Classical Unified Field Theory of Gravitation and Electromagnetism

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Abstract

According to Noether's theorem, for every differentiable symmetry of action, there exists a corresponding conserved quantity. If we assume the stationary condition as a role of symmetry, there is a conserved quantity. By using the definition of the Komar mass, one can calculate the mass in a curved spacetime. If we consider charge as a conserved quantity, it means existence of symmetry, and we can consider one more axis. In this paper, we consider an extension to five dimensions by using results of the ADM formalism. In terms of (4+1) decomposition, with an alternative surface integral, we find the rotating and charged five-dimensional metric solution and check whether it gives the mass, charge, and angular momentum exactly.

1 Introduction

The Kaluza-Klein theory is a classical unified field theory of gravitation and electromagnetism in terms of general relativity extended to five dimensions [1]. In this paper, we use a different but basically equivalent metric ansatz based on ADM formalism, which decomposes spacetime into the time component and three spatial components, and use the concept of extrinsic curvature [2–5]. It plays mathematically important role in the unified field theory, which will be discussed in this paper. The main assumption is that as gravitation can be described geometrically in four dimensions, electromagnetism can also be described geometrically in five dimensions [6]. The main goal of this paper is to find the five-dimensional metric form for the spherical or rotating black hole solutions, which have mass and charge. In addition, we would like to deal with the Lorentz force as well, which is known as the main shortfall of Kaluza-Klein theory. In four dimensions, there are black hole solutions, which have charge and mass, simultaneously known as the Reisner-Nordstrom (RN), Kerr-Newman (KN) black holes [7]. To obtain a 5D solution, we assume two symmetries of the metric. One is cylindrical condition, and the other is the stationary condition.

Under the assumption that electromagnetism is an effect of pure geometry, the expected results will be similar to that of Kaluza-Klein theory [8–10]. After we obtain the 5D metric, which is exactly the 5D vacuum solution, we also obtain the mass, charge and angular momentum of the object.

2 Metric Ansatz

In the following equations, Greek indices refer to spacetime components (0,123) and the index 5 refers to the fifth dimension. Roman indices (a,b) span (5,0,123). Similar to Kaluza's ansatz, we consider the ansatz

$$ {}^{5}g_{\mu\nu} = {}^{4}g_{\mu\nu}, \quad {}^{5}g_{5\mu} = \beta_{\mu}, \quad {}^{5}g_{55} = -N^2 + \beta^{\lambda}\beta_{\lambda}. \tag{1} $$

Then the inverse metric is given by

$$ {}^{5}g^{\mu\nu} = {}^{4}g^{\mu\nu} - \frac{\beta^{\mu}\beta^{\nu}}{N^2}, \quad {}^{5}g^{5\mu} = \frac{\beta^{\mu}}{N^2}, \quad {}^{5}g^{55} = -\frac{1}{N^2}. \tag{2} $$

In other words,

$$ {}^{5}g_{ab} = \begin{bmatrix} -N^2 + \beta^{\lambda}\beta_{\lambda} & \beta_{\nu} \\ & \\ \beta_{\mu} & {}^{4}g_{\mu\nu} \end{bmatrix}, \quad {}^{5}g^{ab} = \begin{bmatrix} -\frac{1}{N^2} & \frac{\beta^{\nu}}{N^2} \\ & \\ \frac{\beta^{\mu}}{N^2} & {}^{4}g^{\mu\nu} - \frac{\beta^{\mu}\beta^{\nu}}{N^2} \end{bmatrix}, \tag{3} $$

where ${}^{5}g_{ab}$ is the 5D metric and ${}^{4}g_{\mu\nu}$ is the standard four-dimensional (4D) metric with the Lorentzian signature, $(-,+++)$. The noticeable point is ${}^{5}g_{55} = -N^2 + \beta^{\lambda}\beta_{\lambda}$. We set the fifth dimension to timelike and there is no problem at this stage (see the first item in the discussion section).

3 5D Christoffel Symbol and 5D Ricci Tensor

We assumed the two symmetries of the metric.

$$\frac{\partial}{\partial \omega}\,{}^{5}g_{ab} = 0, \quad \frac{\partial}{\partial t}\,{}^{5}g_{ab} = 0, \tag{4}$$

where ω is fifth coordinate. Under these conditions, the 5D Christoffel symbols are given by

$${}^{5}\Gamma^{5}_{\mu\nu} = -\frac{K_{\mu\nu}}{N}, \tag{5}$$

$${}^{5}\Gamma^{5}_{\mu 5} = -\frac{1}{N}K_{5\mu} + \frac{1}{N}\partial_{\mu}N, \tag{6}$$

$${}^{5}\Gamma^{5}_{55} = -\frac{\beta^{\sigma}}{2N^{2}}\partial_{\sigma}(-N^{2} + \beta^{\lambda}\beta_{\lambda}), \tag{7}$$

where $K_{\mu\nu}$ is the extrinsic curvature tensor defined as

$$K_{ab} = -\nabla_{b}n_{a} - n_{b}n^{c}\nabla_{c}n_{a} \tag{8}$$

with the unit normal vector $\mathbf{n}$ [5]. This extrinsic curvature comes from (4+1) decomposition, and foliates five dimensions with respect to the fifth coordinate ω [4]. Note that the original ADM formalism is (3+1) decomposition, which foliates spacetime with respect to time t. Then the extrinsic curvature tensor $K_{\mu\nu}$ is given by

$$K_{\mu\nu} = \frac{1}{2N}\left({}^{4}\nabla_{\mu}\beta_{\nu} + {}^{4}\nabla_{\nu}\beta_{\mu} - \frac{\partial}{\partial\omega}\,{}^{4}g_{\mu\nu}\right), \; {}^{4}\nabla_{\mu}\beta_{\nu} \equiv \partial_{\mu}\beta_{\nu} - {}^{4}\Gamma^{\lambda}_{\mu\nu}\beta_{\lambda}. \tag{9}$$

Henceforth, the covariant derivative is related to ${}^{4}g_{\mu\nu}$ and emit ^{4} index for the covariant derivative. In this paper, we assume that the charge identified as $\frac{dx^{5}}{ds} = \frac{q}{m}$ is not changed along the geodesic curve. The fifth component of the geodesic equation is as follows:

$$\frac{d}{ds}\left(\frac{dx^{5}}{ds}\right) + {}^{5}\Gamma^{5}_{ab}\frac{dx^{a}}{ds}\frac{dx^{b}}{ds} = 0. \tag{10}$$

Assuming that the charge does not change along the geodesic curve for any particle, one can set ${}^{5}\Gamma^{5}_{ab} = 0$. Then from eqns(4), (5), (6), (7) and (9), we obtain

$$N = constant, \quad \beta_{\mu} = \alpha\,{}^{4}g_{0\mu}, \tag{11}$$

where α is a constant. Then with eqns(4), (11), ${}^{5}\Gamma^{\mu}_{ab}$ is given by

$${}^{5}\Gamma^{\mu}_{\lambda\rho} = {}^{4}\Gamma^{\mu}_{\lambda\rho}, \tag{12}$$

$${}^{5}\Gamma^{\mu}_{\lambda 5} = {}^{4}g^{\mu\nu}F_{\lambda\nu}, \tag{13}$$

$${}^{5}\Gamma^{\mu}_{55} = -{}^{4}g^{\mu\nu}\partial_{\nu}\phi = -\nabla^{\mu}\phi, \tag{14}$$

where we define

$$F_{\mu\nu} = \partial_\mu \left(\frac{\beta_\nu}{2} \right) - \partial_\nu \left(\frac{\beta_\mu}{2} \right), \tag{15}$$

$$\phi = \frac{1}{2} {}^5 g_{55}. \tag{16}$$

Now, we have all components of Christoffel symbols. Then Ricci tensor is given by

$${}^5 R_{ab} = \begin{bmatrix} -\Box \phi + F_{\lambda\rho} F^{\lambda\rho} & \nabla^\rho F_{\nu\rho} \\ \nabla^\rho F_{\mu\rho} & {}^4 R_{\mu\nu} \end{bmatrix}. \tag{17}$$

4 Lorentz force

In the Lorentz force can be derived from the variation of the 5D geodesic equations. In Kaluza's hypothesis, however, the problem with this is that there is the quadratic term $\frac{dx^5}{ds}$. It was known as the main shortfall of the Kaluza hypothesis.

Although we induce Lorentz force exactly, it is a suggestion for problem-solving rather than assertion. For clarity, we use the relation $-c^2 d\tilde{\tau}^2 = {}^5 g_{ab} dx^a dx^b$ rather than $ds^2 = {}^5 g_{ab} dx^a dx^b$. By adjusting the scale of dx^5, we can write $\frac{dx^5}{d\tilde{\tau}} \equiv \frac{q}{m}$. We consider $\frac{dx^5}{d\tau} = 0$ here because of two reasons. One is that τ is irrelevant to dx^5 and the other is that when $\frac{dx^5}{d\tau}$ is nonzero, this means that the object disappears from our hypersurface, the world we live in. Now, because of $\frac{dx^5}{d\tau} = 0$, $\frac{d\tilde{\tau}}{d\tau} = 1$. Then

$$\frac{dx^\mu}{d\tau} = \frac{d\tilde{\tau}}{d\tau} \frac{dx^\mu}{d\tilde{\tau}} = \frac{dx^\mu}{d\tilde{\tau}}, \tag{18}$$

$$\frac{dx^5}{d\tau} \neq \frac{d\tilde{\tau}}{d\tau} \frac{dx^5}{d\tilde{\tau}} = \frac{dx^5}{d\tilde{\tau}}. \tag{19}$$

With equation

$$\frac{D}{d\tau} \left(\frac{dx^\mu}{d\tilde{\tau}} \right) = \frac{d\tilde{\tau}}{d\tau} \frac{D}{d\tilde{\tau}} \left(\frac{dx^\mu}{d\tilde{\tau}} \right) \tag{20}$$

and eqn(19), we obtain

$$\frac{d}{d\tau} \left(\frac{dx^\mu}{d\tilde{\tau}} \right) \neq \frac{d\tilde{\tau}}{d\tau} \frac{d}{d\tilde{\tau}} \left(\frac{dx^\mu}{d\tilde{\tau}} \right). \tag{21}$$

Therefore, we consider eqn(21) carefully and only use eqn(20). From eqn(20),

$$\frac{D}{d\tau} \left(\frac{dx^\mu}{d\tilde{\tau}} \right) = \frac{d}{d\tau} \left(\frac{dx^\mu}{d\tilde{\tau}} \right) + {}^5\Gamma^\mu_{ab} \frac{dx^a}{d\tilde{\tau}} \frac{dx^b}{d\tau} = 0. \tag{22}$$

With eqn(18), $\frac{dx^5}{d\tau} = 0$, eqn(22) becomes

$$\frac{d}{d\tau}\left(\frac{dx^\mu}{d\tau}\right) + {}^5\Gamma^\mu_{\lambda\rho}\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau} = -{}^5\Gamma^\mu_{5\rho}\frac{dx^5}{d\tilde{\tau}}\frac{dx^\rho}{d\tau}. \tag{23}$$

With eqns(12) and (13), eqn(23) becomes

$$\frac{d}{d\tau}\left(\frac{dx^\mu}{d\tau}\right) + {}^4\Gamma^\mu_{\lambda\rho}\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau} = \frac{q}{m}\,{}^4g^{\mu\nu}F_{\nu\rho}\frac{dx^\rho}{d\tau}. \tag{24}$$

At this stage, by separate several components of eqn(17), ${}^5R_{5\nu}$ contains Maxwell's equations and ${}^4R_{\mu\nu}$ denotes the Einstein equations (see the third item in the discussion section).

5 Correspondence with Classical Dynamics

Unexpectedly, the result of our analysis for the 5D vacuum solution is ${}^4R_{\mu\nu} = 0$. The expected metric solution, ${}^4g_{\mu\nu}$, is the well-known charged 4D black hole, such as the RN or KN black holes, but our results indicate that those are not solutions in our formalism. In this paper, we check why the 4D metric should be a vacuum solution as the Schwarzschild or Kerr black hole, although it is a charged black hole. As we identified $\beta_\mu = \alpha^4 g_{0\mu} = 2A_\mu$, we can fix α as $\alpha = -\frac{Qc}{4\pi\varepsilon_0 GM}$. Then for a rotating charged black hole, the 5D metric solution is given by

$${}^5g_{ab} = \begin{bmatrix} -N^2 - (\frac{Qc}{4\pi\varepsilon_0 GM})^2(1-\frac{2GMr}{\Sigma c^2}) & \frac{Qc}{4\pi\varepsilon_0 GM}(1-\frac{2GMr}{\Sigma c^2}) & 0 & 0 & \frac{2Qr}{4\pi\varepsilon_0 c\Sigma}asin^2(\theta) \\ \frac{Qc}{4\pi\varepsilon_0 GM}(1-\frac{2GMr}{\Sigma c^2}) & -(1-\frac{2GMr}{\Sigma c^2}) & 0 & 0 & -\frac{2GMr}{\Sigma c^2}asin^2(\theta) \\ 0 & 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & 0 & \Sigma & 0 \\ \frac{2Qr}{4\pi\varepsilon_0 c\Sigma}asin^2(\theta) & -\frac{2GMr}{\Sigma c^2}asin^2(\theta) & 0 & 0 & (r^2+a^2)sin^2(\theta) + \frac{2GMr}{\Sigma c^2}a^2 sin^4(\theta) \end{bmatrix}.(*)$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and we used Boyer-Lindquist coordinates. Since we just used condition ${}^4R_{\mu\nu} = 0$ of eqn(17), we cannot assure it is a vacuum solution. In fact, condition of ${}^4R_{\mu\nu} = 0$ is only condition. Thus, this ${}^5g_{ab}$ is vacuum solution. This point will be discussed later. Now, to observe dynamics, we assume the spherical vacuum solution and the initiation of free falling. Then we consider $a \equiv \frac{J}{mc} = 0$ and $\frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau} = 0$. Note that $\frac{dx^0}{d\tau} = c\ {}^{-\frac{1}{2}}\!\!\sqrt{-{}^4g_{00}} = c\ {}^{-\frac{1}{2}}\!\!\sqrt{1-\frac{2GM}{rc^2}}$. From eqn(24), we obtain

$$m\frac{d}{d\tau}\left(\frac{dx^1}{d\tau}\right) = \sqrt{1-\frac{2GM}{rc^2}}\frac{Qq}{4\pi\varepsilon_0 r^2} - \frac{GMm}{r^2}. \tag{25}$$

This describes the known classical dynamics well. Now for the RN black hole, from eqn(24), we obtain

$$m\frac{d}{d\tau}\left(\frac{dx^1}{d\tau}\right) = -\frac{GMm}{r^2} + \frac{GQ^2m}{4\pi\varepsilon_0 c^4 r^3} \tag{26}$$

$$+\sqrt{1-\frac{2GM}{rc^2}+\frac{GQ^2}{4\pi\varepsilon_0c^4r^2}}\left(\frac{Qq}{4\pi\varepsilon_0 r^2}-\frac{Q^3q}{(4\pi\varepsilon_0c^2)^2Mr^3}\right).$$

It is a strange result because there are $\frac{1}{r^3}$ terms. Furthermore, the well-known charged 4D black hole cannot be a solution under 5D vacuum solution. Now, we would like to explain the reason why the 5D vacuum condition gives a 4D vacuum solution rather than the known charged 4D black hole. Accordingly, let us check what was neglected in the well-known 4D charged black hole,

$$S=\int {}^5R\sqrt{{}^5g}\,d^5x. \tag{27}$$

Noting $\det(A)=adj(A_{ij})\frac{1}{A^{-1}_{ij}}$, where A^{-1}_{ij} is a component of i-th row j-th column of inverse matrix of A and $adj(A_{ij})$ is adjoint of A_{ij}, then from eqn(3), $\sqrt{{}^5g}=N\sqrt{-{}^4g}$. Then the action of the integrand can be rewritten as

$$S=\int {}^5R\sqrt{-{}^4g}\,d^4x\,Ndx^5. \tag{28}$$

Since N is a constant and the integrand is independent of x^5, we can ignore the overall quantity Ndx^5. From eqns (3) and (17), we obtain

$${}^5R=({}^4R-\frac{F_{\lambda\rho}F^{\lambda\rho}}{N^2})+\frac{1}{N^2}(\Box\phi-{}^4R_{\lambda\rho}\beta^\lambda\beta^\rho+2\beta^\lambda\nabla^\rho F_{\lambda\rho}). \tag{29}$$

From eqns (9) and (11), β_μ follows the Killing equation, its divergence equals 0, that is

$$\nabla^\rho F_{\lambda\rho}=\nabla^\rho\nabla_\lambda\beta_\rho=(\nabla_\rho\nabla_\lambda-\nabla_\lambda\nabla_\rho)\beta^\rho={}^4R_{\rho\lambda}\beta^\rho, \tag{30}$$

where $\nabla_\lambda\nabla\rho\beta^\rho$ is a hypothetical representation of zero terms. Now, the second parentheses in eqn(29) can be rewritten as

$$\frac{1}{N^2}(\Box\phi+\beta^\lambda\nabla^\rho F_{\lambda\rho}). \tag{31}$$

Now we put $\nabla^\rho F_{\lambda\rho}=\mu_0 J_\lambda$. By neglecting total divergence of eqn(31), from eqn(29) we obtain

$${}^5R={}^4R-(\frac{F_{\lambda\rho}F^{\lambda\rho}}{N^2}-\frac{\mu_0}{N^2}J_\lambda\beta^\lambda). \tag{32}$$

Since we are considering a source-free region, $J_\mu=0$, one can think that by excluding current term, eqn(32) will give the well-known charged 4D black hole solutions. However, we have the relation $\mu_0 J^\lambda\beta_\lambda=-\Box\phi+$

$F_{\lambda\rho}F^{\lambda\rho}$. Now, with eqn(32) and by neglecting total divergence, the matter Lagrangian effectively equals 0. Until now, we just neglected J_λ since we are considering a source-free region. However this result says that this is not the case. Consequently, our action is equivalent to ${}^4R$. To find out solution more clearly, we try to obtain a solution without using variation principle. In fact, to obtain the solution, it is more reasonable to solve eqn(17) rather than solving eqn(32) with the variation principle. From eqns(16), (17), we obtain

$$ {}^5R_{55} = -(\nabla^\rho\beta^\lambda\nabla_\rho\beta_\lambda + \beta^\lambda\nabla^\rho\nabla_\rho\beta_\lambda) + F_{\lambda\rho}F^{\lambda\rho} \tag{33} $$

From eqn(15),

$$ \nabla^\rho\beta^\lambda\nabla_\rho\beta_\lambda = F^{\rho\lambda}F_{\rho\lambda}. \tag{34} $$

With eqn(30),

$$ \beta^\lambda\nabla^\rho\nabla_\rho\beta_\lambda = -\,{}^4R_{\rho\lambda}\beta^\rho\beta^\lambda. \tag{35} $$

Then eqn(17) becomes

$$ {}^5R_{ab} = \begin{bmatrix} {}^4R_{\lambda\rho}\beta^\lambda\beta^\rho & {}^4R_{\nu\rho}\beta^\rho \\ {}^4R_{\mu\rho}\beta^\rho & {}^4R_{\mu\nu}. \end{bmatrix}. \tag{36} $$

Then we can easily obtain the condition that satisfies ${}^5R_{ab} = 0$ under the conditions of symmetry and the conservation of charge along the geodesic curve. It is only the ${}^4R_{\mu\nu} = 0$ condition. In this step, we can say that our ${}^5g_{ab}$ in (*) is a 5D vacuum solution.
In summary, from the 5D perspective RN and KN black hole solutions are incompatible with $J_\mu = 0$ and our dynamic result. In contrast, the 4D vacuum solution is a solution under the 5D vacuum condition with relations describing the classical dynamics for the Lorentz force. The result ${}^4R_{\mu\nu} = 0$ is the same as in [14], although the assumption is different.

6 5D Energy-Momentum Tensor

In Kaluza's hypothesis,

$$ {}^5T_{ab} = \begin{bmatrix} \gamma_s c_5^2\rho_s & \gamma_e c J_\nu \\ \gamma_e c J_\mu & {}^4T_{\mu\nu} \end{bmatrix}. \tag{37} $$

See eqn(84) of [10]. We will induce energy momentum tensor which is consistent with our result, by assuming a perfect fluid source. In 4D, the energy-momentum tensor ${}^4T_{\mu\nu}$ for the perfect fluid is given by [11]

$$ {}^4T_{\mu\nu} = (\rho + \frac{P}{c^2})\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau}\,{}^4g_{\mu\lambda}\,{}^4g_{\nu\rho} + P\,{}^4g^{\lambda\rho}\,{}^4g_{\mu\lambda}\,{}^4g_{\nu\rho}. \tag{38} $$

From Einstein's equations, we have the relation,

$$ {}^{4}R_{\mu\nu} = \kappa \left[(\rho + \frac{P}{c^2}) \frac{dx^\lambda}{d\tau} \frac{dx^\rho}{d\tau} \, {}^{4}g_{\mu\lambda} \, {}^{4}g_{\nu\rho} + \frac{1}{2} (\rho c^2 - P) \, {}^{4}g^{\lambda\rho} \, {}^{4}g_{\mu\lambda} \, {}^{4}g_{\nu\rho} \right]. \quad (39) $$

Now we try to induce ${}^{5}R_{ab}$. With eqn(36):

$$ {}^{5}R_{\mu\nu} = {}^{4}R_{\mu\nu}, \quad (40) $$

$$ {}^{5}R_{\mu 5} = {}^{4}R_{\mu\lambda}\beta^\lambda, \quad (41) $$

$$ {}^{5}R_{55} = {}^{4}R_{\lambda\rho}\beta^\lambda\beta^\rho, \quad (42) $$

we obtain

$$ {}^{5}R_{ab} = \kappa [(\rho + \frac{P}{c^2}) \frac{dx^\lambda}{d\tau} \frac{dx^\rho}{d\tau} \, {}^{5}g_{a\lambda} \, {}^{5}g_{b\rho} + \frac{1}{2} (\rho c^2 - P) \, {}^{4}g^{\lambda\rho} \, {}^{5}g_{a\lambda} \, {}^{5}g_{b\rho}]. \quad (43) $$

Then ${}^{5}T_{ab}$ is

$$ {}^{5}T_{ab} = (\rho + \frac{P}{c^2}) \frac{dx^\lambda}{d\tau} \frac{dx^\rho}{d\tau} \, {}^{5}g_{a\lambda} \, {}^{5}g_{b\rho} + P \, {}^{4}g^{\lambda\rho} \, {}^{5}g_{a\lambda} \, {}^{5}g_{b\rho}. \quad (44) $$

Note that from eqn(37), ${}^{4}T_{\mu\nu}$ is independent of charge. Then by solving eqn(59), ${}^{4}g_{\mu\nu}$ is independent of charge. Then it is consistent with our opinion in section 5.

7 No-hair theorem

We would like to obtain the mass, charge, and angular momentum exactly in 5D. Recall that we assumed the cylindrical condition and stationary condition. We do not consider cosmological constant.

There is a mass quantity in 4D, which is called the Komar mass. Komar mass is defined as

$$ M_K \equiv -\frac{1}{8\pi} \oint_{\mathbf{S_t}} \nabla^\mu \zeta^\nu dS_{\mu\nu} \quad dS_{\mu\nu} = (s_\mu n_\nu - s_\nu n_\mu)\sqrt{q} d^2 y \quad (45) $$

where vector $\mathbf{n}$ is the unit normal to Σ_t, vector $\mathbf{s}$ is the unit normal to S_t, within Σ_t oriented toward the exterior of S_t [5, 12]. However, as we introduced the concept of gravitational and electromagnetic vector potential, we would like to define M and Q as the surface integrals of the gravity field and electric field. Before applying the surface integral, we want to consider integrating in a different way from the Komar mass [16]. Let us consider the following integral:

$$ \oint \mathbf{E} \cdot d\mathbf{S}, \quad (46) $$

where $d\mathbf{S}$ is

$$d\mathbf{S} = \frac{\nabla x^i}{\sqrt{\nabla x^i \cdot \nabla x^i}} \sqrt{-adj(^4 g_{ii})} dS_i, \tag{47}$$

and $dS_1 = dx^2 dx^3$, $dS_2 = dx^1 dx^3$, $dS_3 = dx^1 dx^2$. Thus, it is equivalent to the Komar mass expression. However, it is easier to calculate. Notice, $\frac{1}{\sqrt{\nabla x^i \cdot \nabla x^i}} \sqrt{-adj(^4 g_{ii})} = \sqrt{-^4 g}$ for all i. Now we are ready to calculate M, Q and J. First, we would like to calculate Q. As the electric field is related to $F^{0i} \simeq \frac{\mathbf{E}}{c}$,

$$Q \equiv \varepsilon_0 c \oint F^{0i} \sqrt{-^4 g} dS_i. \tag{48}$$

We want to make sure this is the exact Q. Before the calculation, we want to make sure that it does not matter what surface we choose. From eqn(48),

$$Q \Rightarrow \oint F^{00} \sqrt{-^4 g} dx^1 dx^2 dx^3 + F^{01} \sqrt{-^4 g} dx^0 dx^2 dx^3 \tag{49}$$

$$+ F^{02} \sqrt{-^4 g} dx^0 dx^1 dx^3 + F^{03} \sqrt{-^4 g} dx^0 dx^1 dx^2.$$

In eqn(49), the quantity $F^{00} \sqrt{-^4 g} dx^1 dx^2 dx^3$ is a hypothetical 0 term, dx^0 is a virtual integration. Then eqn(49) reduces to

$$\int \nabla_\nu F^{0\nu} \sqrt{-^4 g}\, d^3 x\, dx^0. \tag{50}$$

Now we obtain

$$Q \equiv \varepsilon_0 c \int \nabla_\nu F^{0\nu} \sqrt{-^4 g}\, d^3 x. \tag{51}$$

For the exterior region, $\nabla_\nu F^{0\nu} = 0$. This guarantees what we want to know. Now we calculate eqn(56) with r=constant surface. Under our 5D metric in (*), we obtain

$$\varepsilon_0 c \oint F^{0i} \sqrt{-^4 g}\, dS_i \tag{52}$$

$$= \varepsilon_0 c \int_0^{2\pi} \int_0^{\pi} \left[\frac{Q}{4\pi\varepsilon_0 c} \frac{(r^2 + a^2)(-r^2 + a^2 \cos^2\theta)}{(r^2 + a^2 \cos^2\theta)^3} \right] (r^2 + a^2 \cos^2\theta) \sin\theta d\theta d\phi$$

$$= \varepsilon_0 c \int_0^{2\pi} \left[\frac{Q}{4\pi\varepsilon_0 c} \frac{-(r^2 + a^2)\cos\theta}{r^2 + a^2 \cos^2\theta} \right] |_0^\pi d\phi = Q.$$

Now we calculate the mass. The gravitational vector potential is $A_\mu^{GM} = \frac{c}{2}\, {}^4 g_{0\mu}$ and let $F_{\mu\nu}^{GM} = \partial_\mu A_\nu - \partial_\nu A_\mu$; then we obtain

$$-\frac{c}{4\pi G} \oint F_{GM}^{0i} \sqrt{-^4 g}\, dS_i \tag{53}$$

$$= -\frac{c}{4\pi G}\int_0^{2\pi}\int_0^{\pi}\frac{c}{2}\left[\frac{-2GM(r^2+a^2)(-r^2+a^2\cos\theta)}{c^2(r^2+a^2\cos^2\theta)^3}\right](r^2+a^2\cos^2\theta)\sin\theta d\theta d\phi$$

$$= -\frac{c}{4\pi G}\int_0^{2\pi}\frac{c}{2}\left[\frac{2GM(r^2+a^2)\cos\theta}{c^2(r^2+a^2\cos^2\theta)}\right]|_0^{\pi}d\phi = M.$$

Finally, we define the angular momentum. The mass and charge are derived from surface integral of E-field. However, as far as we know, there is no physical quantity to obtain the angular momentum through the surface integral. Therefore, let us just carry out the same procedure with the quantities, $\Phi_\mu = c\,{}^4g_{3\mu}$, $\Omega_{\mu\nu} = \partial_\mu\Phi_\nu - \partial_\nu\Phi_\mu$.

$$\frac{c^2}{16\pi G}\oint \Omega^{0i}\sqrt{-{}^4g}\,dS_i \tag{54}$$

$$= \frac{c^2}{16\pi G}\int_0^{2\pi}\int_0^{\pi} c\left[\frac{2GMa\sin^2\theta(-a^4\cos^2\theta+3r^4+2a^2r^2-a^2r^2\sin\theta)}{c^2(r^2+a^2\cos^2\theta)^3}\right](r^2+a^2\cos^2\theta)\sin\theta d\theta d\phi$$

$$= \frac{c^2}{16\pi G}\int_0^{2\pi} c\left[\frac{2GMa\cos\theta((r^2-a^2)\cos^2\theta-3r^2-a^2)}{c^2(r^2+a^2\cos^2\theta)}\right]|_0^{\pi}d\phi = Mac = J.$$

It is already known from the Komar integral that mass is related to stationary symmetry and angular momentum is related to axial symmetry. In this paper, as we start with cylindrical symmetry, we can expect a conserved quantity. From eqn(52), it can be seen that the conserved quantity corresponding to the cylindrical symmetry is charge.

8 Discussion

1. We did not care whether the fifth dimension is timelike or spacelike. We set it to timelike to follow the approach of ADM formalism easily, and if it is spacelike, we can do substitution. However apparently timelike is correct from eqn(32). In the case of spacelike, in other words, ${}^5g_{55} = N^2 + \beta_\lambda\beta^\lambda$, eqn(32) becomes as follows: ${}^5R = {}^4R + (\frac{F_{\lambda\rho}F^{\lambda\rho}}{N^2} - \frac{\mu_0}{N^2}J_\lambda\beta^\lambda)$.
Since these world have two timelike dimensions, ${}^5g_{00} < 0$, ${}^5g_{55} < 0$, someone cannot imagine that we live in these worlds. However even if ${}^5g_{55} < 0$, we do not have to consider this because of $dx^5 = 0$ for electrically neutral body [14].
2. Until now, we had not discussed N. With eqn(32), ${}^5R = {}^4R - (\frac{F_{\lambda\rho}F^{\lambda\rho}}{N^2} - \frac{\mu_0}{N^2}J_\lambda\beta^\lambda)$. It seems like $\frac{1}{N^2} = \frac{\kappa}{2\mu_0}$. where $\kappa = \frac{8\pi G}{c^4}$. By dividing eqn(32) by 2κ,

$$\frac{{}^5R}{2\kappa} = \frac{{}^4R}{2\kappa} - (\frac{1}{4\mu_0}F_{\lambda\rho}F^{\lambda\rho} - \frac{1}{2}J_\lambda A^\lambda). \tag{55}$$

In eqn(55), we used $\beta_\mu = 2A_\mu$. Now we get $\mathfrak{L}_{EH} = \frac{^4R}{2\kappa}, \mathfrak{L}_{EM} = -\frac{1}{4\mu_0}F_{\lambda\rho}F^{\lambda\rho} + \frac{1}{2}J_\lambda A^\lambda$. For $\mathfrak{L}_{EM}$, it satisfies

$$\frac{\partial \mathfrak{L}_{EM}}{\partial A^\mu} - \nabla^\nu \frac{\partial \mathfrak{L}_{EM}}{\partial \nabla^\nu A^\mu} = J_\mu + \frac{1}{\mu_0}\nabla^\nu F_{\nu\mu} = 0. \tag{56}$$

Moreover eqn(56) gives $\nabla^\nu F_{\mu\nu} = \mu_0 J_\mu$. It is noteworthy that these contents were naturally induced from the five dimensions. Note that $J_\lambda = \frac{2}{\mu_0}R_{\rho\lambda}A^\rho$.
3. For a weakly perturbed system, $^5g_{ab} = {}^5\eta_{ab} + {}^5h_{ab}$, the linearized equation is given by

$$^5\Box\left({}^5h_{ab} - \frac{1}{2}\,{}^5\eta_{ab}\,{}^5h\right) = -2\kappa\,{}^5T_{ab}, \qquad {}^5\Box \equiv {}^5\eta^{ab}\partial_a\partial_b. \tag{57}$$

Note that $^5\eta$ is expressed in Cartesian coordinate. By imposing the cylindrical and stationary conditions on $^5h_{ab}$, we obtain

$$\frac{^5h_{ab}(\mathbf{X})}{2} = \frac{1}{4\pi}\int \frac{\kappa({}^5T_{ab}(\mathbf{Y}) - \frac{1}{3}\,{}^5\eta_{ab}\,{}^5T(\mathbf{Y}))}{|\mathbf{X}-\mathbf{Y}|}d^3Y, \tag{58}$$

where $\mathbf{X}, \mathbf{Y}$ are spatial components. For details, see [13]. If $^5h_{5\nu}$ is proportional to electromagnetic vector potential then $^5R_{5\nu}$ should be related to the charge current, $\mu_0 J_\nu$. In section 3 and 4, we identified $^5h_{5\nu} = 2A_\nu$ then $^5R_{5\nu}$ should be related to the charge current. In fact, we got $^5R_{5\nu} = \nabla^\rho F_{\nu\rho}$. Note that $^5R_{ab} = \kappa({}^5T_{ab} - \frac{1}{3}\,{}^5\eta_{ab}\,{}^5T)$ for 5D.
4. Under the stationary condition, $^4R_{00} = -\Box\phi_t + \frac{1}{c^2}F^{GM}_{\lambda\rho}F^{\lambda\rho}_{GM}$, $^4R_{0i} = \frac{1}{c}\nabla^\rho F^{GM}_{i\rho}$, where $\phi_t \equiv \frac{1}{2}{}^4g_{00}$. From the equation

$$\frac{^4h_{\mu\nu}(\mathbf{X})}{2} = \frac{1}{4\pi}\int \frac{\kappa({}^4T_{\mu\nu}(\mathbf{Y}) - \frac{1}{2}\,{}^4\eta_{\mu\nu}\,{}^4T(\mathbf{Y}))}{|\mathbf{X}-\mathbf{Y}|}d^3Y, \tag{59}$$

we obtain

$$A_i^{GM} = \frac{1}{4\pi}\int \frac{\nabla^\rho F^{GM}_{i\rho}}{|\mathbf{X}-\mathbf{Y}|}d^3Y, \tag{60}$$

where $\nabla^\rho F^{GM}_{i\rho} = -\frac{4\pi G}{c^2}J_i$ and J_i is matter current, ρU_i. In this step, as we mentioned in section 3, if $^4g_{0\mu}$ is identified as gravitational vector potential, then $^4R_{0i}$ is identified as matter current. In this way, we can develop a theory of gravitomagnetism [10, 15].
Under the azimuthal symmetry condition, $^4R_{33} = -\Box\phi_\phi + \frac{1}{4c^2}\Omega_{\lambda\rho}\Omega^{\lambda\rho}$, $^4R_{3\delta} = \frac{1}{2c}\nabla^\rho\Omega_{\delta\rho}$, where $\phi_\phi \equiv \frac{1}{2}{}^4g_{33}$, δ span 0,1,2. Then we obtain

$$\Phi_\delta = \frac{1}{4\pi}\int \frac{\nabla^\rho\Omega_{\delta\rho}}{|\mathbf{X}-\mathbf{Y}|}d^3Y. \tag{61}$$

5. In eqn(28), since N is constant and the integrand is independent of x^5, we ignored the overall quantity Ndx^5. Note that this is one of the results of the conservation of charge along the geodesic curve.

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