which are approximately sparse with respect to $D$ can be stably recovered by the $l_1$-analysis method. In the case of $D = I$, the bound is sharp, see Cai and Zhang's work [5]. When $t = 1$, the present bound improves the condition $\delta_s < 0.307$ from Lin et al.'s reuslt to $\delta_s < 1/3$. In addition, numerical simulations are conducted to indicate that the $l_1$-analysis method can stably reconstruct the sparse signal in terms of tight frames.

**Keywords.**   Compressed sensing; $l_1$-analysis approach; restricted isometry property; sparse recovery; tight frames.

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## 1   Introduction

In recent years, compressed sensing (CS) has brought about important research activity. Formally, in CS, researchers are interested in the model below

$$b = \Phi f + z,$$

where $\Phi \in \mathbb{R}^{M \times N}$ ($M \ll N$) is a known measurement matrix, observed signal $b \in \mathbb{R}^M$, an unknown signal $f \in \mathbb{R}^N$ and $z \in \mathbb{R}^M$ is a vector of measurement errors. In the standard compressed sensing frame, if the signal is (nearly) sparse in the nature base or other orthonormal bases, it can be accurately or stably recovered in the noiseless case or the noise case respectively under various sufficient conditions on the sensing matrix $\Phi$, for instance a restricted isometry property (RIP) condition, see [1-15].

However, in practical applications, there exist a large number of signals of interest whose sparsity is not expressed in terms of an orthogonal basis. On the contrary, these signals are sparse with

respect to an overcomplete dictionary or a tight frame [16, 17] or a general frame [18]. That is to say, the original signal $f \in \mathbb{R}^N$ can be represented as $f = Dx$ where $D$ is some coherent and redundant dictionary with the size $N \times d$ ($N \leq d$) and $x$ is (approximately) sparse coefficient with $x \in \mathbb{R}^d$.

For a matrix $D$, a tight frame is formed by its $d$ columns $D_1, \cdots, D_d$, namely,

$$\sum_{i=1}^{d} D_i \langle f, D_i \rangle = f \text{ and } \sum_{i=1}^{d} |\langle f, D_i \rangle|^2 = \|f\|_2^2$$

for all $f \in \mathbb{R}^N$, where $\langle \cdot, \cdot \rangle$ stands for the standard Euclidean inner product. For all $q \in [1, \infty)$, $x \in \mathbb{R}^N$, let $\|x\|_q = \sqrt[q]{\sum_{i=1}^{N} |x_i|^q}$ and $\|x\|_\infty = \bigvee_{i=1}^{N} |x_i|$ denote the maximum of $\{|x_1|, \cdots, |x_N|\}$. One can easily prove that for all $f \in \mathbb{R}^N$,

$$DD^* = I_n, \ \|f\|_2^2 = \|D^*f\|_2^2,$$

where $D^*$ denotes the transpose of the matrix $D$ and $I_n$ is an identity matrix of $n$ order.

Our goal in the present paper is to recover the true signal $f \in \mathbb{R}^N$ from $M$ linear measurements $b = \Phi f + z$ in the case that the signal is sparse or approximately sparse in terms of $D$. The $l_1$-analysis approach below is utilized for the recovery of such signal

$$\tilde{f} = \arg \min_{f \in \mathbb{R}^N} \|D^*f\|_1 \text{ subject to } \Phi f - b \in \mathcal{B}, \tag{1.1}$$

where $\mathcal{B}$ is a bounded set relied on the noise structure. In this paper, we discuss two types of bounded noise: $l_2$ bounded $\mathcal{B}_0(\varepsilon) = \{z : \|z\|_2 \leq \varepsilon\}$ and Dantzig selector bounded $\mathcal{B}_1(\zeta) = \{z : \|D^*\Phi^*z\|_\infty \leq \zeta\}$. In order to discuss the performance of the above method, we will present the concept of $D$-RIP of a sensing matrix, which was first proposed by Candés et al. [17]. In fact, it is a generalization to the standard RIP.

**Definition 1.1.** *(D-RIP) Let $D$ be a matrix with the size $N \times d$ ($N \leq d$). One says that a measurement matrix $\Phi$ satisfies the restricted isometry property adapted to $D$ (D-RIP for short) of $s$ order with constant $\delta_s$ if*

$$\sqrt{1 - \delta_s}\|Dx\|_2 \leq \|\Phi Dx\|_2 \leq \sqrt{1 + \delta_s}\|Dx\|_2 \tag{1.2}$$

*holds for all $s$-sparse vector $x$ in $\mathbb{R}^d$. We call a vector $x \in \mathbb{R}^d$ $s$-sparse if $|\sup(x)| \leq s$, where $\sup(x) = \{i : x_i \neq 0\}$ represents the support of the vector $x$. Define the $D$-RIP constant $\delta_s$ as the smallest number $\delta_s$ obeying (1.2) for any $s$-sparse vector $x \in \mathbb{R}^d$.*

By the way, we give the definition of $l_0$ based on the notion of support that $\|x\|_0 = |\sup(x)|$, that is, $\|x\|_0$ counts the number of non-zero entries for vector $x$. In the case of that $D$ is an identity matrix, i.e., $D = I$, the Definition 1.1 degenerates to the definition of standard RIP.

It has been shown that sparse signals in terms of $D$ can be recovered though $l_1$-analysis approach under a variety of conditions on the $D$-RIP. Candés et al. [17] proved that a sufficient condition $\delta_{2s} < 0.08$ can guarantee the reconstruction of sparse signals. In [19, 20] respectively improved the upper bound to $\delta_{2s} < 0.4931$ and $\delta_s < 0.307$. Liu et al. [18] employed the assumption $9\delta_{2s} + 4\delta_{4s} < 5$ to assure recovery under the general frame. Zhang and Li [21] refined the bound to $\delta_{2s} < \sqrt{2}/2 \approx 0.707$ and $\delta_s < 1/3 \approx 0.333$. Furthermore, Chen and Li [22] gave a high order condition on the $D$-RIP for the recovery of signal.

In this paper, we provide a sufficient condition on D-RIP constant $\delta_{ts}$ under which signals from undersampled data that are approximately sparse in terms of $D$ are guaranteed to be stably reconstructed in the noise situation or exactly recovered in the noiseless situation via the $l_1$-analysis method. We prove that under the condition $\delta_{ts} < t/(4-t)$ for $0 < t < 4/3$, any signals $f$ which are sparse with respect to a tight frame $D$ can be accurately and stably recovered via (1.1). When $t = 1$, our main results are consistent with Theorem 3.1 and Theorem 4.1 in [21]. Besides, in the situation of $D = I$ (for the standard compressing sensing), we obtain same results as the main results in [23] and the bound on the constant $\delta_{ts}$ is sharp, referred to see [5]. Observe that our new bound $\delta_{ts} < t/(4-t)$ ($t \in (0, 4/3)$) improves the condition $\delta_s < 0.307$ given in [20] in the case when $t$ is equal to 1. Weakening the $D$-RIP condition brings a few advantages [20]. First of all, it allows more sensing matrices to be utilized in compressed sensing. Second, it provides better error estimation in a usual issue to reconstruct signals with noise. Last, improving the $D$-RIP condition allows estimating a sparse signal with more nonzero entries.

The remainder of this paper is organized as following. Some key lemmas and notations are presented in section 2. The main results are establish in section 3. In section 4, we provide proofs of theorems. In section 5, some experiments are carried out to indicate $l_1$-analysis method is robust to the signal reconstruction. Conclusions are given in the section 6.

## 2    Some technical lemmas

Define $D$ as a tight frame with the size $N \times d$ ($N \le d$). This means that all row vectors are orthonormal. Denote by $D^\perp$ its orthonormal complement, then it is also a tight frame. Therefore, for all $x \in \mathbb{R}^d$,

$$\sqrt{\|Dx\|_2^2 + \|D^\perp x\|_2^2} = \|x\|_2.$$

The above equality implies that the property of a tight frame $D$: $\|Dx\|_2 \le \|x\|_2$. It is straightforward to check that if the measurement matrix $\Phi$ obeys D-RIP with constant $\delta_s$, then

$$\sqrt{1 - \delta_s}\|x\|_2 \le \sqrt{\|\Phi Dx\|_2^2 + \|D^\perp x\|_2^2} \le \sqrt{1 + \delta_s}\|x\|_2 \tag{2.1}$$

holds for all vectors $x \in \mathbb{R}^d$ with $\|x\|_0 \le s$.

Throughout of this article, we utilize the following notations. Set $\tilde{f} = f + h$ be a solution to (1.1), where $f$ is the unknown signal we wish to recover. For $T \subset \{1, 2, \cdots, d\}$, $D_T$ indicates the matrix $D$ limited to the columns indexed by $T$ and $T^c$ represents the complement of $T$ in $\{1, 2, \cdots, d\}$, i.e., $T^c = \{1, 2, \cdots, d\}\backslash T$. Set $D_T^* = (D_T)^*$ and $D^*h = (x_1, x_2, \cdots, x_d)$. Suppose that $\{|x_i|\}_{i=1}^d$ is a non-increasing sequence, rearranging the indices if necessary. Let $T_0 = \{1, 2, \cdots, s\}$. Let $S_0$ stand for the index set of the largest $s$ components of $D^*f$ in amplitude. Denote by $x_{[s]}$ the vector comprising the $s$ largest absolute-value coefficients of $x \in \mathbb{R}^d$: $x_{[s]} = \arg\min_{\|y\|_0 \le s} \|x - y\|_2$.

The following lemmas are used in the proofs of main results.

**Lemma 2.1.** *(Lemma 1 in [23]) Suppose $\Pi$ is an index set with $|\Pi| = s$. Given vectors $\{u_j : j \in \Pi\}$ are in a vector space $X$ with inner product $\langle, \cdot, \rangle$. Select all subset $\Pi_i \subset \Pi$ satisfying $|\Pi_i| = k, i \in U$ with $|U| = C_s^k$ ($C_s^k = \binom{s}{k}$), then*

$$\sum_{i \in U} \sum_{j \in \Pi_i} u_j = C_{s-1}^{k-1} \sum_{i \in \Pi} u_i, \tag{2.2}$$

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*holds for $k \geq 1$.*

In [5], Cai and Zhang introduced a crucial technical tool, which expresses points in a polytope by convex combinations of sparse vectors. The proofs of theorems for this paper depend on this new technique which is stated as the following lemma.

**Lemma 2.2.** *Assume that $\alpha$ is a positive number and $s$ is a positive integer. The polytope $\tau(\alpha, s) \subset \mathbb{R}^d$ is defined by*

$$\tau(\alpha, s) = \{x \in \mathbb{R}^d : \|x\|_\infty \leq \alpha, \|x\|_1 \leq s\alpha\}.$$

*The set of sparse vectors $\mathcal{U}(\alpha, s, x) \subset \mathbb{R}^d$ is defined by*

$$\mathcal{U}(\alpha, s, x) = \{u \in \mathbb{R}^d : \ \mathrm{sup}(u) \subseteq \mathrm{sup}(x), \|u\|_0 \leq s, \|u\|_1 = \|x\|_1, \|u\|_\infty \leq \alpha\}$$

*for all $x \in \mathbb{R}^d$. Then we can represent any $x \in \tau(\alpha, s)$ as*

$$x = \sum_i \lambda_i u_i,$$

*where $u_i \in \mathcal{U}(\alpha, s, x)$, $0 \leq \lambda_i \leq 1$, $\sum_i \lambda_i = 1$, and $\sum_i \lambda_i \|u_i\|_2^2 \leq s\alpha^2$.*

**Lemma 2.3.** *We get that*

$$\|D_\Pi^* h\|_1 + 2\|D_{\Pi^c}^* f\|_1 \geq \|D_{\Pi^c}^* h\|_1 \tag{2.3}$$

*for any subset $\Pi \subset \{1, \cdots, d\}$.*

**Proof** According to the minimality of $\tilde{f}$, one implies that

$$\|D^* f\|_1 \geq \|D^* \tilde{f}\|_1.$$

That is,

$$\|D_\Pi^* f\|_1 + \|D_{\Pi^c}^* f\|_1 \geq \|D_\Pi^* \tilde{f}\|_1 + \|D_{\Pi^c}^* \tilde{f}\|_1.$$

Applying the inverse triangular inequality to the above inequality, we get

$$\|D_\Pi^* f\|_1 + \|D_{\Pi^c}^* f\|_1 \geq \|D_\Pi^* f\|_1 - \|D_\Pi^* h\|_1 + \|D_{\Pi^c}^* h\|_1 - \|D_{\Pi^c}^* f\|_1,$$

which deduces the desired result.

$\square$

# 3　Main results

In the present section, we will establish new $D$-RIP conditions for the stable reconstruction of sparse signals in terms of $D$ via the $l_1$-analysis method.

A. Bounded Noise

In this subsection, we discuss recovery of signals which are sparse with respect to $D$ in the noisy situation. Specially, we think over two types of bounded noise: $l_2$ bound $\mathcal{B}_0(\varepsilon) = \{z : \|z\|_2 \leq \varepsilon\}$ and Dantzig selector bound $\mathcal{B}_1(\zeta) = \{z : \|D^* \Phi^* z\|_\infty \leq \zeta\}$. For simplicity, let $\tilde{f}^{ABP}$ represent the solution of (1.1) obeying $\mathcal{B} = \mathcal{B}_0(\varepsilon)$ and $\tilde{f}^{ADS}$ stand for the solution of (1.1) such that $\mathcal{B} = \mathcal{B}_1(\zeta)$.

**Theorem 3.1.** *Suppose that $D$ denotes an arbitrary tight frame. If the measurement matrix $\Phi$ fulfils the D-RIP with constant $\delta_{ts} < t/(4-t)$ for $0 < t < 4/3$, then*

$$\|\tilde{f}^{ABP} - f\|_2 \le A\varepsilon + B\frac{\|D^*f - (D^*f)_{[s]}\|_1}{\sqrt{s}},$$

*and*

$$\|\tilde{f}^{ADS} - f\|_2 \le C\zeta + B\frac{\|D^*f - (D^*f)_{[s]}\|_1}{\sqrt{s}}, \tag{3.1}$$

*where*

$$A = \frac{2\sqrt{2}\sqrt{1+\delta_{ts}}\tilde{t}}{t + (t-4)\delta_{ts}},$$
$$B = \frac{4\sqrt{2}\delta_{ts} + 2\sqrt{2}\sqrt{(t + (t-4)\delta_{ts})\delta_{ts}}}{t + (t-4)\delta_{ts}} + \frac{\sqrt{2}}{2},$$

*and*

$$C = \frac{2\sqrt{2s}\tilde{t}}{t + (t-4)\delta_{ts}}.$$

**Remark 3.1.** *When $D$ is an identity matrix, namely for the standard compressed sensing, we obtain same results as Theorem 2 in [23] and the condition is the weakest, for more information, see [5].*

**Remark 3.2.** *In the case when $t$ is equal to 1, Theorem 3.1 coincides with Theorem 4.1 in [21]. The sufficient condition $\delta_{ts} < t/(4-t)$ $(t \in (0, 4/3))$ extends the condition $\delta_s < 1/3$ to a more general context. Furthermore, the error estimations (3.1) is much better than (4.1) and (4.2) of Theorem 4.1 in [21].*

**Remark 3.3.** *It is known that Lin et al. [20] gave a sufficient condition $\delta_s < 0.307$, which guarantees that s-sparse signals in term of $D$ can be stably reconstructed. When $t = 1$, the result of Theorem 3.1 improves the bound to $\delta_s < 1/3$, for more details, see Remark III.2.(d) in [20].*

**Remark 3.4.** *In the proof of Theorem 3.1, by applying Lemma 5.3 in [4] to the estimation of $\|D^*_{T^c}h\|_2$, we obtain different error estimations as follows:*

$$\|\tilde{f}^{ABP} - f\|_2 \le A\varepsilon + \tilde{B}\frac{\|D^*f - (D^*f)_{[s]}\|_1}{\sqrt{s}},$$

*and*

$$\|\tilde{f}^{ADS} - f\|_2 \le C\zeta + \tilde{B}\frac{\|D^*f - (D^*f)_{[s]}\|_1}{\sqrt{s}}, \tag{3.2}$$

*where*

$$\tilde{B} = \frac{4\sqrt{2}\delta_{ts} + 2\sqrt{2}\sqrt{(t + (t-4)\delta_{ts})\delta_{ts}}}{t + (t-4)\delta_{ts}} + 2,$$

*and $A$, $C$ are defined by Theorem 3.1. Obviously, note that the upper bounds of error estimations determined by (3.1) is much better than those presented by (5.1). In addition, (5.1) is the same as Theorem 4.1 in [21] in the situation that $t$ is equal to 1.*

In the following corollary, we consider exact reconstruction of sparse (in terms of $D$) signals with noiseless via the approach (1.1) with $\mathcal{B} = \{0\}$.

**Corollary 3.1.** *Assume that one observes $b = \Phi f$ with $\|D^*f\|_0 \leq s$. The solution to (1.1) satisfying $\mathcal{B} = \{0\}$ estimates the unknown signal $f$, viz, $\tilde{f} = f$ provided that the matrix $\Phi$ meets the D-RIP with constant $\delta_{ts} < t/(4-t)$ for $0 < t < 4/3$.*

**Remark 3.5.** *In the noise-free situation when noise vector $z = 0$ and the unknown signal $f$ is $s$-sparse with respect to $D$, the result directly follows from Theorem 3.1.*

**Remark 3.6.** *When $D$ is an identity matrix, the result is the same as Theorem 1 in [23].*

B. Gaussian Noise

In statistics and signal processing, it is of interest to study the signal reconstruction with the error vector obeying Gaussian noise (i.e., $z \sim \mathbb{N}(0, \sigma^2 I)$). Define two bounded sets

$$\mathcal{B}_2 = \left\{ z : \|z\|_2^2 \leq \sigma^2 \left( M + 2\sqrt{M \log M} \right) \right\}$$

and

$$\mathcal{B}_3 = \left\{ z : \|D^* \Phi^* z\|_\infty^2 \leq 8\sigma^2 \log d \right\}.$$

The following lemma shows that the Gaussian error is bounded.

**Lemma 3.1.** *(Lemma III.3 in [20]) Suppose a matrix $\Phi \in \mathbb{R}^{M \times N}$ fulfils the D-RIP with real number $\delta_1 \in (0, 1)$, then Gaussian noise $z$ obeys*

$$1 - \mathcal{P}(z \in \mathcal{B}_2) \leq \frac{1}{M}$$

*and*

$$1 - \mathcal{P}(z \in \mathcal{B}_3) \leq \frac{1}{d\sqrt{2\pi \log d}},$$

*where the noise vector $z \sim \mathbb{N}(0, \sigma^2 I)$.*

**Theorem 3.2.** *We suppose that $D$ is an arbitrary tight frame. Let $\tilde{f}_1$ represent the solution of (1.1) obeying $\mathcal{B} = \mathcal{B}_2$ and $\tilde{f}_2$ stand for the solution of (1.1) such that $\mathcal{B} = \mathcal{B}_3$. If the measurement matrix $\Phi$ fulfils the D-RIP with constant $\delta_{ts} < t/(4-t)$ for $0 < t < 4/3$, then*

$$\|\tilde{f}_1 - f\|_2 \leq A\sigma\sqrt{M + 2\sqrt{M \log M}} + B\frac{\|D^*f - (D^*f)_{[s]}\|_1}{\sqrt{s}}$$

*with probability not less than $1 - 1/M$ and*

$$\|\tilde{f}_2 - f\|_2 \leq 2\sqrt{2}C\sigma\sqrt{\log d} + B\frac{\|D^*f - (D^*f)_{[s]}\|_1}{\sqrt{s}}$$

*with probability greater or equal to $1 - 1/(d\sqrt{2\pi \log d})$. Here, the constants $A$, $B$ and $C$ are defined by Theorem 3.1.*

**Remark 3.7.** *Since in this situation noise vector $z$ is in some bounded set with high probability, the theorem follows from Theorem 3.1 and Lemma 3.1.*

**Remark 3.8.** *In the situation when $D$ is equal to an identity matrix $I$, Theorem 3.2 is the same as Corollary 2 in [23].*

# 4 Proofs of main results

**Proof of Theorem** 3.1.

We suppose that $ts$ is an integer. Observe that $\|D_{S_0}^* h\|_1 \leq \|D_{T_0}^* h\|_1$, $\|D_{T_0^c}^* h\|_1 \leq \|D_{S_0^c}^* h\|_1$. It thus follows from Lemma 2.3 that

$$\|D_{T_0}^* h\|_1 + 2\|D_{S_0^c}^* f\|_1 \geq \|D_{T_0^c}^* h\|_1. \tag{4.1}$$

Set

$$\frac{\|D_{T_0}^* h\|_1 + 2\|D_{S_0^c}^* f\|_1}{s} = r.$$

Then

$$r^2 = \frac{\|D_{T_0}^* h\|_1^2 + 4\|D_{T_0}^* h\|_1\|D_{S_0^c}^* f\|_1 + 4\|D_{S_0^c}^* f\|_1^2}{s^2}.$$

By employing $\|D_{T_0}^* h\|_1^2 \leq s\|D_{T_0}^* h\|_2^2$ to the above equality, we get

$$r^2 \leq \frac{\|D_{T_0}^* h\|_2^2}{s} + \frac{4\|D_{T_0}^* h\|_2\|D_{S_0^c}^* f\|_1}{s^{\frac{3}{2}}} + \frac{4\|D_{S_0^c}^* f\|_1^2}{s^2}. \tag{4.2}$$

Pick out positive integers $m$ and $n$ contenting $n \leq m \leq s$ and $(m+n)/s = t$. We use subsets $T_i, S_j \subset \{1, \cdots, s\}$ to represent all probable index sets with $|T_i| = m$ ($i \in U$), $|S_j| = n$ ($j \in V$) that $|U| = C_s^m$ and $|V| = C_s^n$.

From the definition of $T_0$, we have

$$\|D_{T_0^c}^* h\|_\infty \leq \frac{\|D_{T_0}^* h\|_1}{s} \leq \frac{\|D_{T_0}^* h\|_1 + 2\|D_{S_0^c}^* f\|_1}{s} \leq \frac{s}{n} r. \tag{4.3}$$

Moreover, it is known that the $l_1$ norm of $D_{T_0^c}^* h$ is bounded, i.e.,

$$\|D_{T_0^c}^* h\|_1 \leq n\frac{s}{n} r.$$

Hence, by Lemma 2.2, we imply that

$$D_{T_0^c}^* h = \sum_i \lambda_i u_i,$$

where $u_i$ is $n$-sparse, i.e.,

$$\|u_i\|_0 = |\sup(u_i)| \leq n,$$

and $\sum_i \lambda_i = 1$ with for each $i$, $\lambda_i \in (0, 1]$,

$$\sup(u_i) \subset \sup(D_{T_0^c}^* h), \ \|u_i\|_1 = \|D_{T_0^c}^* h\|_1, \ \|u_i\|_\infty \leq \frac{s}{n} r,$$

$$\sum_i \lambda_i \|u_i\|_2^2 \leq n\left(\frac{s}{n} r\right)^2 = \frac{s^2 r^2}{n}. \tag{4.4}$$

Analogously, we can also decompose $D_{T_0^c}^* h$ into

$$D_{T_0^c}^* h = \sum_i \mu_i v_i,$$

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and
$$D^*_{T_0^c}h = \sum_i \nu_i w_i.$$

Here $|\sup(v_i)| \le m$, $|\sup(w_i)| \le (t-1)s$ $(t > 1)$ and

$$\sum_i \mu_i \|v_i\|_2^2 \le \frac{s^2 r^2}{m}, \tag{4.5}$$

$$\sum_i \nu_i \|w_i\|_2^2 \le \frac{sr^2}{t-1}. \tag{4.6}$$

By the feasibility of $\tilde{f}^{ABP}$, we get that

$$\|\Phi h\|_2 \le \|\Phi(\tilde{f}^{ABP} - f)\|_2 \le \|\Phi\tilde{f}^{ABP} - b\|_2 + \|\Phi f - b\|_2 \le 2\varepsilon. \tag{4.7}$$

Thus, it follows that

$$\begin{aligned}
|\langle \Phi D D^*_{T_0} h, \Phi h \rangle|^2 &\le \|\Phi D D^*_{T_0} h\|_2^2 \|\Phi h\|_2^2 \\
&\le 4\varepsilon^2 (1 + \delta_s) \|D D^*_{T_0} h\|_2^2 \\
&\le 4\varepsilon^2 (1 + \delta_{ts}) \|D^*_{T_0} h\|_2^2
\end{aligned} \tag{4.8}$$

for $t \in [1, 4/3)$, where for the third inequality, we used the fact that $\delta_s \le \delta_{s_1}$ if $s \le s_1$[20];

$$\begin{aligned}
|\langle \Phi D D^*_{T_0} h, \Phi h \rangle|^2 &\le 4\varepsilon^2 (1 + \delta_s) \|D D^*_{T_0} h\|_2^2 \\
&\le 4\varepsilon^2 \left(1 + \left(\frac{2}{t} - 1\right) \delta_{ts}\right) \|D^*_{T_0} h\|_2^2 \\
&\le \frac{4\varepsilon^2 (1 + \delta_{ts})}{t} \|D^*_{T_0} h\|_2^2
\end{aligned} \tag{4.9}$$

for $t \in (0, 1)$, where for the second inequality follows from Lemma 4.1 in [4].

For notational simplicity, set

$$\begin{aligned}
\Delta_0 = &\sum_{i \in U, k} \frac{s - n}{mC_s^m} \lambda_k \left(m^2 \|E(D^*_{T_i} h + \frac{n}{s} u_k)\|_2^2 - n^2 \|E(D^*_{T_i} h - \frac{m}{s} u_k)\|_2^2\right) \\
&+ \sum_{j \in V, k} \frac{s - m}{nC_s^n} \mu_k \left(n^2 \|E(D^*_{S_j} h + \frac{m}{s} v_k)\|_2^2 - m^2 \|E(D^*_{S_j} h - \frac{n}{s} v_k)\|_2^2\right),
\end{aligned} \tag{4.10}$$

where $E \in \mathbb{R}^{L \times d}$ and $L$ is an arbitrary integer. Let $\rho(m, n) = (m - n)^2 + 2(t - 2)mn$. The construction of following two identities make use of ideals from equalities (14) and (15) in [23]. One can verify that

$$\begin{aligned}
\frac{\rho(m, n)(t - 1)}{mnC_s^m C_{s-m}^n} &\sum_{T_i \cap S_j = \phi} \left(\frac{mn}{t - 1} \|E(D^*_{T_i} h + D^*_{S_j} h)\|_2^2 + \|E(nD^*_{T_i} h - mD^*_{S_j} h)\|_2^2\right) \\
&= t\Delta_0 + 2mn(t - 2)t^2 \langle ED^*_{T_0} h, ED^* h \rangle
\end{aligned}$$

holds for $t \in (0, 1)$, and

$$\rho(m, n) \sum_k \nu_k \left\{\|E(D^*_{T_0} h + (t - 1)w_k)\|_2^2 - \|(t - 1)E(D^*_{T_0} h - w_k)\|_2^2\right\}$$

$$= -(3t - 4)\Delta_0 + 2\{(t-1)s^2 - mn\}t^3 \left\langle ED_{T_0}^* h, ED^* h \right\rangle$$

holds for $t \in [1, 4/3)$. It thus follows that

$$\frac{\rho(m,n)(t-1)}{mnC_s^m C_{s-m}^n} \sum_{T_i \bigcap S_j = \phi} \left( \frac{mn}{t-1} \|\Phi D(D_{T_i}^* h + D_{S_j}^* h)\|_2^2 + \|\Phi D(nD_{T_i}^* h - mD_{S_j}^* h)\|_2^2 \right.$$

$$+ \frac{mn}{t-1} \|D^\perp(D_{T_i}^* h + D_{S_j}^* h)\|_2^2 + \|D^\perp(nD_{T_i}^* h - mD_{S_j}^* h)\|_2^2 \Big)$$

$$= t\tilde{\Delta}_0 + 2mn(t-2)t^2 \left\langle \Phi DD_{T_0}^* h, \Phi h \right\rangle \tag{4.11}$$

holds for $t \in (0, 1)$, and

$$\rho(m,n) \sum_k \nu_k \left( \|\Phi D(D_{T_0}^* h + (t-1)w_k)\|_2^2 - \|(t-1)\Phi D(D_{T_0}^* h - w_k)\|_2^2 \right.$$

$$+ \|D^\perp(D_{T_0}^* h + (t-1)w_k)\|_2^2 - \|(t-1)D^\perp(D_{T_0}^* h - w_k)\|_2^2 \Big)$$

$$= -(3t-4)\tilde{\Delta}_0 + 2\{(t-1)s^2 - mn\}t^3 \left\langle \Phi DD_{T_0}^* h, \Phi h \right\rangle \tag{4.12}$$

holds for $t \in [1, 4/3)$, where

$$\tilde{\Delta}_0 = \sum_{i \in U,k} \frac{s-n}{mC_s^m} \lambda_k \left( m^2 \|\Phi D(D_{T_i}^* h + \frac{n}{s}u_k)\|_2^2 - n^2 \|\Phi D(D_{T_i}^* h - \frac{m}{s}u_k)\|_2^2 \right.$$

$$+ m^2 \|D^\perp(D_{T_i}^* h + \frac{n}{s}u_k)\|_2^2 - n^2 \|D^\perp(D_{T_i}^* h - \frac{m}{s}u_k)\|_2^2 \Big)$$

$$+ \sum_{j \in V,k} \frac{s-m}{nC_s^n} \mu_k \left( n^2 \|\Phi D(D_{S_j}^* h + \frac{m}{s}v_k)\|_2^2 - m^2 \|\Phi D(D_{S_j}^* h - \frac{n}{s}v_k)\|_2^2 \right.$$

$$+ n^2 \|D^\perp(D_{S_j}^* h + \frac{m}{s}v_k)\|_2^2 - m^2 \|D^\perp(D_{S_j}^* h - \frac{n}{s}v_k)\|_2^2 \Big), \tag{4.13}$$

and we used the fact that $\left\langle D^\perp D_{T_0}^* h, D^\perp D^* h \right\rangle = 0$. For $\rho(m,n)$, if $ts$ is odd, then let $m = n + 1 = (ts+1)/2$; if $ts$ is even, set $m = n = ts/2$. Easily check that $\rho(m,n) < 0$ for both situations.

According to $\|D_{T_i}^* h\|_0 = \|v_k\|_0 \leq m$, $\|D_{S_j}^* h\|_0 = \|u_k\|_0 \leq n$ $(m + n = ts)$ and combining with (2.1), we imply that

$$\tilde{\Delta}_0 \geq \sum_{i \in U,k} \frac{s-n}{mC_s^m} \lambda_k \left( m^2(1-\delta_{ts}) \|D_{T_i}^* h + \frac{n}{s}u_k\|_2^2 - n^2(1+\delta_{ts}) \|D_{T_i}^* h - \frac{m}{s}u_k\|_2^2 \right)$$

$$+ \sum_{j \in V,k} \frac{s-m}{nC_s^n} \mu_k \left( n^2(1-\delta_{ts}) \|D_{S_j}^* h + \frac{m}{s}v_k\|_2^2 - m^2(1+\delta_{ts}) \|D_{S_j}^* h - \frac{n}{s}v_k\|_2^2 \right).$$

Notice that $\left\langle D_{T_i}^* h, u_k \right\rangle = \left\langle D_{S_j}^* h, v_k \right\rangle = 0$ for each $i, j, k$. Then

$$\tilde{\Delta}_0 \geq \sum_{i \in U,k} \frac{s-n}{mC_s^m} \lambda_k \left( m^2(1-\delta_{ts}) \|D_{T_i}^* h\|_2^2 + \frac{m^2 n^2}{s^2}(1-\delta_{ts}) \|u_k\|_2^2 - n^2(1+\delta_{ts}) \|D_{T_i}^* h\|_2^2 \right.$$

$$- \frac{m^2 n^2}{s^2}(1+\delta_{ts}) \|u_k\|_2^2 \Big)$$

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$$+ \sum_{j \in V, k} \frac{s-m}{nC_s^n} \mu_k \left( n^2(1-\delta_{ts})\|D_{S_j}^*h\|_2^2 + \frac{m^2n^2}{s^2}(1-\delta_{ts})\|v_k\|_2^2 - m^2(1+\delta_{ts})\|D_{S_j}^*h\|_2^2 \right.$$

$$\left. - \frac{m^2n^2}{s^2}(1+\delta_{ts})\|v_k\|_2^2 \right)$$

$$= \frac{s-n}{mC_s^m}\{(m+n)(m-n) - (m^2+n^2)\delta_{ts})\} \sum_{i \in U} \|D_{T_i}^*h\|_2^2 - \frac{2(s-n)mn^2\delta_{ts}}{s^2} \sum_k \lambda_k \|u_k\|_2^2$$

$$+ \frac{s-m}{nC_s^n}\{-(m+n)(m-n) - (m^2+n^2)\delta_{ts}\} \sum_{j \in V} \|D_{S_j}^*h\|_2^2 - \frac{2(s-m)m^2n\delta_{ts}}{s^2} \sum_k \mu_k \|v_k\|_2^2.$$

By exploiting (2.2) to the above inequality, we get

$$\tilde{\Delta}_0 \geq \{(m+n)(m-n) - (m^2+n^2)\delta_{ts}\}\frac{s-n}{mC_s^m}C_{s-1}^{m-1}\|D_{T_0}^*h\|_2^2 - \frac{2(s-n)mn^2\delta_{ts}}{s^2} \sum_k \lambda_k \|u_k\|_2^2$$

$$+ \{-(m+n)(m-n) - (m^2+n^2)\delta_{ts}\}\frac{s-m}{nC_s^n}C_{s-1}^{n-1}\|D_{T_0}^*h\|_2^2 - \frac{2(s-m)m^2n\delta_{ts}}{s^2} \sum_k \mu_k \|v_k\|_2^2.$$

By utilizing (4.4) and (4.5) to the above inequality, we get that

$$\tilde{\Delta}_0 \geq \{(m+n)(m-n) - (m^2+n^2)\delta_{ts}\}\frac{s-n}{s}\|D_{T_0}^*h\|_2^2 - \frac{2(s-n)mn^2\delta_{ts}}{s^2}\frac{s^2r^2}{n}$$

$$+ \{-(m+n)(m-n) - (m^2+n^2)\delta_{ts}\}\frac{s-m}{s}\|D_{T_0}^*h\|_2^2 - \frac{2(s-m)m^2n\delta_{ts}}{s^2}\frac{s^2r^2}{m}$$

$$= \left( \frac{(m+n)(m-n)^2}{s} - \frac{(m^2+n^2)(2s-m-n)\delta_{ts}}{s} \right)\|D_{T_0}^*h\|_2^2$$

$$- 2mn(2s-m-n)\delta_{ts}r^2$$

$$= \{(m-n)^2t + (m^2+n^2)(t-2)\delta_{ts}\}\|D_{T_0}^*h\|_2^2 + 2mns(t-2)\delta_{ts}r^2. \tag{4.14}$$

For convenience, denote

$$\mathcal{F} = \frac{\rho(m,n)(t-1)}{mnC_s^mC_{s-m}^n} \sum_{T_i \bigcap S_j = \phi} \left( \frac{mn}{t-1}\|\Phi D(D_{T_i}^*h + D_{S_j}^*h)\|_2^2 + \|\Phi D(nD_{T_i}^*h - mD_{S_j}^*h)\|_2^2 \right.$$

$$\left. + \frac{mn}{t-1}\|D^\perp(D_{T_i}^*h + D_{S_j}^*h)\|_2^2 + \|D^\perp(nD_{T_i}^*h - mD_{S_j}^*h)\|_2^2 \right),$$

$$\mathcal{G} = t\tilde{\Delta}_0 + 2mn(t-2)t^2 \left\langle \Phi DD_{T_0}^*h, \Phi h \right\rangle,$$

$$\mathcal{H} = \rho(m,n) \sum_k \nu_k \left( \|\Phi D(D_{T_0}^*h + (t-1)w_k)\|_2^2 - \|(t-1)\Phi D(D_{T_0}^*h - w_k)\|_2^2 \right.$$

$$\left. + \|D^\perp(D_{T_0}^*h + (t-1)w_k)\|_2^2 - \|(t-1)D^\perp(D_{T_0}^*h - w_k)\|_2^2 \right),$$

$$\mathcal{I} = -(3t-4)\tilde{\Delta}_0 + 2\{(t-1)s^2 - mn\}t^3 \left\langle \Phi DD_{T_0}^*h, \Phi h \right\rangle,$$

where $\tilde{\Delta}_0$ is determined by (4.13).

We firstly take into the situation of $t \in (0,1)$.

Since $D_{T_i}^*h + D_{S_j}^*h$ are $ts$-sparse for each $i, j$, we make use of (2.1), then it gives

$$\mathcal{F} \leq \frac{\rho(m,n)(t-1)}{mnC_s^mC_{s-m}^n} \sum_{T_i \bigcap S_j = \phi} \left( \frac{mn}{t-1}(1-\delta_{ts})\|D_{T_i}^*h + D_{S_j}^*h\|_2^2 + (1+\delta_{ts})\|nD_{T_i}^*h - mD_{S_j}^*h\|_2^2 \right)$$

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$$= \frac{\rho(m,n)(t-1)}{mnC_s^m C_{s-m}^n} \left\{ \frac{mn}{t-1}(1-\delta_{ts})\left( C_{s-m}^n \sum_{i\in U} \|D_{T_i}^* h\|_2^2 + C_{s-n}^m \sum_{j\in V} \|D_{S_j}^* h\|_2^2 \right) \right.$$

$$\left. + (1+\delta_{ts})\left( n^2 C_{s-m}^n \sum_{i\in U} \|D_{T_i}^* h\|_2^2 + m^2 C_{s-n}^m \sum_{j\in V} \|D_{S_j}^* h\|_2^2 \right) \right\}.$$

Applying Lemma 2.2 to the above inequality, we get that

$$\mathcal{F} \leq \frac{\rho(m,n)(t-1)}{mnC_s^m C_{s-m}^n} \left\{ \frac{mn}{t-1}(1-\delta_{ts})\left( C_{s-m}^n C_{s-1}^{m-1} \|D_{T_0}^* h\|_2^2 + C_{s-n}^m C_{s-1}^{n-1} \|D_{T_0}^* h\|_2^2 \right) \right.$$

$$+ (1+\delta_{ts})\left( n^2 C_{s-m}^n C_{s-1}^{m-1} \|D_{T_0}^* h\|_2^2 + m^2 C_{s-n}^m C_{s-1}^{n-1} \|D_{T_0}^* h\|_2^2 \right) \Bigg\}$$

$$= \rho(m,n)\{t + (t-2)\delta_{ts}\}t\|D_{T_0}^* h\|_2^2. \tag{4.15}$$

A combination of (4.9) and (4.14), we obtain

$$\mathcal{G} \geq \{t(m-n)^2 + (m^2+n^2)(t-2)\delta_{ts}\}t\|D_{T_0}^* h\|_2^2 + 2mnst(t-2)\delta_{ts}r^2$$

$$+ 4\varepsilon mn(t-2)t\sqrt{(1+\delta_{ts})t}\|D_{T_0}^* h\|_2. \tag{4.16}$$

Combine the above two inequalities (4.15) and (4.16), which leads to

$$-2mnst(t-2)\delta_{ts}r^2 \geq 2mn(t-2)t[-t+(3-t)\delta_{ts}]\|D_{T_0}^* h\|_2^2 + 4\varepsilon mn(t-2)t\sqrt{(1+\delta_{ts})t}\|D_{T_0}^* h\|_2.$$

Plug (4.2) into the above inequality, we get

$$2mnt(t-2)\left( (4-t)\left[ \frac{t}{4-t} - \delta_{ts} \right] \|D_{T_0}^* h\|_2^2 \right.$$

$$\left. - \left( \frac{4\delta_{ts}\|D_{S_0^c}^* f\|_1}{\sqrt{s}} + 2\varepsilon\sqrt{(1+\delta_{ts})t} \right) \|D_{T_0}^* h\|_2 - \frac{4\delta_{ts}\|D_{S_0^c}^* f\|_1^2}{s} \right) \geq 0, \tag{4.17}$$

which is a second-order inequality for $\|D_{T_0}^* h\|_2$ with the coefficient of square term is less than zero under the condition $\delta_{ts} < t/(4-t)$.

We now consider the situation of $t \in [1, 4/3)$.

Combining with $\rho(m,n) < 0$, the definition of the $D$-RIP of $ts$ order and (2.1), we imply

$$\mathcal{H} \leq \rho(m,n) \sum_k \nu_k \left( (1-\delta_{ts})\|D_{T_0}^* h + (t-1)w_k\|_2^2 - (1+\delta_{ts})\|(t-1)(D_{T_0}^* h - w_k)\|_2^2 \right).$$

Since the support of $D_{T_0}^* h$ does not intersect with that of $w_k$, one gains

$$\mathcal{H} \leq \rho(m,n) \sum_k \nu_k \left( (1-\delta_{ts})\|D_{T_0}^* h\|_2^2 + (1-\delta_{ts})(t-1)^2\|w_k\|_2^2 \right.$$

$$- (t-1)^2(1+\delta_{ts})\|D_{T_0}^* h\|_2^2 - (t-1)^2(1+\delta_{ts})\|w_k\|_2^2 \Bigg)$$

$$= \rho(m,n) \sum_k \nu_k \left( [1 - (t-1)^2 - (1+(t-1)^2)\delta_{ts}]\|D_{T_0}^* h\|_2^2 \right.$$

$$- 2(t-1)^2\delta_{ts}\|w_k\|_2^2 \Bigg).$$

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Substituting (4.6) into the above inequality, we get that

$$\mathcal{H} \le \rho(m,n)\left\{[1-(t-1)^2-(1+(t-1)^2)\delta_{ts}]\|D_{T_0}^*h\|_2^2 - 2(t-1)s\delta_{ts}r^2\right\}. \tag{4.18}$$

Under the condition of Theorem 3.1, we can check that for $t \in [1, 4/3)$,

$$(t-1)s^2 < \frac{(2-t)^2s^2-1}{4} + (t-1)s^2 = \frac{t^2s^2-1}{4} \le mn. \tag{4.19}$$

By (4.8), (4.14) and (4.19), we derive

$$\mathcal{I} \ge -(3t-4)\Big([t(m-n)^2+(m^2+n^2)(t-2)\delta_{ts}]\|D_{T_0}^*h\|_2^2 + 2mns(t-2)\delta_{ts}r^2\Big)$$
$$+ 4\varepsilon t^3\sqrt{1+\delta_{ts}}((t-1)s^2-mn)\|D_{T_0}^*h\|_2. \tag{4.20}$$

Combining with (4.18) and (4.20), one gets

$$2t^2s\left[-(t-1)s^2+mn\right]\delta_{ts}r^2$$
$$\ge \left\{-2s^2t^3(t-1)+2mnt^2+2t^2(t-3)[-(t-1)s^2+mn]\delta_{ts}\right\}\|D_{T_0}^*h\|_2^2$$
$$+ 4\varepsilon t^3\sqrt{1+\delta_{ts}}[(t-1)s^2-mn]\|D_{T_0}^*h\|_2.$$

By applying (4.2) to the above inequality, we have

$$2[(t-1)s^2-mn]t^2\left\{(4-t)\left(\frac{t}{4-t}-\delta_{ts}\right)\|D_{T_0}^*h\|_2^2\right.$$
$$\left. - \left(\frac{4\delta_{ts}}{\sqrt{s}}\|D_{S_0^c}^*f\|_1 + 2\varepsilon t\sqrt{1+\delta_{ts}}\right)\|D_{T_0}^*h\|_2 - \frac{4\delta_{ts}}{s}\|D_{S_0^c}^*f\|_1^2\right\} \ge 0, \tag{4.21}$$

which is a second-order inequality for $\|D_{T_0}^*h\|_2$. Combining with (4.17) and (4.21), we get

$$\|D_{T_0}^*h\|_2 \le \left(\frac{4\delta_{ts}\|D_{S_0^c}^*f\|_1}{\sqrt{s}} + 2\varepsilon\sqrt{1+\delta_{ts}}\tilde{t}\right)\frac{1}{2(t+(t-4)\delta_{ts})}$$
$$+ \left\{\left(\frac{4\delta_{ts}\|D_{S_0^c}^*f\|_1}{\sqrt{s}} + 2\varepsilon\sqrt{1+\delta_{ts}}\tilde{t}\right)^2 + \frac{16(t+(t-4)\delta_{ts})\delta_{ts}\|D_{S_0^c}^*f\|_1^2}{s}\right\}^{\frac{1}{2}}$$
$$\times \frac{1}{2(t+(t-4)\delta_{ts})},$$

where $\tilde{t} = \max\{t, \sqrt{t}\}$ denotes the maximum of $\{t, \sqrt{t}\}$. By applying the fact that $(x^2+y^2)^{1/2} \le x+y$ for $x, y \ge 0$ to the above inequality, we get

$$\|D_{T_0}^*h\|_2 \le \frac{2\varepsilon\sqrt{1+\delta_{ts}}\tilde{t}}{t+(t-4)\delta_{ts}} + \frac{2\left(2\delta_{ts}+\sqrt{(t+(t-4)\delta_{ts})\delta_{ts}}\right)\|D_{S_0^c}^*f\|_1}{\sqrt{s}(t+(t-4)\delta_{ts})}. \tag{4.22}$$

By the common inequality

$$\sum_{j=1}^{N}|v_j|^2 \le \max_{1\le j\le N}|v_j|\sum_{j=1}^{N}|v_j|,$$

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we get

$$\|D^*_{T_0^c}h\|_2 \leq \sqrt{\|D^*_{T_0^c}h\|_\infty \|D^*_{T_0^c}h\|_1}.$$

By applying $\|D^*_{T_0^c}h\|_\infty \leq \|D^*_{T_0}h\|_1/s$ and (4.1) to the above inequality, we get

$$
\begin{aligned}
\|D^*_{T_0^c}h\|_2 &\leq \sqrt{\frac{\|D^*_{T_0}h\|_1^2}{s} + \frac{2\|D^*_{T_0}h\|_1\|D^*_{S_0^c}f\|_1}{s}} \\
&\leq \sqrt{\|D^*_{T_0}h\|_2^2 + \frac{2\|D^*_{T_0}h\|_2\|D^*_{S_0^c}f\|_1}{\sqrt{s}}},
\end{aligned}
\tag{4.23}
$$

where the last inequality, we have utilized $\|D^*_{T_0}h\|_1 \leq \sqrt{s}\|D^*_{T_0}h\|_2$.

Finally, a combination of (4.22) and (4.23), we obtain

$$
\begin{aligned}
\|h\|_2^2 = \|D^*h\|_2^2 &= \|D^*_{T_0}h\|_2^2 + \|D^*_{T_0^c}h\|_2^2 \\
&\leq 2\|D^*_{T_0}h\|_2^2 + \frac{2\|D^*_{T_0}h\|_2\|D^*_{S_0^c}f\|_1}{\sqrt{s}} \\
&\leq \left\{ \sqrt{2}\|D^*_{T_0}h\|_2 + \frac{\|D^*_{S_0^c}f\|_1}{\sqrt{2s}} \right\}^2 \\
&\leq \left\{ \frac{2\sqrt{2}\varepsilon\sqrt{1+\delta_{ts}\tilde{t}}}{t+(t-4)\delta_{ts}} + \left( \frac{4\sqrt{2}\delta_{ts} + 2\sqrt{2}\sqrt{(t+(t-4)\delta_{ts})\delta_{ts}}}{(t+(t-4)\delta_{ts})} + \frac{1}{\sqrt{2}} \right) \frac{\|D^*_{S_0^c}f\|_1}{\sqrt{s}} \right\}^2.
\end{aligned}
$$

If $ts$ is not an integer, denote $\hat{t}s = [ts]$, then $\hat{t}s$ is an integer obeying $\hat{t} > t$. For $\hat{t} \in (0, 4/3)$, we have $\delta_{\hat{t}s} = \delta_{ts} < t/(4-t) < \hat{t}/(4-\hat{t})$. Similar to the above proof, one can prove the result by dealing with $\delta_{\hat{t}s}$.

For the situation of Dantzig selector bounded noise:

According to the feasibility of $\tilde{f}^{ADS}$:

$$
\begin{aligned}
\|D^*\Phi^*\Phi h\|_\infty &\leq \|D^*\Phi^*(b - \Phi\tilde{f}^{ADS})\|_\infty + \|D^*\Phi^*(\Phi f - b)\|_\infty \\
&\leq 2\zeta
\end{aligned}
\tag{4.24}
$$

It consequently follows that

$$
\begin{aligned}
|\langle \Phi h, \Phi DD^*_{T_0}h\rangle|^2 &\leq |\langle D^*\Phi^*\Phi h, D^*_{T_0}h\rangle|^2 \\
&\leq \|D^*\Phi^*\Phi h\|_\infty^2 \|D^*_{T_0}h\|_1^2 \\
&\leq 4s\zeta^2\|D^*_{T_0}h\|_2^2.
\end{aligned}
\tag{4.25}
$$

The rest of proof is similar to the $l_2$ bounded noise situation. The proof of Theorem 3.1 is complete.

$\square$

# 5 Numerical simulation

In this section, we present a series of numerical experiments to illustrate our obtained theoretical results. In order to solve the optimization problem (1.1), we employ the dictionary-iteratively

re-weighted least squares (D-IRLS) approach. We rewrite problem (1.1) as the unconstrained regularization problem as follows:

$$\min_{x \in \mathbb{R}^d} \|x\|_{2,\epsilon} + \frac{1}{2\lambda} \|\Phi Dx - b\|_2^2, \tag{5.1}$$

where a regularization parameter $\lambda > 0$, a smoothing parameter $\epsilon > 0$ and $\|x\|_{2,\epsilon} = \sum_{i=1}^{d} [\epsilon^2 + |x_i|^2]^{1/2}$. We begin with $\epsilon_0 = 1$ and an initial $x^{(0)}$ such that $\Phi Dx^{(0)} = b$. Then, the iterative solution $x^{(t+1)}$ to (5.1) can be provided as follows:

$$\left( \frac{x_i^{(t+1)}}{[\epsilon_t^2 + |x_i^{(t)}|^2]^{1/2}} \right)_{1 \le i \le d} + \frac{1}{\lambda} (\Phi D)^* (\Phi Dx^{(t+1)} - b) = 0. \tag{5.2}$$

The parameter $\epsilon_t$ equals the minimum of $\{\epsilon_{t-1}, r(x^{(t)})_{s+1}\}$, where $r(x^{(t)})_{s+1}$ denotes the $(s+1)$-th maximum of $x_i^{(t)}$ in magnitude. The iteration is continued until $\|x^{(t+1)} - x^{(t)}\|_2 \le 10^{-5}$ or the iteration times more than 700.

In our simulations, the sensing matrix is produced as a $128 \times 256$ matrix, whose entries obey independent identically distribution (i.i.d.) Gauss distribution and the tight frame $D$ is generated by $256 \times 1024$ discrete cosine transform (DCT) dictionary or wavelet transform (WT) dictionary. The original signal $f$ is synthesized by utilizing $f = Dx$ and the linear measurements $b = \Phi Dx + z$ where noise vector $z$ satisfies a Gaussian distribution with the mean value 0 and deviation 0.05. $\|f - f^*\|_2 / \|f\|_2$ stands for the relative error between the unknown signal $f$ and the recovered signal $f^*$.

## 5.1 Signal based on WT dictionary

In this, we take WT as a dictionary, see Figure 5.1(a) and $x \in \mathbb{R}^{1024}$ is a 15-sparse vector. Figure 5.1(b), (c) carve the reconstruction of the signal $f$ via $l_1$-analysis approach with noise level $\varepsilon = 0.01$. The results indicate that the unknown signal can stably be recovered by $l_1$-analysis approach. Figure 5.1(d) depicts the relationship between the sparsity, the noise level and the relative error of the recovery for the signal. The results exhibit that the smaller sparsity, the better the effect for the signal recovery. Besides, the noise level hardly influences on the recovery effect.
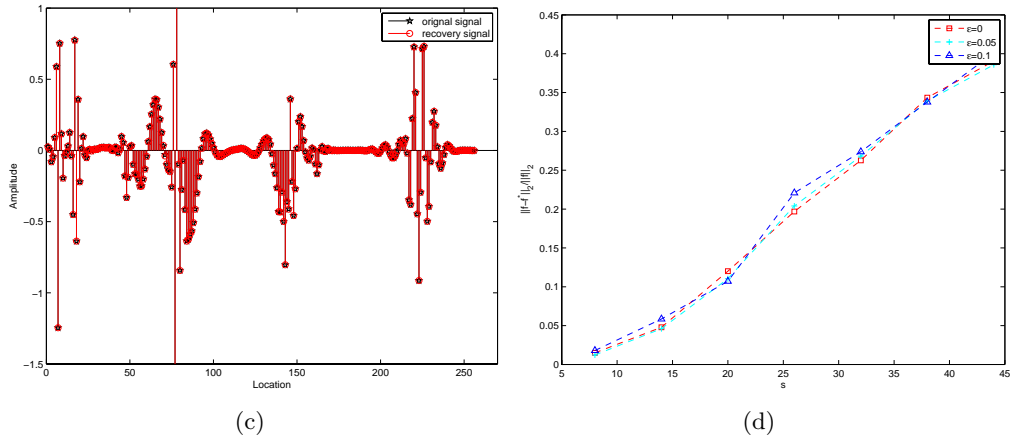


(a)

(b)

(c)



(d)
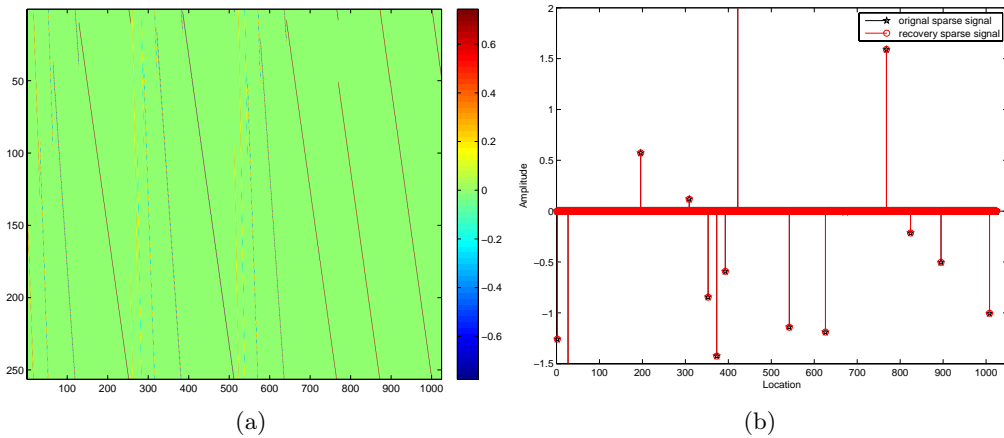
**Fig.** 5.1: The recovery of signal on the basis of WT dictionary

## 5.2 Signal based on DCT dictionary

In this part, select DCT as a dictionary, see Figure 5.2($a$). Figure 5.2($b$), ($c$) present the signal recovery by using $l_1$-analysis method with noise size $\varepsilon = 0.01$. Observe from the figures that the $l_1$-analysis approach can robustly reconstruct the original signal. Figure 5.2($d$) shows the relation between the sparsity, the noise size and the relative error of the recovery for the signal. It mainly indicates that the sparsity affects for the reconstruction larger than the noise level.



(a)



(b)
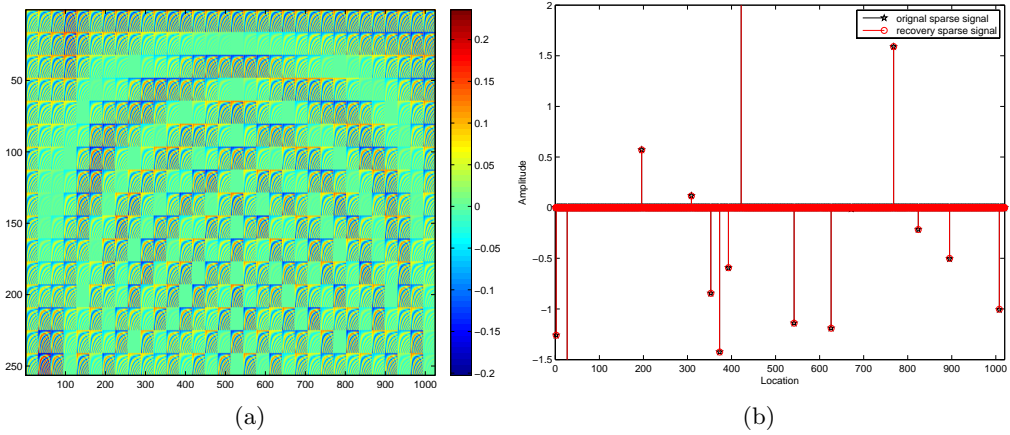
## 5.3 Contrast based on different dictionary

Figure 5.3($a$) illustrates the relationship between the average normalized reconstruction error and the number of samples $M$ in both WT and DCT dictionaries. It can be observed that when the dictionary is DCT, the recovery effect is better.

In Figure 5.3($b$), the relative error is plotted versus the regularization parameter $\lambda$ for two different dictionaries. The figure shows that the parameter $\lambda = 1 \times 10^{-3}$ is a nice choice.

Figure 5.3($c$) describes the relation between the the average normalized reconstruction error and the sparsity of the signal with respect to two types of dictionaries. The result indicates that as
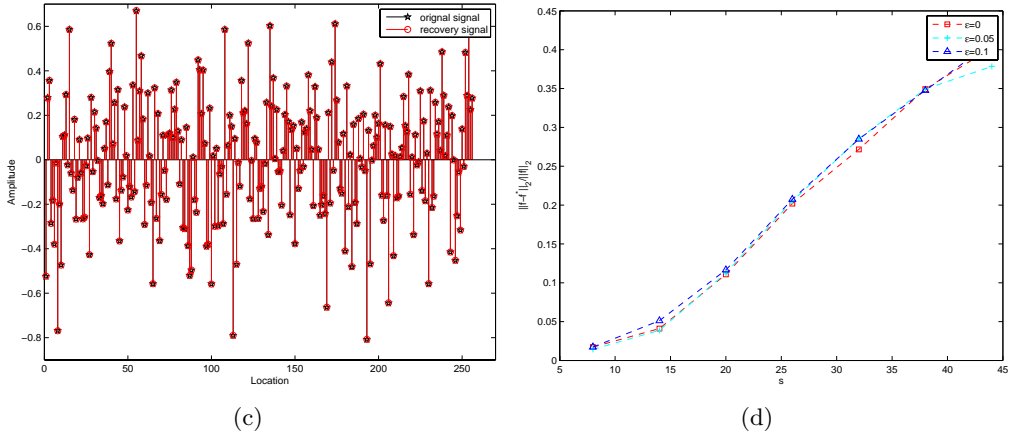
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(c)



(d)

**Fig.** 5.2: The recovery of signal on the basis of DCT dictionary

the sparsity reduces, the relative error becomes small. Furthermore, the DCT dictionary is a good choice in the case.

In Figure 5.3(d), the average normalized reconstruction error is plotted versus the noise level in terms of WT and DCT dictionaries. Obviously observe that as the noise size reduces, the relative error tends to small. In addition, both two dictionaries can be chosen in this setting.



(a)



(b)
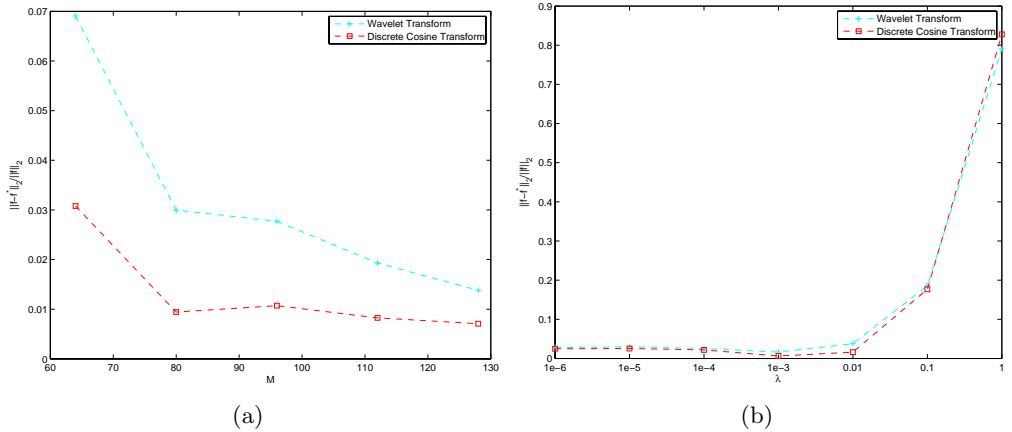
# 6    Conclusions

This paper considers a sufficient condition concerning the restricted isometry property adapted to a tight frame $D$ for the reconstruction of signals. We show that the $D$-RIP constant $\delta_{ts}$ from the measurement matrix $\Phi$ obeys $\delta_{ts} < t/(4-t)$ for $t \in (0, 4/3)$, signals that are nearly $s$-sparse with respect to $D$ can be stably estimated by $l_1$-analysis methods. When $t = 1$, our main results coincide with Theorems 3.1, 4.1 in [21] and the error estimates are smaller than those given in their work. Moreover, we derive a much weaker sufficient condition than $\delta_s < 0.307$ provided by [20]. In the case of $D = I$, our main results return to these results in [23]. Meanwhile, the bound is sharp in the case, for more details, see [5]. Finally, we conduct a series of numerical experiments to illustrate the performance of $l_1$-analysis method and our theoretical results.
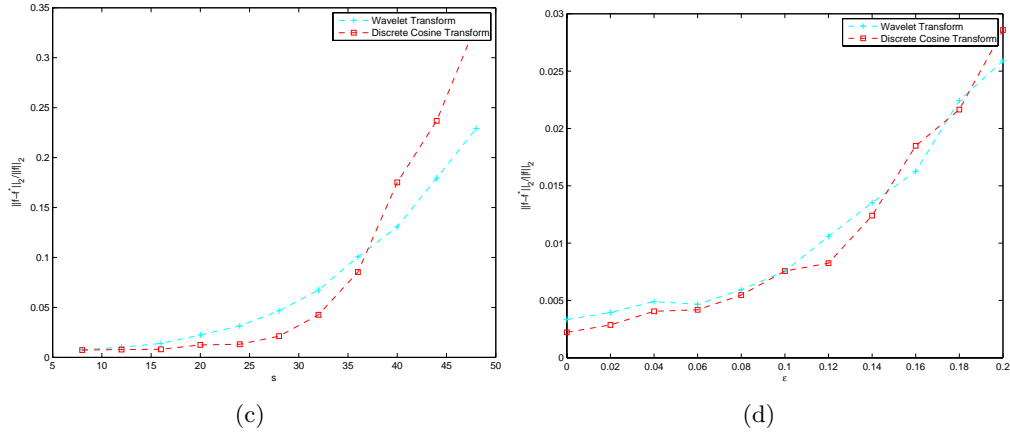
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**Fig.** 5.3: Reconstruction effect versus number of samples, regularization parameter, sparsity and noise level

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