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Neutrosophic Commutative \mathcal{N} -Ideals in BCK -Algebras

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Abstract: The notion of a neutrosophic commutative \mathcal{N} -ideal in BCK -algebras is introduced, and several properties are investigated. Relations between a neutrosophic \mathcal{N} -ideal and a neutrosophic commutative \mathcal{N} -ideal are discussed. Characterizations of a neutrosophic commutative \mathcal{N} -ideal are considered.

Keywords: neutrosophic \mathcal{N} -structure; neutrosophic \mathcal{N} -ideal; neutrosophic commutative \mathcal{N} -ideal

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1. Introduction

As a generalization of fuzzy sets, Atanassov [1] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set.

Smarandache proposed the term “neutrosophic” because “neutrosophic” etymologically comes from “neutrosophy” [French *neutre* < Latin *neuter*, neutral, and Greek *sophia*, skill/wisdom] which means knowledge of neutral thought, and this third/neutral represents the main distinction between “fuzzy”/“intuitionistic fuzzy” logic/set and “neutrosophic” logic/set, i.e., the *included middle* component (Lupasco–Nicolescu’s logic in philosophy), i.e., the neutral/indeterminate/unknown part (besides the “truth”/“membership” and “falsehood”/“non-membership” components that both appear in fuzzy logic/set). Smarandache introduced the degree of indeterminacy/neutrality (i) as an independent component in 1995 (published in 1998) and defined the neutrosophic set on three components

$$(t, i, f) = (\text{truth, indeterminacy, falsehood}).$$

For more details, refer to the site <http://fs.gallup.unm.edu/FlorentinSmarandache.htm>.

Jun et al. [2] introduced a new function which is called negative-valued function, and constructed \mathcal{N} -structures. Khan et al. [3] introduced the notion of neutrosophic \mathcal{N} -structure and applied it to a semigroup. Jun et al. [4] applied the notion of neutrosophic \mathcal{N} -structure to BCK/BCI -algebras. They introduced the notions of a neutrosophic \mathcal{N} -subalgebra and a (closed) neutrosophic \mathcal{N} -ideal in a BCK/BCI -algebra, and investigated related properties. They also considered characterizations of a neutrosophic \mathcal{N} -subalgebra and a neutrosophic \mathcal{N} -ideal, and discussed relations between a neutrosophic \mathcal{N} -subalgebra and a neutrosophic \mathcal{N} -ideal. They provided conditions for a neutrosophic \mathcal{N} -ideal to be a closed neutrosophic \mathcal{N} -ideal. BCK -algebras entered into mathematics in 1966 through the work of Imai and Iséki [5], and have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. Such algebras generalize Boolean rings as well as Boolean D -posets (= MV -algebras). Also, Iséki introduced the notion of a BCI -algebra which is a generalization of a BCK -algebra (see [6]).

In this paper, we introduce the notion of a neutrosophic commutative \mathcal{N} -ideal in BCK-algebras, and investigate several properties. We consider relations between a neutrosophic \mathcal{N} -ideal and a neutrosophic commutative \mathcal{N} -ideal. We discuss characterizations of a neutrosophic commutative \mathcal{N} -ideal.

2. Preliminaries

By a BCI-algebra, we mean a system $X := (X, *, 0) \in K(\tau)$ in which the following axioms hold:

- (I) $((x * y) * (x * z)) * (z * y) = 0,$
- (II) $(x * (x * y)) * y = 0,$
- (III) $x * x = 0,$
- (IV) $x * y = y * x = 0 \Rightarrow x = y$

for all $x, y, z \in X$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a BCK-algebra.

We can define a partial ordering \preceq by

$$(\forall x, y \in X) (x \preceq y \Rightarrow x * y = 0).$$

In a BCK/BCI-algebra X , the following hold:

$$(\forall x \in X) (x * 0 = x), \tag{1}$$

$$(\forall x, y, z \in X) ((x * y) * z = (x * z) * y). \tag{2}$$

A BCK-algebra X is said to be *commutative* if it satisfies the following equality:

$$(\forall x, y \in X) (x * (x * y) = y * (y * x)). \tag{3}$$

A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies

$$0 \in I, \tag{4}$$

$$(\forall x, y \in X) (x * y \in I, y \in I \Rightarrow x \in I). \tag{5}$$

A subset I of a BCK-algebra X is called a *commutative ideal* of X if it satisfies (4) and

$$(\forall x, y, z \in X) ((x * y) * z \in I, z \in I \Rightarrow x * (y * (y * x)) \in I). \tag{6}$$

Lemma 1. *An ideal I is commutative if and only if the following assertion is valid.*

$$(\forall x, y \in X) (x * y \in I \Rightarrow x * (y * (y * x)) \in I). \tag{7}$$

We refer the reader to the books [7,8] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee \{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge \{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, \mathcal{N} -function on X). By an \mathcal{N} -structure, we mean an ordered pair (X, f) of X and an \mathcal{N} -function f on X (see [2]). A *neutrosophic \mathcal{N} -structure* over a nonempty universe of discourse X (see [3]) is defined to be the structure

$$X_N := \frac{X}{(T_N, I_N, F_N)} = \left\{ \frac{x}{(T_N(x), I_N(x), F_N(x))} \mid x \in X \right\} \tag{8}$$

where T_N , I_N and F_N are \mathcal{N} -functions on X which are called the *negative truth membership function*, the *negative indeterminacy membership function* and the *negative falsity membership function*, respectively, on X .

Note that every neutrosophic \mathcal{N} -structure X_N over X satisfies the condition:

$$(\forall x \in X) (-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0).$$

3. Neutrosophic Commutative \mathcal{N} -Ideals

In what follows, let X denote a BCK-algebra unless otherwise specified.

Definition 1 ([4]). A neutrosophic \mathcal{N} -structure X_N over X is called a *neutrosophic \mathcal{N} -ideal of X* if the following assertion is valid.

$$(\forall x, y \in X) \left(\begin{array}{l} T_N(0) \leq T_N(x) \leq \bigvee \{T_N(x * y), T_N(y)\} \\ I_N(0) \geq I_N(x) \geq \bigwedge \{I_N(x * y), I_N(y)\} \\ F_N(0) \leq F_N(x) \leq \bigvee \{F_N(x * y), F_N(y)\} \end{array} \right). \tag{9}$$

Definition 2. A neutrosophic \mathcal{N} -structure X_N over X is called a *neutrosophic commutative \mathcal{N} -ideal of X* if the following assertions are valid.

$$(\forall x \in X) (T_N(0) \leq T_N(x), I_N(0) \geq I_N(x), F_N(0) \leq F_N(x)), \tag{10}$$

$$(\forall x, y, z \in X) \left(\begin{array}{l} T_N(x * (y * (y * x))) \leq \bigvee \{T_N((x * y) * z), T_N(z)\} \\ I_N(x * (y * (y * x))) \geq \bigwedge \{I_N((x * y) * z), I_N(z)\} \\ F_N(x * (y * (y * x))) \leq \bigvee \{F_N((x * y) * z), F_N(z)\} \end{array} \right). \tag{11}$$

Example 1. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the Cayley table which is given in Table 1.

Table 1. Cayley table for the binary operation “*”.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	1
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

The neutrosophic \mathcal{N} -structure

$$X_N = \left\{ \frac{0}{(-0.8, -0.2, -0.9)}, \frac{1}{(-0.3, -0.9, -0.5)}, \frac{2}{(-0.7, -0.7, -0.4)}, \frac{3}{(-0.3, -0.6, -0.7)}, \frac{4}{(-0.5, -0.3, -0.1)} \right\}$$

over X is a neutrosophic commutative \mathcal{N} -ideal of X .

Theorem 1. Every neutrosophic commutative \mathcal{N} -ideal is a neutrosophic \mathcal{N} -ideal.

Proof. Let X_N be a neutrosophic commutative \mathcal{N} -ideal of X . For every $x, z \in X$, we have

$$\begin{aligned}
 T_N(x) &= T_N(x * (0 * (0 * x))) \leq \bigvee \{T_N((x * 0) * z), T_N(z)\} = \bigvee \{T_N(x * z), T_N(z)\}, \\
 I_N(x) &= I_N(x * (0 * (0 * x))) \geq \bigwedge \{I_N((x * 0) * z), I_N(z)\} = \bigwedge \{I_N(x * z), I_N(z)\}, \\
 F_N(x) &= F_N(x * (0 * (0 * x))) \leq \bigvee \{F_N((x * 0) * z), F_N(z)\} = \bigvee \{F_N(x * z), F_N(z)\}
 \end{aligned}$$

by putting $y = 0$ in (11) and using (1). Therefore, X_N is a neutrosophic commutative \mathcal{N} -ideal of X . \square

The converse of Theorem 1 is not true in general as seen in the following example.

Example 2. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the Cayley table which is given in Table 2.

Table 2. Cayley table for the binary operation “*”

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

The neutrosophic \mathcal{N} -structure

$$X_N = \left\{ \frac{0}{(-0.8, -0.1, -0.7)}, \frac{1}{(-0.7, -0.6, -0.6)}, \frac{2}{(-0.6, -0.2, -0.4)}, \frac{3}{(-0.3, -0.8, -0.4)}, \frac{4}{(-0.3, -0.8, -0.4)} \right\}$$

over X is a neutrosophic \mathcal{N} -ideal of X . But it is not a neutrosophic commutative \mathcal{N} -ideal of X since $F_N(2 * (3 * (3 * 2))) = F_N(2) = -0.4 \not\leq -0.7 = \bigvee \{F_N((2 * 3) * 0), F_N(0)\}$.

We consider characterizations of a neutrosophic commutative \mathcal{N} -ideal.

Theorem 2. Let X_N be a neutrosophic \mathcal{N} -ideal of X . Then, X_N is a neutrosophic commutative \mathcal{N} -ideal of X if and only if the following assertion is valid.

$$(\forall x, y \in X) \left(\begin{array}{l} T_N(x * (y * (y * x))) \leq T_N(x * y), \\ I_N(x * (y * (y * x))) \geq I_N(x * y), \\ F_N(x * (y * (y * x))) \leq F_N(x * y) \end{array} \right). \tag{12}$$

Proof. Assume that X_N is a neutrosophic commutative \mathcal{N} -ideal of X . The assertion (12) is by taking $z = 0$ in (11) and using (1) and (10).

Conversely, suppose that a neutrosophic \mathcal{N} -ideal X_N of X satisfies the condition (12). Then,

$$(\forall x, y \in X) \left(\begin{array}{l} T_N(x * y) \leq \bigvee \{T_N((x * y) * z), T_N(z)\} \\ I_N(x * y) \geq \bigwedge \{I_N((x * y) * z), I_N(z)\} \\ F_N(x * y) \leq \bigvee \{F_N((x * y) * z), F_N(z)\} \end{array} \right). \tag{13}$$

It follows that the condition (11) is induced by (12) and (13). Therefore, X_N is a neutrosophic commutative \mathcal{N} -ideal of X . \square

Lemma 2 ([4]). For any neutrosophic \mathcal{N} -ideal X_N of X , we have

$$(\forall x, y, z \in X) \left(x * y \preceq z \Rightarrow \left\{ \begin{array}{l} T_N(x) \leq \bigvee \{T_N(y), T_N(z)\} \\ I_N(x) \geq \bigwedge \{I_N(y), I_N(z)\} \\ F_N(x) \leq \bigvee \{F_N(y), F_N(z)\} \end{array} \right. \right). \tag{14}$$

Theorem 3. In a commutative BCK-algebra, every neutrosophic \mathcal{N} -ideal is a neutrosophic commutative \mathcal{N} -ideal.

Proof. Let $X_{\mathcal{N}}$ be a neutrosophic \mathcal{N} -ideal of a commutative BCK-algebra X . For any $x, y, z \in X$, we have

$$\begin{aligned} & ((x * (y * (y * x))) * ((x * y) * z)) * z \\ &= ((x * (y * (y * x))) * z) * ((x * y) * z) \\ &\preceq (x * (y * (y * x))) * (x * y) \\ &= (x * (x * y)) * (y * (y * x)) = 0, \end{aligned}$$

that is, $(x * (y * (y * x))) * ((x * y) * z) \preceq z$. It follows from Lemma 2 that

$$\begin{aligned} T_{\mathcal{N}}(x * (y * (y * x))) &\leq \bigvee \{T_{\mathcal{N}}((x * y) * z), T_{\mathcal{N}}(z)\}, \\ I_{\mathcal{N}}(x * (y * (y * x))) &\geq \bigwedge \{I_{\mathcal{N}}((x * y) * z), I_{\mathcal{N}}(z)\}, \\ F_{\mathcal{N}}(x * (y * (y * x))) &\leq \bigvee \{F_{\mathcal{N}}((x * y) * z), F_{\mathcal{N}}(z)\}. \end{aligned}$$

Therefore, $X_{\mathcal{N}}$ is a neutrosophic commutative \mathcal{N} -ideal of X . \square

Let $X_{\mathcal{N}}$ be a neutrosophic \mathcal{N} -structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Consider the following sets.

$$\begin{aligned} T_{\mathcal{N}}^{\alpha} &:= \{x \in X \mid T_{\mathcal{N}}(x) \leq \alpha\}, \\ I_{\mathcal{N}}^{\beta} &:= \{x \in X \mid I_{\mathcal{N}}(x) \geq \beta\}, \\ F_{\mathcal{N}}^{\gamma} &:= \{x \in X \mid F_{\mathcal{N}}(x) \leq \gamma\}. \end{aligned}$$

The set

$$X_{\mathcal{N}}(\alpha, \beta, \gamma) := \{x \in X \mid T_{\mathcal{N}}(x) \leq \alpha, I_{\mathcal{N}}(x) \geq \beta, F_{\mathcal{N}}(x) \leq \gamma\}$$

is called the (α, β, γ) -level set of $X_{\mathcal{N}}$. It is clear that

$$X_{\mathcal{N}}(\alpha, \beta, \gamma) = T_{\mathcal{N}}^{\alpha} \cap I_{\mathcal{N}}^{\beta} \cap F_{\mathcal{N}}^{\gamma}.$$

Theorem 4. If $X_{\mathcal{N}}$ is a neutrosophic \mathcal{N} -ideal of X , then $T_{\mathcal{N}}^{\alpha}$, $I_{\mathcal{N}}^{\beta}$ and $F_{\mathcal{N}}^{\gamma}$ are commutative ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$ whenever they are nonempty.

We call $T_{\mathcal{N}}^{\alpha}$, $I_{\mathcal{N}}^{\beta}$ and $F_{\mathcal{N}}^{\gamma}$ level commutative ideals of $X_{\mathcal{N}}$.

Proof. Assume that $T_{\mathcal{N}}^{\alpha}$, $I_{\mathcal{N}}^{\beta}$ and $F_{\mathcal{N}}^{\gamma}$ are nonempty for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then, $x \in T_{\mathcal{N}}^{\alpha}$, $y \in I_{\mathcal{N}}^{\beta}$ and $z \in F_{\mathcal{N}}^{\gamma}$ for some $x, y, z \in X$. Thus, $T_{\mathcal{N}}(0) \leq T_{\mathcal{N}}(x) \leq \alpha$, $I_{\mathcal{N}}(0) \geq I_{\mathcal{N}}(y) \geq \beta$, and $F_{\mathcal{N}}(0) \leq F_{\mathcal{N}}(z) \leq \gamma$, that is, $0 \in T_{\mathcal{N}}^{\alpha} \cap I_{\mathcal{N}}^{\beta} \cap F_{\mathcal{N}}^{\gamma}$. Let $(x * y) * z \in T_{\mathcal{N}}^{\alpha}$ and $z \in T_{\mathcal{N}}^{\alpha}$. Then, $T_{\mathcal{N}}((x * y) * z) \leq \alpha$ and $T_{\mathcal{N}}(z) \leq \alpha$, which imply that

$$T_{\mathcal{N}}(x * (y * (y * x))) \leq \bigvee \{T_{\mathcal{N}}((x * y) * z), T_{\mathcal{N}}(z)\} \leq \alpha,$$

that is, $x * (y * (y * x)) \in T_{\mathcal{N}}^{\alpha}$. If $(a * b) * c \in I_{\mathcal{N}}^{\beta}$ and $c \in I_{\mathcal{N}}^{\beta}$, then $I_{\mathcal{N}}((a * b) * c) \geq \beta$ and $I_{\mathcal{N}}(c) \geq \beta$.

Thus

$$I_{\mathcal{N}}(a * (b * (b * c))) \geq \bigwedge \{I_{\mathcal{N}}((a * b) * c), I_{\mathcal{N}}(c)\} \geq \beta,$$

and so $a * (b * (b * c)) \in I_N^\beta$. Finally, suppose that $(u * v) * w \in F_N^\gamma$ and $w \in F_N^\gamma$. Then, $F_N((u * v) * w) \leq \gamma$ and $F_N(w) \leq \gamma$. Thus,

$$F_N(u * (v * (v * w))) \leq \bigvee \{F_N((u * v) * w), F_N(w)\} \leq \gamma,$$

that is, $u * (v * (v * w)) \in F_N^\gamma$. Therefore, T_N^α, I_N^β and F_N^γ are commutative ideals of X . \square

Corollary 1. Let X_N be a neutrosophic \mathcal{N} -structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If X_N is a neutrosophic commutative \mathcal{N} -ideal of X , then the nonempty (α, β, γ) -level set of X_N is a commutative ideal of X .

Proof. Straightforward. \square

Lemma 3 ([4]). Let X_N be a neutrosophic \mathcal{N} -structure over X and assume that T_N^α, I_N^β and F_N^γ are ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then X_N is a neutrosophic \mathcal{N} -ideal of X .

Theorem 5. Let X_N be a neutrosophic \mathcal{N} -structure over X and assume that T_N^α, I_N^β and F_N^γ are commutative ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then, X_N is a neutrosophic commutative \mathcal{N} -ideal of X .

Proof. If T_N^α, I_N^β and F_N^γ are commutative ideals of X , then they are ideals of X . Hence, X_N is a neutrosophic \mathcal{N} -ideal of X by Lemma 3. Let $x, y \in X$ and $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$ such that $T_N(x * y) = \alpha, I_N(x * y) = \beta$ and $F_N(x * y) = \gamma$. Then, $x * y \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$. Since $T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$ is a commutative ideal of X , it follows from Lemma 1 that $x * (y * (y * x)) \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$.

Hence

$$\begin{aligned} T_N(x * (y * (y * x))) &\leq \alpha = T_N(x * y), \\ I_N(x * (y * (y * x))) &\geq \beta = I_N(x * y), \\ F_N(x * (y * (y * x))) &\leq \gamma = F_N(x * y). \end{aligned}$$

Therefore, X_N is a neutrosophic commutative \mathcal{N} -ideal of X by Theorem 2. \square

Theorem 6. Let $f : X \rightarrow X$ be an injective mapping. Given a neutrosophic \mathcal{N} -structure X_N over X , the following are equivalent.

- (1) X_N is a neutrosophic commutative \mathcal{N} -ideal of X , satisfying the following condition.

$$(\forall x \in X) \left(\begin{array}{l} T_N(f(x)) = T_N(x) \\ I_N(f(x)) = I_N(x) \\ F_N(f(x)) = F_N(x) \end{array} \right). \tag{15}$$

- (2) T_N^α, I_N^β and F_N^γ are commutative ideals of X_N , satisfying the following condition.

$$f(T_N^\alpha) = T_N^\alpha, f(I_N^\beta) = I_N^\beta, f(F_N^\gamma) = F_N^\gamma. \tag{16}$$

Proof. Let X_N be a neutrosophic commutative \mathcal{N} -ideal of X , satisfying the condition (15). Then, T_N^α, I_N^β and F_N^γ are commutative ideals of X_N by Theorem 4. Let $\alpha \in \text{Im}(T_N), \beta \in \text{Im}(I_N), \gamma \in \text{Im}(F_N)$ and $x \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$. Then $T_N(f(x)) = T_N(x) \leq \alpha, I_N(f(x)) = I_N(x) \geq \beta$ and $F_N(f(x)) = F_N(x) \leq \gamma$. Thus, $f(x) \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$, which shows that $f(T_N^\alpha) \subseteq T_N^\alpha, f(I_N^\beta) \subseteq I_N^\beta$ and $f(F_N^\gamma) \subseteq F_N^\gamma$. Let $y \in X$ be such that $f(y) = x$. Then, $T_N(y) = T_N(f(y)) = T_N(x) \leq \alpha, I_N(y) = I_N(f(y)) = I_N(x) \geq \beta$

and $F_N(y) = F_N(f(y)) = F_N(x) \leq \gamma$, which imply that $y \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$. Thus, $x = f(y) \in f(T_N^\alpha) \cap f(I_N^\beta) \cap f(F_N^\gamma)$, and so $T_N^\alpha \subseteq f(T_N^\alpha)$, $I_N^\beta \subseteq f(I_N^\beta)$ and $F_N^\gamma \subseteq f(F_N^\gamma)$. Therefore (16) is valid.

Conversely, assume that T_N^α, I_N^β and F_N^γ are commutative ideals of X_N , satisfying the condition (16). Then, X_N is a neutrosophic commutative \mathcal{N} -ideal of X by Theorem 5. Let $x, y, z \in X$ be such that $T_N(x) = \alpha, I_N(y) = \beta$ and $F_N(z) = \gamma$. Note that

$$\begin{aligned} T_N(x) = \alpha &\iff x \in T_N^\alpha \text{ and } x \notin T_N^{\tilde{\alpha}} \text{ for all } \alpha > \tilde{\alpha}, \\ I_N(y) = \beta &\iff y \in I_N^\beta \text{ and } y \notin I_N^{\tilde{\beta}} \text{ for all } \beta < \tilde{\beta}, \\ F_N(z) = \gamma &\iff z \in F_N^\gamma \text{ and } z \notin F_N^{\tilde{\gamma}} \text{ for all } \gamma > \tilde{\gamma}. \end{aligned}$$

It follows from (16) that $f(x) \in T_N^\alpha, f(y) \in I_N^\beta$ and $f(z) \in F_N^\gamma$. Hence, $T_N(f(x)) \leq \alpha, I_N(f(y)) \geq \beta$ and $F_N(f(z)) \leq \gamma$. Let $\tilde{\alpha} = T_N(f(x)), \tilde{\beta} = I_N(f(y))$ and $\tilde{\gamma} = F_N(f(z))$. If $\alpha > \tilde{\alpha}$, then $f(x) \in T_N^{\tilde{\alpha}} = f(T_N^{\tilde{\alpha}})$, and thus $x \in T_N^{\tilde{\alpha}}$ since f is one to one. This is a contradiction. Hence, $T_N(f(x)) = \alpha = T_N(x)$. If $\beta < \tilde{\beta}$, then $f(y) \in I_N^{\tilde{\beta}} = f(I_N^{\tilde{\beta}})$ which implies from the injectivity of f that $y \in I_N^{\tilde{\beta}}$, a contradiction. Hence, $I_N(f(y)) = \beta = I_N(y)$. If $\gamma > \tilde{\gamma}$, then $f(z) \in F_N^{\tilde{\gamma}} = f(F_N^{\tilde{\gamma}})$. Since f is one to one, we have $z \in F_N^{\tilde{\gamma}}$ which is a contradiction. Thus, $F_N(f(z)) = \gamma = F_N(z)$. This completes the proof. \square

For any elements $\omega_t, \omega_i, \omega_f \in X$, we consider sets:

$$\begin{aligned} X_N^{\omega_t} &:= \{x \in X \mid T_N(x) \leq T_N(\omega_t)\}, \\ X_N^{\omega_i} &:= \{x \in X \mid I_N(x) \geq I_N(\omega_i)\}, \\ X_N^{\omega_f} &:= \left\{x \in X \mid F_N(x) \leq F_N(\omega_f)\right\}. \end{aligned}$$

Obviously, $\omega_t \in X_N^{\omega_t}, \omega_i \in X_N^{\omega_i}$ and $\omega_f \in X_N^{\omega_f}$.

Lemma 4 ([4]). Let ω_t, ω_i and ω_f be any elements of X . If X_N is a neutrosophic \mathcal{N} -ideal of X , then $X_N^{\omega_t}, X_N^{\omega_i}$ and $X_N^{\omega_f}$ are ideals of X .

Theorem 7. Let ω_t, ω_i and ω_f be any elements of X . If X_N is a neutrosophic commutative \mathcal{N} -ideal of X , then $X_N^{\omega_t}, X_N^{\omega_i}$ and $X_N^{\omega_f}$ are commutative ideals of X .

Proof. If X_N is a neutrosophic commutative \mathcal{N} -ideal of X , then it is a neutrosophic \mathcal{N} -ideal of X and so $X_N^{\omega_t}, X_N^{\omega_i}$ and $X_N^{\omega_f}$ are ideals of X by Lemma 4. Let $x * y \in X_N^{\omega_t} \cap X_N^{\omega_i} \cap X_N^{\omega_f}$ for any $x, y \in X$. Then, $T_N(x * y) \leq T_N(\omega_t), I_N(x * y) \geq I_N(\omega_i)$ and $F_N(x * y) \leq F_N(\omega_f)$. It follows from Theorem 2 that

$$\begin{aligned} T_N(x * (y * (y * x))) &\leq T_N(x * y) \leq T_N(\omega_t), \\ I_N(x * (y * (y * x))) &\geq I_N(x * y) \geq I_N(\omega_i), \\ F_N(x * (y * (y * x))) &\leq F_N(x * y) \leq F_N(\omega_f). \end{aligned}$$

Hence, $x * (y * (y * x)) \in X_N^{\omega_t} \cap X_N^{\omega_i} \cap X_N^{\omega_f}$, and therefore $X_N^{\omega_t}, X_N^{\omega_i}$ and $X_N^{\omega_f}$ are commutative ideals of X by Lemma 1. \square

Theorem 8. Any commutative ideal of X can be realized as level commutative ideals of some neutrosophic commutative \mathcal{N} -ideal of X .

Proof. Let A be a commutative ideal of X and let X_N be a neutrosophic \mathcal{N} -structure over X in which

$$\begin{aligned}
 T_N : X \rightarrow [-1, 0], \quad x \mapsto & \begin{cases} \alpha & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases} \\
 I_N : X \rightarrow [-1, 0], \quad x \mapsto & \begin{cases} \beta & \text{if } x \in A, \\ -1 & \text{otherwise,} \end{cases} \\
 F_N : X \rightarrow [-1, 0], \quad x \mapsto & \begin{cases} \gamma & \text{if } x \in A, \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

where $\alpha, \gamma \in [-1, 0)$ and $\beta \in (-1, 0]$. Division into the following cases will verify that X_N is a neutrosophic commutative \mathcal{N} -ideal of X .

If $(x * y) * z \in A$ and $z \in A$, then $x * (y * (y * x)) \in A$. Thus,

$$\begin{aligned}
 T_N((x * y) * z) &= T_N(z) = T_N(x * (y * (y * x))) = \alpha, \\
 I_N((x * y) * z) &= I_N(z) = I_N(x * (y * (y * x))) = \beta, \\
 F_N((x * y) * z) &= F_N(z) = F_N(x * (y * (y * x))) = \gamma,
 \end{aligned}$$

and so (11) is clearly verified.

If $(x * y) * z \notin A$ and $z \notin A$, then $T_N((x * y) * z) = T_N(z) = 0$, $I_N((x * y) * z) = I_N(z) = -1$ and $F_N((x * y) * z) = F_N(z) = 0$. Hence

$$\begin{aligned}
 T_N(x * (y * (y * x))) &\leq \bigvee \{T_N((x * y) * z), T_N(z)\}, \\
 I_N(x * (y * (y * x))) &\geq \bigwedge \{I_N((x * y) * z), I_N(z)\}, \\
 F_N(x * (y * (y * x))) &\leq \bigvee \{F_N((x * y) * z), F_N(z)\}.
 \end{aligned}$$

If $(x * y) * z \in A$ and $z \notin A$, then $T_N((x * y) * z) = \alpha$, $T_N(z) = 0$, $I_N((x * y) * z) = \beta$, $I_N(z) = -1$, $F_N((x * y) * z) = \gamma$ and $F_N(z) = 0$. Therefore,

$$\begin{aligned}
 T_N(x * (y * (y * x))) &\leq \bigvee \{T_N((x * y) * z), T_N(z)\}, \\
 I_N(x * (y * (y * x))) &\geq \bigwedge \{I_N((x * y) * z), I_N(z)\}, \\
 F_N(x * (y * (y * x))) &\leq \bigvee \{F_N((x * y) * z), F_N(z)\}.
 \end{aligned}$$

Similarly, if $(x * y) * z \notin A$ and $z \in A$, then (11) is verified. Therefore, X_N is a neutrosophic commutative \mathcal{N} -ideal of X . Obviously, $T_N^\alpha = A$, $I_N^\beta = A$ and $F_N^\gamma = A$. This completes the proof. \square

4. Conclusions

In order to deal with the negative meaning of information, Jun et al. [2] have introduced a new function which is called negative-valued function, and constructed \mathcal{N} -structures. The concept of neutrosophic set (NS) has been developed by Smarandache in [9,10] as a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. In this article, we have introduced the notion of a neutrosophic commutative \mathcal{N} -ideal in BCK-algebras, and investigated several properties. We have considered relations between a neutrosophic \mathcal{N} -ideal and a neutrosophic commutative \mathcal{N} -ideal. We have discussed characterizations of a neutrosophic commutative \mathcal{N} -ideal.

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