In search of Schrödinger's electron —and Einstein's atom too!

Jean Louis Van Belle, Drs, MA*Ec*, BA*Ec*, B*Phil*

Email: jeanlouisvanbelle@outlook.com

18 October 2018

Abstract: This article continues to explore a possible physical interpretation of the wavefunction that has been elaborated in previous papers (see http://vixra.org/author/jean_louis_van_belle). It zooms in on the physical model it implies for an electron in free space, but then also discusses the wavefunction for particles (including non-charged particles) in general. While it basically concludes that the mainstream interpretation of quantum physics (the Copenhagen interpretation) is and remains the most parsimonious explanation, it also argues that one or two extra assumptions – the wavefunction as a two-dimensional self-sustaining electromagnetic or gravitational oscillation in space – would make more frivolous explanations (many worlds, pilot wave, etcetera) redundant.

Contents

In search of Schrödinger's electron —and Einstein's atom too!

Introduction

The title obviously refers, jokingly, to John Gribbins' *In Search of Schrödinger's Cat: Quantum Physics and Reality* (1984) – the book that got me started a very long time ago. I have come some way since then. More in particular, I recently re-took the MIT 8.04 course[1] (Sept-Oct 2018) to try to check at what point *exactly* our ideas about a possible physical interpretation of the wavefunction become incompatible with the mainstream interpretation of the math – if they do at all. Indeed, one should highlight that most amateurs (which includes me) are *not* interested in non-mainstream interpretations of quantum mechanics. We only try to push the boundaries of the Copenhagen interpretation. We do *not* like to entertain the idea of many worlds or pilot waves, for example, because… Well… Occam's Razor: a simpler explanation is better than a more complicated one. 😉

We are humble amateurs: most of us do not venture beyond quantum electrodynamics (the behavior of electrons and photons) because we know QCD is a *very* different world. In fact, we like to stay in the QED world because we know the striking mathematical resemblance between QED and classical electromagnetism is what it is because both theories are based on the same mathematical symmetry group U(1). Hence, the transformations under which the objects remain invariant or change feel very similar because – from a purely mathematical point of view – they *are* the same.

Hence, we duly note and explore the striking resemblance between a circularly polarized electromagnetic wave and the elementary wavefunction and try to think of how a wavefunction could possibly carry energy and (linear or angular) momentum – or, at the very least, how it can possibly contain all of the information about these quantities.[2] However, when we enroll in formal courses we are told that electromagnetic waves are real – but wavefunctions are not. We ask a lot of questions and get mixed answers – some convincing, some less so. This paper explores some of these answers.

The basics of geometric interpretations

As mentioned above, the structural similarities between classical electromagnetics and QED inspires many amateurs to try to think of the real and imaginary part of the elementary wavefunction $a \cdot e^{i\theta}$ as *real* perpendicular field vectors with the same amplitude – and driven by the same function (but for a phase difference of 90 degrees):

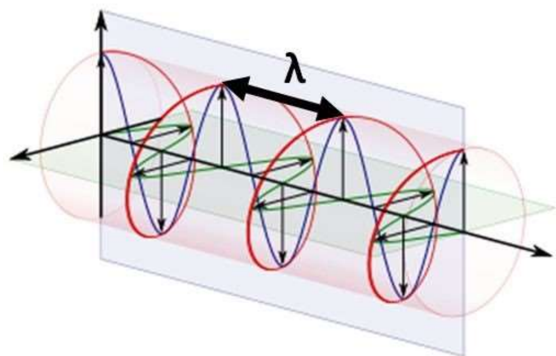
$$a \cdot e^{i\theta} = a \cdot (\cos\theta + i \cdot \sin\theta) = a \cdot \sin(\theta + \pi/2) + i \cdot a \cdot \sin\theta$$

[1] Both are available online free of charge. MIT offers its 8.04.1*x* interactive course through the edX MOOC platform.

[2] The expected values of the various operators give us all the information we can understand *classically*. If the information is in the wavefunction, why not the physics?

The visualization below – which shows a propagating circularly polarized electric field – is common but probably off the mark: it is not compatible with the idea of our particle having some magnetic moment or *spin*. We will come back to this. The point here is that easy visualizations like this strongly encourage us to think of a geometric representation of the wavefunction—if only because, conversely, one may also adopt the convention that the imaginary unit should be interpreted as a unit vector pointing in a direction that is perpendicular to the direction of propagation of the wave and one may then write the magnetic field vector as $\mathbf{B} = -i\cdot\mathbf{E}/c$.

Figure 1: A complex wave?[3]



Note the minus sign in the $\mathbf{B} = -i\cdot\mathbf{E}/c$.[4] It is there because of consistency: we must combine a classical *physical* right-hand rule for **E** and **B** here as well as the *mathematical* convention that multiplication with the imaginary unit amounts to a *counter*clockwise rotation by 90 degrees. The point is: we can re-write Maxwell's equations using complex numbers. Hence, the *geometry* may not be the same, but we can think of something similar, can't we?

We can. However, we should make a few remarks first. First, we think of the amplitude *a* as some real-valued number here. That may be problematic because, in quantum mechanics, we do not exclude linear operations using complex-valued coefficients. For example, when using the framework of *state vectors*, we may write something like $|\mathrm{X}\rangle = \alpha|\mathrm{A}\rangle + \beta|\mathrm{B}\rangle$, and α and β would be complex numbers.

It is a common objection to what we are trying to model here, but a trivial one – not fundamental at all. The mathematical framework of state vectors[5] can be related to the representation of what we think of as a particle – say, an electron, to be specific – but is different *in practice*. For example, the electron orbitals of the hydrogen atom – the correct description of which is one of the main feats of quantum mechanics – are described by wavefunctions with real-valued coefficients.

---

[3] Credit: https://commons.wikimedia.org/wiki/User:Dave3457. The author only added the wavelength, which can be interpreted as the *de Broglie* wavelength for a particle. For more details, see http://vixra.org/pdf/1709.0390v5.pdf.

[4] Boldface letters represent geometric vectors – the electric and magnetic field vectors, in this case.

[5] Your quantum physics course will also highlight that state vectors have nothing to do with *geometric* vectors (i.e. those vectors in three dimensions – e.g. ***x*** or ***L*** – that we are used to). They are rather abstract mathematical objects that represent a physical state. From a mathematical point of view, they are vectors in a Hilbert space. Depending on what you know about Hilbert spaces, this may or may not help you to understand what they are or, equally important, what they are not.

[…] Let me be *very* precise here, because it is an important point. ☺ Yes. We all know the wavefunction for an electron orbital will include a factor like $-\frac{1}{\sqrt{2}} \cdot \sin\theta \cdot e^{i\phi}$, so there is a complex number there.[6] However, note how the complex factor appears: it is just a phase shift. The *envelope* for our function, so to speak, is some real number. It always is. Let me give another example that, *in practice*, we do often use simple real-valued coefficients in front of our complex-valued wavefunctions. We know we should be thinking of wave packets which, using the Fourier transform (and freezing time), we can write as:

$$\psi(x,0) = \int_{-\infty}^{+\infty} \Phi(k) e^{ikx} dk$$

The $\Phi(k)$ function gives us the *weight* factors for each of the waves that make up the packet.[7] Here also, we think of $\Phi(k)$ as a real-valued function, centered around some value $k_0 = \frac{p_0}{\hbar}$ and width Δk. In short, the objection that the coefficients in front of wavefunctions are generally complex is *not* valid and should, therefore, not deter us from trying to find some physical or geometric interpretation.

OK. I think I dealt with this – for the time being, that is.

Second, we should note that, when re-writing Maxwell's equations using complex numbers, the plus or minus sign in the exponent of $a \cdot e^{i\theta}$ or $a \cdot e^{-i\theta}$ would *not* be a matter of convention: one has left- and the other has right-handed polarization. In quantum mechanics, that is not the case. For example, in the mentioned MIT course[8], it is shown that only $\psi = \exp(i\theta) = \exp[i(kx-\omega t)]$ *or* $\psi = \exp(-i\theta) = \exp[i(\omega t-kt)]$ would be acceptable waveforms for a particle that is propagating in the *x*-direction – as opposed to, say, some real-valued sinusoid. We would then think some proof should follow of why one would be better than the other, or some discussion on why they might be different. But, no, the professor happily concludes that "the choice is a matter of convention and, happily, most physicists use the same convention."

I find this quite surprising. We *know*, from *experience*, that theoretical or mathematical possibilities often turn out to represent real things. Think of the experimental verification of the existence of the positron (or of anti-matter in general) after Dirac had predicted its existence based on the mathematical possibility only. So why would that *not* be the case here? In fact, it raises an intriguing question on quantum-mechanical conventions: if there is a physical interpretation of the wavefunction, then we should, perhaps, not have to choose between the two mathematical possibilities but use the sign to denote the direction of quantum-mechanical spin.

The point is to be investigated, because wavefunctions of spin-1/2 particles (which is what we are thinking of here) have a weird 720° symmetry.[9] This weird symmetry is not there for spin-1 particles. Hence, intuition tells us that it should disappear when we would use the two mathematical possibilities for describing the wavefunction of a particle to distinguish between two particles that are identical but have opposite spin. If our intuition is correct (but I have no proof for that yet), the most important objection to a physical interpretation of the wavefunction would no longer be valid.

[6] The formula gives us the angular dependence of the amplitude for the orbital angular momentum number *l* = 1.
[7] You will usually see a $\frac{1}{\sqrt{2\pi}}$ factor in front of the integral, and it should be there, but we left it out for clarity.
[8] Course 8.04.1*x*, Lecture Notes, Chapter 4, Section 3.
[9] See, for example, Feynman (III-6).

The issue triggers another question: because most mainstream physicists – there are, fortunately, some wonderful exceptions[10] – refuse to entertain the notion of a physical interpretation of a physical interpretation of the wavefunction and, therefore, also refuse to enter a discussion of the physical interpretation of its complex conjugate. The common interpretation of the complex conjugate of the (elementary) $\psi = \exp(i\theta) = \exp[i(kx-\omega t)]$ function – so that is $\psi^* = \exp(-i\theta) = \exp[i(\omega t-kt)]$ – is that it is just another *mathematical* possibility but… Well… Physicists chose to not to think about the other possibility and adopted some *convention.*

The idea of associating the complex conjugate of a wavefunction with a particle that's identical except for its (opposite) spin might be outlandish[11], but a much simpler idea might be palatable: the complex conjugate of a wavefunction obviously reverses the trajectory of the particle in space and in time: $\boldsymbol{x}$ becomes $-\boldsymbol{x}$ and t becomes $-t$. So, what happens? What are we doing? We may relate this discussion to the *Hermiticity* of (many) operators. If A is an operator[12], then it could operate on some state $|\psi\rangle$. We write this operation as:

$$A|\psi\rangle$$

Now, we can then think of some (probability) amplitude that this operation produces some other state $|\varphi\rangle$, which we would write as:

$$\langle\varphi|A|\psi\rangle$$

We can now take the complex conjugate:

$$\langle\varphi|A|\psi\rangle^* = \langle\psi|A^\dagger|\varphi\rangle$$

$A^\dagger$ is, of course, the conjugate transpose of A: $A^\dagger_{ij}=(A_{ji})^*$, and we will call the operator (and the matrix) Hermitian if the conjugate transpose of this operator (or the matrix) gives us the same operator matrix, so that is if $A^\dagger = A$. Many operators are Hermitian. Why? Well… What is the meaning of $\langle\varphi|A|\psi\rangle^* = \langle\psi|A^\dagger|\varphi\rangle = \langle\psi|A|\varphi\rangle$? Well… In the $\langle\varphi|A|\psi\rangle$ we go from some state $|\psi\rangle$ to some other state $\langle\varphi|$. Conversely, the $\langle\psi|A|\varphi\rangle$ expression tells us we were in state $|\varphi\rangle$ but now we are in the state $\langle\psi|$.

So, is there some meaning to the complex conjugate of an amplitude like $\langle\varphi|A|\psi\rangle$? Read up on time reversal and CPT symmetry. 😊 Quantum mechanics is full of easy *physical* interpretations, so it is surprising we get so few of them in a formal course. It may be difficult to prove that this or that interpretation is correct (we probably can't because they are *interpretations* only) but it is sad we don't explore them – if only for didactic purposes. So, let us try harder. 😊

---

[10] *How to understand quantum mechanics* (2017) from John P. Ralston, Professor of Physics and Astronomy at the University of Texas, is one of such welcome exceptions. I love the self-criticism: "Quantum mechanics is the only subject in physics where teachers traditionally present haywire axioms they don't really believe, and regularly violate in research." (p. 1-10)

[11] It sure wouldn't be if we would be able to prove it gets rid of the weird 720-degree symmetry of the wavefunction for spin-1/2 particles, but we are not there (yet).

[12] I should use the *hat* because the symbol without the hat is reserved for the *matrix* that does the operation and, therefore, A already assumes a representation, i.e. some chosen set of base states. However, let's skip the niceties here.

Schrödinger's electron

Besides the obvious similarity in the geometries of the electromagnetic wave and the quantum-mechanical wavefunction, there is also an easy analog between energy densities and quantum-mechanical probability densities:

$$u = \frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E} + \frac{\epsilon_0 \cdot c^2}{2}\mathbf{B}\cdot\mathbf{B} = \frac{\epsilon_0}{2}\mathrm{E}^{\mathbf{2}} + \frac{\epsilon_0 \cdot c^2}{2}\mathrm{B}^{\mathbf{2}} = \frac{\epsilon_0}{2}\mathrm{E}^{\mathbf{2}} + \frac{\epsilon_0 \cdot c^2}{2}\frac{\mathrm{E}^2}{c^2} = \epsilon_0 \cdot \mathrm{E}^2$$

$$P = |\psi|^2 \; = \; |ae^{-i\theta}|^2 = |a|^2|\cos\theta - i\sin\theta|^2 = a^2(\cos^2\theta + \sin^2\theta) = a^2$$

Of course, the second equation assumes the wavefunction has been normalized to ensure all probabilities add up to 1: $\int P \, d^3\boldsymbol{x} = 1$. If not, we need to replace the equality sign by a ∝ sign:

$$P \propto |\psi|^2 \; = \; a^2$$

The E and *a* are both amplitudes, and the formulas differ only by ε_0, which is just a scaling coefficient. The suggested interpretation is very intuitive: we would expect to have more of a chance to find the particle where the energy or mass ($E = mc^2$) densities are larger.

The next question then becomes: what explains the energy and the mass? For that, we would need to associate a physical dimension with the real and imaginary part of the wavefunction. The obvious candidate is the same as for the electromagnetic wave: force per unit charge (N/C). As for the mass, if we effectively think of the electron as a pointlike *charge* only, with no internal structure, then its mass can be electromagnetic mass only.

This is not new, of course. The description above is that of Schrödinger's electron, who described it in terms of a local circulatory motion: the *Zitterbewegung*. So what *exactly* is being proposed here? The theoretical model should give us some easy formulas that are consistent with experimentally determined values of measurables, and it does. So what are we doing here? What's Schrödinger's electron?[13]

We may think of Schrödinger's electron as a pointlike *charge* with no internal structure which can, therefore, only have some electromagnetic mass[14]. If we then think of the Zitterbewegung as an oscillation in two orthogonal directions (let us refer to them as the *x* and *y* direction[15]), we can add the energies of both oscillations and, perhaps, write something like this:

$$E_{total} = E_{kinetic} + E_{potential} = T + U = E_y + E_z = (T_y + U_y) + (T_z + U_z)$$

[13] What follows is a rather loose and creative interpretation of what Schrödinger may or may not have written. For a more accurate overview of the idea of the *Zitterbewegung*, see: David Hestenes, the *Zitterbewegung* in Quantum Mechanics: A Research Program, 2008 (, last accessed on 17 Sept 2018).

[14] Terminology might be a bit confusing here. Feynman (*Lectures*, II-28-3) opposes electromagnetic mass to what he refers to as mechanical mass. The term rest mass is ambiguous because it will still designate the rest mass of the electron even if we look at it as a pointlike charge that is always moving around. In fact, that is the whole point of the model: it wants to explain the rest mass of the electron in terms of its electromagnetic mass and its motion (the *Zitterbewegung*) only.

[15] The *x*-direction would then be the direction of propagation of the wave. This follows the usual convention in quantum mechanics, according to which we will measure something (e.g. angular momentum) along the *z*-direction, which is perpendicular to the direction of propagation, i.e. the *x*-direction. The *y*-direction is then determined by the right-hand rule. We may say this establishes a reference frame that combines the object and the subject (the measurement apparatus).

$$= (1/2)\cdot m\cdot\omega^2\cdot a^2\cdot[\sin^2(\omega\cdot t + \Delta) + \cos^2(\omega\cdot t + \Delta)] + = (1/2)\cdot m\cdot\omega^2\cdot a^2\cdot[\cos^2(\omega\cdot t + \Delta) + \sin^2(\omega\cdot t + \Delta)]$$

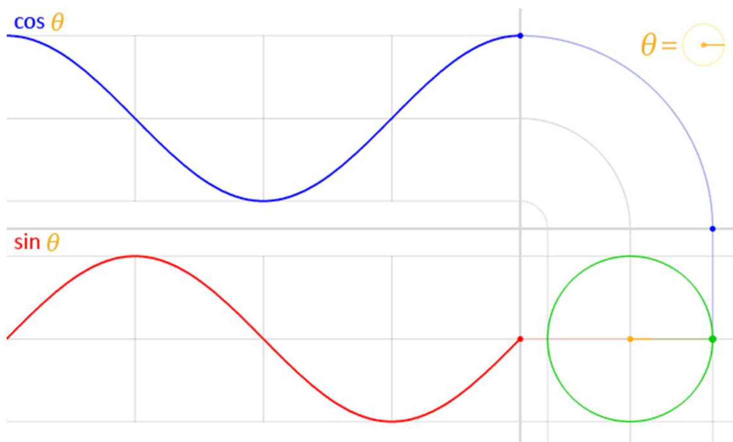
$$= m\cdot a^2\cdot\omega^2/2 + m\cdot a^2\cdot\omega^2/2 = m\cdot a^2\cdot\omega^2$$

This may look uninteresting but when combining this equation for the energy with Einstein's mass-energy equivalence relation ($E = mc^2$) and *de Broglie*'s relation for the energy of a particle ($E = \hbar\omega$), we can calculate a as the Compton radius of an electron:

$$E = m\cdot c^2 = m\cdot a^2\cdot\omega^2 = m\cdot a^2\cdot(E/\hbar)^2 \Leftrightarrow a = \hbar/mc$$

Hence, we may now think of the Compton wavelength as an effective scattering radius.[16] To assist the reader, we may visualize Euler's formula[17]: the combination of the sinuisoidal and cosinusoidal motion makes the pointlike charge (the green dot) go around in a circle.

Figure 2: Euler's formula



Of course, the Compton radius is, obviously, not the only measured quantity that is to be explained in a simple physical electron model such as this one. What can we say about the angular momentum of an electron? The easy classical and – importantly – non-relativistic analysis (see, for example, Feynman, *Lectures*, II-34-2) uses the $L = m\cdot v\cdot r$ and $\mu = I\cdot A$ formulas for the (orbital) angular momentum and the magnetic moment respectively[18].

However, these formulas envisage a pointlike charge in orbit around some nucleus at a non-relativistic speed. In fact, we should probably be precise here: the non-relativistic speed is given by Sommerfeld's fine-structure constant, which he interpreted as the relative velocity of an electron in the first circular

---

[16] The Compton radius is usually referred to as the Compton wavelength, and it is relevant in the context of photon scattering. However, in the context of this model, an interpretation in terms of an effective radius is, obviously, much more natural. In fact, that is the whole point of the interpretation.

[17] Papers don't do animations, unfortunately. For the animation (for which credit goes to https://commons.wikimedia.org/wiki/User:LucasVB), see https://en.wikipedia.org/wiki/Sine#/media/File:Circle_cos_sin.gif.

[18] The I in $\mu = I\cdot A$ formula is not the rotational mass but the (effective) current, and A is the surface area of the loop of current. As for denoting the angular momentum by L instead of J, we are discussing *orbital* angular momentum here. Since Feynman published his lectures, the J seems to have been reserved to denote the more general concept of angular momentum, which combines *spin* as well as *orbital* angular momentum. One should think of the $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ formula here or, even more generally, the theory of angular momentum coupling.

orbit of the Bohr model of the atom: $v/c = \alpha \approx 1/137$. We are not presenting the Bohr model of an atom here: the model is one based on Schrödinger's concept of a *Zitterbewegung*, and it aims to explain an electron as such – one in free space, that is. So what can we reasonably say about it? If the model is any good, then it tells us the orbital or tangential velocity (v) here will be equal to the speed of light: $v = a \cdot \omega = (\hbar/mc) \cdot (E/\hbar) = (\hbar/mc) \cdot (mc^2/\hbar) = c$.

Is that possible at all? It should be because we assume our pointlike charge has no mechanical mass: its only mass is electromagnetic, and that mass is zero if it doesn't move. But how should we think of the angular momentum here? What is the form factor for the angular momentum? Are we thinking of some disk? Let us first calculate the magnetic moment. If our pointlike charge is, effectively, going around in a loop, the effective current will be equal to the charge (q_e) divided by the period (T) of the orbital revolution: : $I = q_e/T$. Now, the period of the orbit is the time that is needed for the electron to complete one loop, so T is equal to the circumference of the loop ($2\pi \cdot a$) divided by the tangential velocity (v). Using our results, we should substituting v for c and a for the Compton radius $a = \hbar/(m \cdot c)$, and the formula for the area is what it is: $A = \pi \cdot a^2$. Hence, we get:

$$\mu = I \cdot A = (q_e /T) \cdot (\pi \cdot a^2) = [(q_e \cdot c)/(2\pi \cdot a)] \cdot (\pi \cdot a^2) = [(q_e \cdot c)/2] \cdot a = [(q_e \cdot c)/2] \cdot [\hbar/(m \cdot c)] = (q_e/2m) \cdot \hbar$$

This formula for the magnetic moment looks surprisingly good, but we need to combine this result with some formula for the angular momentum so as to get the hoped-for value for the angular momentum, which is $\pm \hbar/2$. The $J = m \cdot v \cdot r$ is obviously useless because (1) we do not have a pointlike *mass* here and (2) the orbital speed is as relativistic as it can be ($v = c$). Let us first tackle the first objection.

If the *rest* mass of our electron in this *Zitterbewegung* model is all electromagnetic, then the form factor is more likely to represent some rotating disk. However, because of the precession of a magnet in a magnetic field, the plane of this rotary disk is, most likely, not stationary. Hence, let us use the non-relativistic formula for the angular momentum written in terms of angular mass (which I will denote by m_L instead of the I that you are used to[19]) and angular frequency (ω):

$$L = m_L \cdot \omega$$

So what is the form factor that separates (or distinguishes, I should say) the *effective* or *mechanical* mass of our electron (which is just its *rest* mass m) and its *angular* mass in this *Zitterbewegung* model of an electron in free space? Well… This is what experiment should tell us. Up to this point, we just had a crazy theory of some pointlike charge oscillating in two dimensions – in a plane that may or, more likely, may not be stationary because of precession[20] – but now we need to answer an experiment: something – an electron, let's say[21] – with, presumably, some magnetic moment and some angular momentum, enters the Stern-Gerlach apparatus, and comes out of it in one of two very different *physical* states: *up* or *down*.

[19] I do so for two reasons. First, it avoids confusion with the symbol I used for the effective current (I). Second, it allows me to distinguish between the rest mass of the electron (or its *effective* mass as used in Schrödinger's equation) and its angular mass, which depends on the form factor.

[20] It is interesting that precession applies to both mechanical masses (in a gravitational field) as well as to electromagnetic masses (as modeled here) in an (electro)magnetic field.

[21] If we can explain the electron, we can – perhaps – explain a lot more. 😉

Let me be precise what I mean by that: the electron hits the detector here, *or* there – not at some place inbetween.[22] So what happens here? Let us be very precise. We *interpret* the measurement as the angular momentum of an electron being equal to $\pm\hbar/2$ based on a preconceived notion of the angular momentum, and the magnetic moment, being equal to this or that (formula). But… Well… All we can reasonably conclude is that the ratio of (1) whatever concept of the angular momentum and (2) whatever concept we have of the magnetic moment might be equal to:

$$\frac{L}{\mu} = \frac{\frac{\hbar}{2}}{\frac{q_e}{2m}\hbar} = \frac{q_e}{m}$$

But what can we *measure* here? The only *observable* variable is the magnetic moment, which is revealed by measuring the (extra) magnetic energy our electron would get in a magnetic field: $U = -\boldsymbol{\mu}\cdot\mathbf{B}$. Hence, is the angular momentum really equal to $\hbar/2$? What is the form factor? Should we write $m_L = ma^2$ (for a hoop) or $m_L = ma^2/2$ (for a disk)? Our model suggests the effective mass of our electron (m) is spread over a disk, so let us try the second formula:

$$L = m_L\cdot\omega = m\cdot a^2\cdot(c/a)/2 = m\cdot a\cdot c/2 = m\cdot(\hbar/mc)\cdot c/2 = \hbar/2$$

This is what we think we measure when doing the Stern-Gerlach experiment, so our model is right on. We don't even need to assume its direction is fuzzy because of precession, or whatever its quantum-mechanical equivalent might be. In fact, we seem to have solved a deep mystery: we explained the mysterious *g*-factor for the pure spin moment of an electron: our model tells us it must be equal to 2, which is what it is.[23]

Propagation mechanisms

All of the above is, obviously, too nice to be true. The professional physicist may reluctantly want to envisage a model with some electromagnetic mass and some electromagnetic oscillation in plane – especially in light of the remarks on a possible interpretation of probality densities as energy densities – but he or she should ask: what is the propagation mechanism here? Electromagnetic waves propagate. If the real and imaginary part of the wavefunction are to be interpreted as electromagnetic field vectors, how do they propagate?

Here one should further explore the idea of Schrödinger's equation as a diffusion equation, which Feynman (III-16-1) summarized as follows: "We can think of Schrödinger's equation as describing the diffusion of the probability amplitude from one point to the next. […] But the imaginary coefficient in front of the derivative makes the behavior completely different from the ordinary diffusion such as you

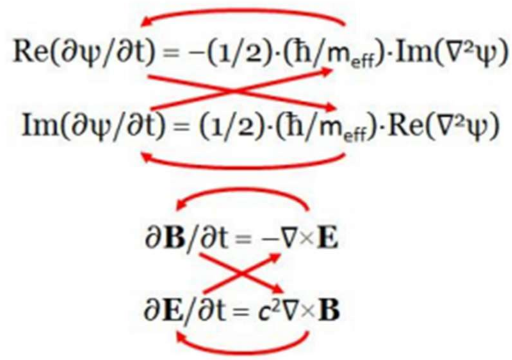
[22] The spread around the 'up' or 'down' position is easily explained by the unavoidable randomness in the experiment itself. See my comments on the MIT's standard lab experiment: https://readingfeynman.org/2017/03/03/comments-on-the-mits-stern-gerlach-paper/.

[23] See Feynman's magisterial *exposé* the impreciseness of directions in quantum mechanics (Feynman, II-34-7). Also see Feynman (II-34-2) for a good conceptual discussion of the *g*-factor in classical and quantum mechanics. The reader will know quantum physicists calculate a slightly different *g*-factor (about 2.0023193) but that's because they use a formula based on the fine-structure constant. In other words, their model is "deep down in relativistic quantum mechanics", as Feynman would put it.

would have for a gas spreading out along a thin tube. Ordinary diffusion gives rise to real exponential solutions, whereas the solutions of Schrödinger's equation are complex waves."

But let us look at how different that behavior really is. Schrödinger's equation is one equation but, because the wavefunction is complex, one gets two equations for the price of one, as shown below.[24]

Figure 3: Propagation mechanisms



The red arrows visualize the propagation mechanism. For an electromagnetic wave, we have Maxwell's equations in free space (no charges, no potential): a changing electric field (∂**E**/∂t) will cause some (infinitesimal) circulation of **E**, so that is some curl (∇×**E**), and that causes a change in the magnetic field (∂**B**/∂t), and so we have some curl of a magnetic field (∇×**B**), and so that is equivalent to some ∂**E**/∂t again. Let us analyze the mechanism for Schrödinger's equations now. A curl operator (∇×) is, obviously, very different from a Laplacian (∇^2), but one needs to combine the Laplacian operator with the real and imaginary 'operators' Re(*z*) and Im(*z*). How can we think of this? Laplacians pop up in energy diffusion equations, like the heat diffusion equation, which is easy to represent geometrically. However, here we have cyclical functions. Let us do the calculations for the elementary wavefunction $ae^{i\theta} = ae^{i(kx-\omega t)}$:

$$Re\left(\frac{\partial\psi}{\partial t}\right) = Re\left(-ia\omega e^{i\theta}\right) = Re\left(-ia\omega(cos\theta + isin\theta)\right) = Re(-ia\omega cos\theta - i^2 a\omega sin\theta) = a\omega sin\theta$$

$$Im(\nabla^2\psi) = Im(\frac{\partial\left(\frac{\partial\left(ae^{i\theta}\right)}{\partial x}\right)}{\partial x}) = Im(\frac{\partial\left(iake^{i\theta}\right)}{\partial x}) = Im(-ak^2e^{i\theta}) = Im(-ak^2(cos\theta + isin\theta)) = -ak^2sin\theta$$

Does this make sense? It is interesting to see that – using de Broglie's relations – we can get the usual energy concept that is used in Schrödinger's equation:

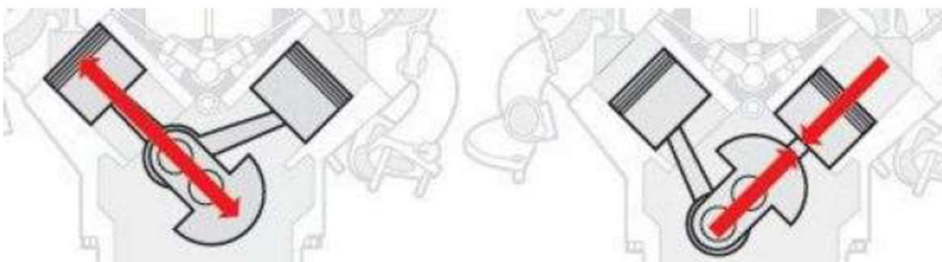
$$Re\left(\frac{\partial\psi}{\partial t}\right) = -\frac{\hbar}{2m}Im(\nabla^2\psi) \Leftrightarrow a\omega sin\theta = \frac{\hbar}{2m}ak^2sin\theta \Leftrightarrow \frac{\hbar}{2m}k^2 = \omega \Leftrightarrow \frac{\hbar}{2m}\frac{p^2}{\hbar^2} = \frac{E}{\hbar} \Leftrightarrow E = \frac{p^2}{2m}$$

---

[24] Two complex numbers are *a*+i*b* and *c*+i*d* are equal if their real and imaginary parts are equal. One also needs to use the i(*c*+i*d*)=i*c*+i²*d* = −*d*+i*c* equality here. The Schrödinger equation that is used is the equation in free space (no potential): $\frac{\partial\psi}{\partial t} = i\frac{\hbar}{2m}\nabla^2\psi$. It is non-relativistic but will do to illustrate what we could be thinking of.

But that is not the point here: our model should be valid for p = 0. In fact, Schrödinger's *Zitterbewegung* is a model of a stationary electron: all its energy is in its rest mass, which we interpret as electromagnetic mass. The mechanism is, in fact, like that of an electromagnetic wave, but let us use a metaphor to visualize it. Think of a V-2 engine with the pistons at a 90-degree angle, as illustrated below. The 90° angle makes it possible to perfectly balance the counterweight and the pistons, thereby ensuring smooth travel always. With permanently closed valves, the air inside the cylinder compresses and decompresses as the pistons move up and down. It provides, therefore, a restoring force. As such, it will store potential energy, just like a spring. In fact, the motion of the pistons will also reflect that of a mass on a spring: it is described by a sinusoidal function, with the zero point at the center of each cylinder. We can, therefore, think of the moving pistons as harmonic oscillators, just like mechanical springs.[25]

Figure 4: Propagation and energy conservation: the V2 metaphor



Instead of two cylinders with pistons, one may also think of connecting two springs with a crankshaft. The analogy can also be extended to include two *pairs* of springs or pistons, in which case the springs or pistons in each pair would help drive each other. The point is: we have a great *metaphor* here. Somehow, in this beautiful interplay between linear and circular motion, energy is borrowed from one place and then returns to the other, cycle after cycle. While transferring kinetic energy from one piston to the other, the crankshaft will rotate with a constant angular velocity: linear motion becomes circular motion, and vice versa. Let us look at our *set* of Schrödinger equations again:

$$Re\left(\frac{\partial\psi}{\partial t}\right) = a\omega sin\theta = \frac{\hbar}{2m}ak^2 sin\theta = -\frac{\hbar}{2m}Im(\nabla^2\psi)$$

$$Im\left(\frac{\partial\psi}{\partial t}\right) = a\omega cos\theta = \frac{\hbar}{2m}ak^2 cos\theta = \frac{\hbar}{2m}Re(\nabla^2\psi)$$

We have a propagation mechanism here: the sine and cosine functions represent an oscillation in two dimensions. The sine and cosine are, effectively, the same function but with a 90-degree phase difference. The *kinetic* energy of the first oscillator is equal to $a^2\omega^2 sin^2\theta$ and, hence, the energy *transfer* from this oscillator to the other is given by:

$$2a^2\omega^2\cdot sin\theta\cdot d(sin\theta)/d\theta = 2\, a^2\omega^2\cdot sin\theta\cdot cos\theta$$

This is absorbed by the other oscillator, whose motion is given by the cosθ function, which is equal to sin(θ+π /2). Hence, its kinetic energy is equal to $sin^2(\theta+\pi/2)$, and how it *changes* – as a function of θ – will be equal to:

[25] Instead of two cylinders with pistons, one may also think of connecting two springs with a crankshaft. The analogy can also be extended to include two *pairs* of springs or pistons. The two springs or pistons in each pair

$$2a^2\omega^2\cdot\sin(\theta+\pi/2)\cdot\cos(\theta+\pi/2) = -2a^2\omega^2\cdot\cos\theta\cdot\sin\theta$$

All that remains to be done is to simplify the $2a^2\omega^2$ by substituting a for the Compton radius. We get the following:

$$a^2\omega^2 = \frac{\hbar^2}{m^2c^2}\frac{E^2}{\hbar^2} = \frac{m^2c^4}{m^2c^2} = c^2$$

We get the bold assumption we started out with: E = $m\cdot c^2 = m\cdot a^2\cdot\omega^2$. The total energy that is stored in the system is the sum of the kinetic and potential energies of the two oscillators:

$$E = T + U = ma^2\omega^2 = m\cdot c^2$$

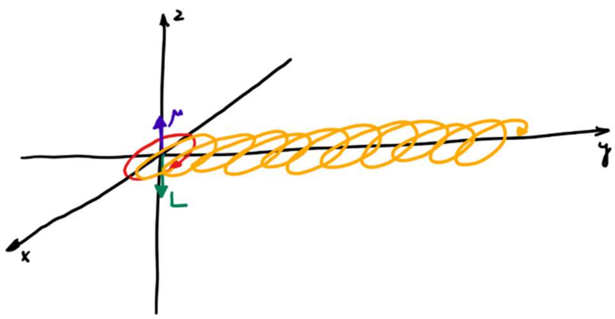
Where is the Uncertainty?

Where is the uncertainty in this *Zitterbewegung* model of a pointlike charged particle? We think of the charge as being *some*where, but where exactly? The analogous question here is: where *is* the propeller of a plane? The answer to this question would be expressed in terms of mass densities and probabilities. The uncertainty in the *Zitterbewegung* model is of the same nature.

Figure 5: Where is the propeller?

Here we need to discuss what may well be the worst *cliché* in quantum mechanics: the elementary wavefunction has no meaning because it does not allow one to localize the particle. That is utter nonsense: one only has to restrict its domain between, say, x_1 and x_2, or – because space is three-dimensional – we can choose some 3D cube or sphere. This raises an interesting question: this Zitterbewegung model is flat: two dimensions only. How can we understand this? The answer is: we probably cannot, but we can think of some ideas. One idea is that the plane of oscillation itself moves about, and that it is only when an external magnetic field is being applied (in a Stern-Gerlach apparatus, for example) that the plane of the oscillation orients itself, as shown below, perhaps.

Figure 6: Is this how an electron travels through a Stern-Gerlach apparatus?



Here we need to re-examine our interpretation of the probabilities we get when taking the absolute square of a wavefunction:

$$P = |\psi|^2 \ = \ |ae^{-i\theta}|^2 = |a|^2|\cos\theta - i\sin\theta|^2 = a^2(\cos^2\theta + \sin^2\theta) = a^2$$

These probabilities should be *proportional* to the energy densities: we need to normalize them. The *Zitterbewegung* model gives us a simple *physical* normalization condition: all the energy in a volume (which we will write as $\int d^3\boldsymbol{x}$) must add up to the total energy E = mc^2 = m$a^2\omega^2$. Denoting the energy density as *u*, we can write:

$$\int P\, d^3\boldsymbol{x} = 1 = \frac{1}{E}\int u\, d^3\boldsymbol{x}$$

Here we need to think of the *distribution* of the energy density *u* because it is, perhaps, *not* so obvious to assume it is uniformly distributed. Indeed, where is the energy? It is not in our pointlike charge. It is in the oscillation, but where *exactly*? That is where the two-spring or two-piston metaphor is, perhaps, not very useful: the energy is in the spring, or in the pressure of the air in the cylinders of our engine, the moving parts, etcetera. But we do not have any moving parts here. When everything is said and done, the *Zitterbewegung* model assumes that space is, somehow, elastic, and its elasticity is given by c^2. How should we think about this? I am not sure, but it has always struck me that Einstein's E = mc^2 equation implies that the ratio between the energy and the mass of *any* particle is always the same. Hence, we can write, for example:

$$\frac{E_{electron}}{m_{electron}} = \frac{E_{proton}}{m_{proton}} = \frac{E_{photo}}{m_{photon}} = \frac{E_{any\ particle}}{m_{any\ particle}} = c^2$$

This reminds us of the $\omega^2 = C^{-1}/L$ or $\omega^2 = k$/m of harmonic oscillators once again.[26] The key difference is that the $\omega^2 = C^{-1}/L$ and $\omega^2 = k$/m formulas introduce *two* or more degrees of freedom.[27] In contrast, c^2=

[26] The ω^2= 1/LC formula gives us the natural or resonant frequency for an electric circuit consisting of a resistor (R), an inductor (L), and a capacitor (C). Writing the formula as $\omega^2 = C^{-1}/L$ introduces the concept of elastance, which is the equivalent of the mechanical stiffness (*k*) of a spring. It is important to note that the *k* in the $\omega^2 = k$/m equation is the mechanical stiffness. It is very different from the wavenumber k = p/ħ in the wavefunction.

[27] The resistance in an electric circuit introduces a damping factor. When analyzing a mechanical spring, one may also want to introduce a drag coefficient. Both are usually defined as a fraction of the *inertia*, which is the mass for

E/m for *any* particle, *always*. However, that is exactly the point: we can modulate the resistance, inductance and capacitance of electric circuits, and the stiffness of springs and the masses we put on them, but we live in *one* physical space only: *our* spacetime. Hence, the speed of light *c* emerges here as *the* defining property of spacetime – the resonant frequency, so to speak. We have no further degrees of freedom.

Of course, this elasticity – or whatever one may want to call it – should be the same everywhere, and that is why the assumption of a uniformly distributed energy density makes sense. We still need to make sense of the volume, however. Do we think of our pointlike charge following a loop, or can we find it anywhere in its plane of oscillation? In fact, if we think of the plane of oscillation as moving about as well, then we should probably think of a sphere. So what do we do? Perhaps we should go the other way around. If *u* is uniformly distributed, and if our interpretation of the wavefunction is correct, then *u* should be equal to $u = \epsilon_0 a^2$. Using the Compton radius for *a*, we can then calculate the volume from the $\int P d\boldsymbol{x} = 1 = \frac{1}{E}\int u d\boldsymbol{x}$ integral:

$$\int d^3\boldsymbol{x} = \frac{E}{u} = \frac{mc^2}{\epsilon_0 a^2} = \frac{mc^2}{\epsilon_0}\frac{\hbar^2}{m^2c^2} = \frac{\hbar^2}{\epsilon_0 m}$$

Does that make sense? Maybe. The units do: [E]/[*u*] (energy over energy density) gives us (N·m)/(N/m^2) = m^3, so that is a volume unit. The more important point that is made here is that the probabilities are proportional to the energy densities, and that the constant of proportionality involves the same electric constant we used to calculate energy densities for electromagnetic waves. In fact, we can write the following *physical* equation for the probabilities:

$$P = \frac{u}{E} = \frac{\epsilon_0 a^2}{E} = \frac{\epsilon_0}{mc^2}\frac{m^2c^2}{\hbar^2} = \frac{\epsilon_0}{\hbar^2}m = \frac{\epsilon_0}{\hbar^2 c^2}E$$

The volume and the probability are each other's inverse. They must be because of the mathematical normalization: $\int P\, d^3\boldsymbol{x} = P\int d^3\boldsymbol{x} = 1$. Let's quickly think about the implication of these two equations. If the energy of our electron increases, the probability to find it somewhere will increase as well. Is that logical? It should be because its volume in space will *decrease*. Having said that, we are looking at a free electron. Hence, the question of different energies or masses does not arise.

The key question here is: is the assumption of some uniformly distributed probability (or mass or energy) density reasonable. When we calculate the value for the volume using the formula above, it is clear it doesn't. We get a very different volume from what we would expect using the familiar volume ($V = \frac{4}{3}\pi r^3$) or surface ($S = \pi r^2$) formulas (calculate it for *r* = *a*). The order of magnitude is way out of whack. Hence, we must be doing something wrong here, but I will leave it to the reader to find out. 😉

a spring and the inductance for an electric circuit. Hence, we would write the resistance for a spring as γ*m* and as R = γ*L* respectively.

Particles with mass: Einstein's atom

We had no uncertainty in our model for the electron – no uncertainty in the quantum-mechanical sense, that is: we associated the electron with a precise mass, which is a precise energy.

We should now think of other particles – particles that are not pointlike. Think of a proton, or an entire nucleus. Can we do anything with the *Zitterbewegung* model? Maybe. Maybe not. The first and most obvious remark that must be made is the following: when thinking of the real and imaginary part of our wavefunction as an electromagnetic oscillation – just like an EM wave but with a different geometry – then we also think of our particle as having some (electric) charge. What about atoms or neutrons? They are neutral. There is no real answer to this question. We may say we should think of their constituents – which are quarks and, therefore, do have charge – but, while we might try to think a while along those lines, an argument along these lines will probably not quite cut it.

A more obvious idea is to change the physical dimension we have using so far: instead of a force per unit charge, we may want to think of a force per unit mass. That would turn the wavefunction into some gravitational wave. I think this idea could have more traction. In fact, my very first thoughts on the reality of the wavefunction were that it should be force per unit mass, partly because N/kg simplifies to the dimension of acceleration (m/s^2), which is purely geometric (it only involves distance and time units). I published my very first paper on this exactly one year ago[28] – and I should probably revisit some of the ideas and suggestions in it – but let us not waste space and time here.

The point is that – when discussing larger or non-elementary particles – we should be thinking of the wave packets we presented above[29]:

$$\psi(x,0) = \int_{-\infty}^{+\infty} \Phi(k) e^{i \cdot k \cdot x} dk$$

We need to put the time back into this function, of course, which can be done very simply:

$$\psi(x,t) = \int_{-\infty}^{+\infty} \Phi(k) e^{i \cdot k \cdot x} e^{-i \cdot \omega(k) \cdot t} dk = \int_{-\infty}^{+\infty} \Phi(k) e^{i \cdot (k \cdot x - \omega(k) \cdot t)} dk$$

Now we need to say something about those $\Phi(k)$ *weight* factors for the waves that make up the packet, but first note that we are writing ω as a function of k: ω = ω(k). This is a simple dispersion relation, and its nature will determine whether the wave packet stays together as a group. Of course, we have the *de Broglie* relations that relate the wavenumber (k) and the (angular) frequency (ω) to the particle momentum and energy respectively: ω = E/ħ and k = p/ħ. The dispersion relation is then determined by the relation between energy and momentum. Most introductory courses (and Schrödinger himself) take the (non-relativistic) kinetic energy for the energy: E = mv^2/2 = p^2/2m. We then get:

$$\omega = \frac{\mathrm{E}}{\hbar} = \frac{\mathrm{p}^2}{2\mathrm{m}\hbar} = \frac{\hbar^2 \mathrm{k}^2}{2\mathrm{m}\hbar} = \frac{\hbar \mathrm{k}^2}{2\mathrm{m}} = \omega(\mathrm{k})$$

This approach is simple but, in my humble view, conceptually misleading. First, we leave the largest chunk of energy out of the picture: the particle's rest mass or rest energy ($E_0 = m_0c^2$). This leads to

[28] See: *The Quantum-Mechanical Wavefunction as a Gravitational Wave*, (http://vixra.org/pdf/1709.0390v5.pdf).

[29] I left the $\frac{1}{\sqrt{2\pi}}$ factor out for clarity of discussion.

inconsistencies when we think of the wavefunction in the reference frame of the particle itself, in which it has no momentum and no kinetic energy, as a result of which the wavefunction would collapse to one ($e^0 = 1$). Fortunately, the Uncertainty Principle saves us here: the momentum and the energy will be distributed *around* zero, rather than being *exactly* equal to nil.

More fundamentally, if we believe the wavefunction packs energy in its oscillations, then most of that energy will be rest mass energy. In fact, we could possibly think of the rest mass as a two-dimensional oscillation of spacetime itself. The *Zitterbewegung* theory assumes a pointlike charge with no mechanical mass: all the mass is electromagnetic mass. Here we would have a theory that does not even require us to think of a pointlike something.

Second, we get a group velocity that is a function of k itself:

$$\frac{\partial\omega}{\partial \mathrm{k}} = \frac{\partial(\frac{\hbar \mathrm{k}^2}{2\mathrm{m}})}{\partial \mathrm{k}} = \frac{\hbar}{\mathrm{m}}\mathrm{k}$$

The consequence is that our wave packet dissipates – or disperses, as we say in the context of waves. Of course, real-life particles stay together. They don't spread out in space: they are particles. This is a conceptual problem.[30] The problem can easily be solved by noting that, if k varies, we should – perhaps – also allow *m* to vary. To be precise: we could plug in the relativistically correct p = m*v* formula to get what we should get for the group velocity:

$$\frac{\partial\omega}{\partial \mathrm{k}} = \frac{\hbar}{\mathrm{m}}\mathrm{k} = \frac{\hbar}{\mathrm{m}}\frac{\mathrm{p}}{\hbar} = \frac{\mathrm{m}v}{\mathrm{m}} = v$$

What do we get for the phase velocity? Using the non-relativistic kinetic energy only, we find $\frac{\omega}{\mathrm{k}} = \frac{\mathrm{E}}{\mathrm{p}} = \frac{\mathrm{m}v^2}{2\mathrm{m}} = \frac{v}{2}$. This value looks plain weird: half of the group velocity? Why? We get a better answer when using the relativistically correct energy and momentum formulas:

$$\frac{\omega}{\mathrm{k}} = \frac{\mathrm{E}}{\mathrm{p}} = \frac{\mathrm{m}c^2}{\mathrm{m}v} = \frac{c^2}{v} = \frac{c}{\beta}$$

The β is the *relative* velocity *v*/*c*: some number between 0 and 1. The phase velocity is therefore always superluminal, which is OK because it is *not* the speed of the traveling *group*.[31] That group travels at the speed we want to see it travel, and that's the particle's classical velocity *v*.

[30] The mentioned MIT course (8.04.1*x*) tries to explain this away in Chapter 7 but does a bad job at it. The exercises on it are easy but nonsensical. What is the maximum time for an electron to be localized in an area Δx = 1 cm? The answer is easy to calculate (the formula is $t \approx \frac{m}{\hbar}(\Delta x)^2$ so it's about one second) but the theory behind the formula is conceptually unsound.

[31] The phase velocity cannot represent any signal, so its speed can be superluminal. Also, we should remind ourselves that the phase velocity is a *mathematical* concept: it is not like something *real* is traveling at the phase velocity.

OK. This was a rather long introduction to the topic we wanted to discuss here, and that's these contributions (or amplitudes, *weights* or coefficients – whatever term is more appropriate[32]) $\Phi(k)$. We already noted that we should always think of $\Phi(k)$ as a real-valued function, centered around some value $k_0 = \frac{p_0}{\hbar}$ and width Δk. But what should we say about them when interpreting our wavefunction – and its component – as actual waves, i.e. waves that carry some energy in their oscillations.

The energy of a wave depends on its frequency or wavelength (think of the Planck-Einstein relation for electromagnetic waves here: $E = hf = hc/\lambda$, for example) and – oft forgotten – its *maximum* amplitude. To be precise, the energy of a wave will be proportional to the *square* of its amplitude: $E \propto a^2$. What's the proportionality coefficient? That depends on the nature of the wave. For an electromagnetic wave, we wrote $E = \varepsilon_0 a^2$. What should we write here?[33]

Let us first think about that frequency: the ω for our component waves will all be centered around some some value $\omega_0 = \frac{E_0}{\hbar}$ and width $\Delta\omega$. That's just an expression for the uncertainty of the energy (ΔE) – just like we had an expression for the uncertainty in the momentum: $\Delta E = \hbar\cdot\Delta\omega$ and $\Delta p = \hbar\cdot\Delta k$. So, whether we think of our component waves as being real or not does not matter: their frequency is what it is.

However, if each component wave has a real amplitude and, therefore, carries some real energy, then any sum of a finite or infinite number of wavefunctions will need to acknowledge just that: each component wave its own energy $E_i = \hbar\omega_i$ and will contribute this energy to the total energy of the wave packet. This energy can be associated with an equivalent mass, of course, which we will write as $m_i = E_i/c^2$. If we only have one or two component waves, then these E_i and m_i will be substantial. Conversely, if we think of an infinite number of component waves (as we do when doing a Fourier transform), then these quantities become infinitesimally small. However, even then they will have to add up to the total energy of the packet. In short, a formula like the one below may make sense:

$$\mathrm{E} = \sum \mathrm{m}_i \cdot a_i^2 \cdot \omega_i^2 = \sum \frac{\mathrm{E}_i}{c^2} \cdot a_i^2 \cdot \frac{{\mathrm{E}_i}^2}{\hbar^2}$$

We can re-write this as:

$$c^2\hbar^2 = \frac{\sum a_i^2 \cdot \mathrm{E}_i^3}{\mathrm{E}} \Leftrightarrow c^2\hbar^2\mathrm{E} = \sum a_i^2 \cdot \mathrm{E}_i^3$$

What is the meaning of this equation? We may look at it as some sort of *physical* normalization condition when building up the *Fourier* sum.

But… Oh, yes. You're right: we forget our ε_0 coefficient. Should it be there? Should it not be there? It depends. On what? It depends on your theory of the reality of the wavefunction: is it an electromagnetic

---

[32] The term amplitude is probably the most confusing, because it is used as a shorthand for the *probability* amplitude – so that is the wave*function* $ae^{i\theta}$ – as well as for its *maximum* amplitude, which is the coefficient in front (*a*).

[33] We wrote $u = \epsilon_0 \cdot \mathrm{E}^2$, so we wrote the energy as an energy *density.* Also note that the E in this equation is not the energy but the magnitude of the electromagnetic field vector (**E**), so we replaced it by *a*.

oscillation? Or do you like the gravitational theory better? That will determine if and what scaling constant you should use.[34]

Uncertainty revisited

So what is the nature of the uncertainty here? How should we understand the $\Delta p \Delta x \geq \hbar$ and $\Delta E \Delta t \geq \hbar$ expressions? Before making any philosophical statements here, I warmly recommend going to Kennard's formal proof of it. The Wikipedia article on the Uncertainty Principle has a version on it, which I find particularly clear and straightforward.[35] It basically proofs the following:

(1) *If* the position and momentum wavefunction are each other's Fourier transform, and

(2) *If* we believe that the momentum is, effectively, uncertain for some reason that this mathematical proof does not specify (it may be something fundamental, but it may also just be our own lack of information or knowledge), and

(3) *If* we are able to express this uncertainty because our distribution for the momentum happens to be a well-behaved symmetrical function that is centered around some mean value p_0 and, therefore, allows us to calculate the variance σ_p^2 (so that's the square of the standard deviation) as a measure for this uncertainty, then we can show that:

(i) The distribution of *x* will be a similarly well-behaved function and we will be able to calculate the corresponding variance σ_x^2.

(ii) We can then multiply σ_p^2 and σ_x^2 and apply some theorems (most notably the Cauchy-Schwarz inequality) and do some calculations (just some absolute squares and working out the resulting equations) to show that $\sigma_p^2\sigma_x^2 \geq \hbar^2/4$ or – taking the square root – that $\sigma_p\sigma_x \geq \hbar/2$.

Now, you would usually measure the width of a function by taking *twice* the standard deviation[36], so we can effectively write the ubiquitous $\Delta p \Delta x \geq \hbar$ expression and then repeat the calculations for the other two conjugate variables E and t to find the complementary $\Delta E \Delta t \geq \hbar$ expression.

In short, there is nothing fundamental about this. We get Planck's constant back because we put it there when we wrote the wavenumber as $k = p/\hbar$. That's all. If we would write the uncertainty in terms of k (as opposed to p), our uncertainty relation reduces to $\Delta p \Delta x = \Delta(\hbar k)\Delta x \geq \hbar$, which is equivalent to $\Delta k \Delta x \geq 1$. The uncertainty is built into our model, so to speak. Hence, the question remains: is the uncertainty fundamental or not?

---

[34] As a reward for the reader who got this far, I should say the discrepancies we get between the volume we calculated using the $u = \epsilon_0 a^2$ formula and the volume we get when using the $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi a^3$ formula (their respective order of magnitudes is hugely different) is also due to my non-consistent use of ε_0. In fact, we should not exclude the real coefficient is some weird mix of the electric and gravitational constant. 😉

[35] See https://en.wikipedia.org/wiki/Uncertainty_principle, accessed on 18 Oct 2018. Note you must click the *proof of the Kennard inequality using wave mechanics* link to see the proof.

[36] If the distribution is normal, then the [μ–σ, μ+σ] interval will contain almost 70% of all values of the variable (which is *x* or p in this case). The [μ–2σ, μ+2σ] interval will capture 95% of all values.

Of course, I would be foolish to believe it is *not* fundamental. It comes out of experiments and observations, and none of the great scientists we admire so much take issue with it. Let me, therefore, conclude with Feynman's own rather enigmatic summary of what the Uncertainty Principle is and, equally important, what it is *not*:

> "Making an observation affects the phenomenon. *The point is that the effect cannot be disregarded or minimized or decreased arbitrarily by rearranging the apparatus. When we look for a certain phenomenon we cannot help but disturb it in a certain minimum way.*"[37]

So what *is* the uncertainty then? Well… Think about it. The Uncertainty Principle basically relates position and momentum, and also energy and time. Let's start with position. That is - supposedly - a continuous variable, because spacetime is continuous – or so we assume, because all of our math (integrals over some time or space interval, for example) are based on that assumption.

However, for mass or velocity, we are – perhaps – not so sure. Of course, they are also continuous variables in our theories, but we know we have Planck's constant – the quantum of action. So, should we worry here? What *is* action? In German, they call it *Wirkung* – and I think the German sounds better in this case. 😉 Its physical dimension is *energy times time* (J·s = N·m·s) or, alternatively, it's also *momentum* (kg·m/s = N·s) *times distance* (m). So, if you want to think of what the action concept really means, think of it as some amount of energy - let's say the Sun that's shining light on us - that is available over some time - till it has burnt... Well... All its energy. 😉 Alternatively - but it's a bit more difficult to imagine what it represents - a *Wirkung* is also some momentum that's available over some distance.[38]

In both cases, Planck's constant tells us that a *Wirkung* comes in a discrete packet. The reality of Nature is *not* continuous. But so, we should look at both expressions of the Uncertainty Principle as a pair, just like the two de Broglie relations, which must also be presented as a *set* of two complementary relations, rather than individually.[39] But, of course, it is not easy to think that way. We like to freeze time and then look at how our wave looks like in space or – less common – we look at one point in space and see what happens there over time. That's what we do in courses. Let us think about what we are really doing to, perhaps, better understand the nature of Planck's quantum of action.

When we freeze time (cf. the $\psi(x, t=0)$ wavefunction we used above), it is just like we are projecting Planck's quantum (h or $\hbar$ if you want to talk angles rather than distances) in space, and so we talk about the $\Delta p \Delta x = \hbar$ expression of the Uncertainty Principle. Conversely, when we think of space as being 'frozen' somehow (imagine you can only see what happens at one point only), then it is only time that moves, so to speak[40], and then we have its $\Delta E \Delta t = \hbar$ expression.

In short, we should write the Uncertainty Principle simply as follows:

---

[37] See: Feynman's *Lectures*, III-2-6. The italics are mine.

[38] It is, of course, also the dimension of angular momentum, but I prefer the physical dimensions for an exploration of the nature of Planck's constant.

[39] I know that's what most courses or introductory books are doing: they separate the two, start talking about one, and then forget about the other.

[40] We don't think of time moving, of course. We think of it as ticking or sliding away – because it is one-dimensional and, importantly, it goes in one direction only.

Uncertainty Principle = ħ

The ΔEΔt = ħ and the ΔpΔx = ħ expressions represent just two complementary *aspects* of it. They are projections of ħ in space and in time respectively. Does that make sense? I'd be grateful for comments here !

Nelly Furtado told us that all good things must come to an end, so I should be getting there. 😊 Let me, therefore, conclude this paper with some philosophical remarks.

The metaphysics of physics

Let me first tackle the number one question: do I personally believe this physical interpretation of a particle wavefunction makes sense? My honest answer is: yes and no. Yes, because all of the above sounds logical and reasonable. But also… No. Why? I think the wavefunction is a very smart mathematical tool. I still feel that the idea of uncertainty in quantum mechanics remains broadly undefined – and it is undefined for a good reason: our uncertainty is likely to always mix our lack of information or knowledge about the external world as well as the more fundamental uncertainty that is enshrined in that simple symbol: ħ. We should, perhaps, refer to that as the *discrete* nature of the world – but that would cause a lot of confusion because it is quite particular: we are talking about the *discreteness* of physical action only, so that's a *Wirkung*: some force over some distance during some time.

Hence, I think the wavefunction allows us to integrate both types of uncertainty. As such, it is like a statistical link function between actual events – that are discrete and random – to our probability distributions (Poisson distributions and what have you). The probability distributions organize the discrete and random nature of the events into continuous functions that incorporate the concepts of periodicity and/or regularity – which are necessary if we want to say something meaningful on predictability, which is the object of any science.

So, what am I saying? Can we conclude anything? It is a *cliché* to say that quantum-mechanical concepts and principles are very non-intuitive. However, it explains why several *interpretations* of quantum mechanics have emerged and continue to vie for attention – even if the basic math has been formulated (almost) a century ago. I think that is a bad situation but I have ranted about that before and so I will not repeat myself here.[41] Fortunately, there is a mainstream interpretation of quantum mechanics, which we loosely refer as the Copenhagen interpretation. It mainly distinguishes itself from more frivolous interpretations – such as the many-worlds and the pilot-wave interpretations – because it respects Ockham's *lex parsimoniae*: the Copenhagen interpretation makes fewer assumptions.

[41] See: *Why it is Hard to Understand – And, Therefore, Explain – Quantum Math*, http://vixra.org/abs/1806.0183 (accessed on 18 Oct 2018).

Unfortunately, the Copenhagen interpretation itself is subject to many interpretations.[42] One such interpretation may be referred to as radical skepticism or empiricism[43]: we can only say something meaningful about Schrödinger's cat if we open the box and observe its state. According to this rather particular viewpoint, we cannot be sure of its reality as long if we don't observe it. All we can do is *describe* its reality by a superposition of the two *possible* states: dead or alive. That's Hilbert's logic[44]: the two states (dead or alive) are mutually exclusive but we add them anyway. If a tree falls in the wood and no one hears it, then it is both standing and not standing. Richard Feynman – who may well be the most eminent representative of mainstream physics – thinks this epistemological position is nonsensical:

> "A real tree falling in a real forest makes a sound, of course, even if nobody is there. Even if no one is present to hear it, there are other traces left. The sound will shake some leaves, and if we were careful enough we might find somewhere that some thorn had rubbed against a leaf and made a tiny scratch that could not be explained unless we assumed the leaf were vibrating." (*Feynman's Lectures*, III-2-6)

It is hard to not agree with him. So, what is the mainstream physicist's interpretation of the Copenhagen interpretation of quantum mechanics then?

We should separate measurement and consciousness here. The Copenhagen interpretation has nothing to do with consciousness. Reality, a measurement, and consciousness are very different things. After having concluded the tree did make a noise, even if no one was there to hear it, Feynman wraps up the philosophical discussion as follows: "We might ask: was there a *sensation* of sound? No, sensations have to do, presumably, with consciousness. And whether ants are conscious and whether there were ants in the forest, or whether the tree was conscious, we do not know. Let us leave the problem in that form." (ibidem)

In short, I think we can all agree that the cat is dead *or* alive, or that the tree is standing or not standing—regardless of the observer. It's a binary situation. Not something in-between. The box just obscures our view. That's all. There is nothing more to it. However, having said that, quantum physicists don't study cats in boxes or trees in forests. Those are big things. They study the behavior of photons and electrons, and smaller things[45], and then the Uncertainty Principle *does* come into play.

The question then becomes: what can we say about the electron – or the photon – before we observe it, or before we make any measurement? Think of the Stein-Gerlach experiment, which tells us that we'll always measure the angular momentum of an electron – along any axis we choose – as either $+\hbar/2$ or,

---

[42] See, for example, Don Howard, *Who Invented the "Copenhagen Interpretation"? A Study in Mythology*, December 2004, https://www3.nd.edu/~dhoward1/Copenhagen%20Myth%20A.pdf, accessed on 17 September 2018

[43] Radical empiricism and radical skepticism are actually very different epistemological positions, but then we are discussing some basic principles in physics here rather than epistemological theories and so I'll let the reader criticize.

[44] The reference to Hilbert's logic refers to Hilbert spaces: a Hilbert space is an abstract vector space. Its properties allow us to work with quantum-mechanical states, which become *state vectors*. You should not confuse them with the real or complex vectors you're used to. The only thing state vectors have in common with real or complex vectors is that (1) we also need a *base* (aka as a *representation* in quantum mechanics) to define them and (2) that we can make linear combinations.

[45] We limit our analysis to quantum electrodynamics. Hence, this article doesn't try to discuss quarks or other *sectors* of the so-called Standard Model of particle physics.

else, as $-\hbar/2$. What is its *reality* before it enters the apparatus? Do we have to assume it has some *definite* angular momentum, and that its value is as binary as the state of our cat: *up* or *down*—no in-between? Let us take that example, perhaps, because it is a significant one.

Physics, fuzziness and reality

To answer our question, we should first explain what we mean by a *definite* angular momentum. It's a concept from classical physics, and it assumes a precise *value* (or magnitude) along some precise *direction*.

We can easily challenge these assumptions. The direction of the angular momentum may be changing all the time, for example. If we think of the electron as a pointlike charge in some regular or irregular orbit, whizzing around in its own three-dimensional space, then the concept of a precise direction of its angular momentum becomes quite fuzzy, because its direction changes all the time. And if its direction is fuzzy, then its value might be fuzzy as well. In classical physics, such fuzziness is not allowed, because angular momentum is conserved: it takes an outside force – or *torque* – to change it.

Presumably, it would also require a force or a torque to change angular momentum in quantum mechanical, right? Yes, and no. We have the Uncertainty Principle in quantum physics: some energy (force over a distance, remember) can be borrowed, so to speak, as long as it's swiftly being returned—within the limits set by the Uncertainty Principle, that is: $\Delta E \cdot \Delta t = \hbar/2$.

Mainstream physicists – including Feynman – do not try to think about this. For them, the electron that enters the Stern-Gerlach apparatus is just like Schrödinger's cat in a box: the only property we're interested in is its spin (up or down), and there's some box obscuring the view. The cat is dead *or* alive. Likewise, the electron spin is up or down, and each of the two states has some probability – both of which must obviously add up to one. In short, they will write the *state* of the electron before it enters the apparatus as the superposition of the *up* and *down* states:

$$|\psi\rangle = C_{\text{up}}\,|\text{up}\rangle + C_{\text{down}}\,|\text{down}\rangle$$

We are all familiar with this[46]: C_{up} is the amplitude for the electron spin to be equal to $+\hbar/2$ along the chosen direction (which we refer to as the *z*-direction because we will choose our reference frame such that the *z*-axis coincides with this chosen direction) and, likewise, C_{up} is the amplitude for the electron spin to be equal to $-\hbar/2$ (along the same direction, obviously). C_{up} and C_{up} will be functions, and the associated probabilities will vary sinusoidally – with a phase difference to make sure both add up to one.

The math is consistent, but most would agree this feels more like a mathematical trick than a true model of the electron. This description of reality – if that's what it is – does not *feel* like a model of a *real*

[46] It is the *Dirac* or *bra-ket* notation. This simple formula also summarizes the essence of what one should know about *Hilbert spaces*: the two states are *Hilbertian* state vectors which we can combine linearly. These state vectors are always defined in terms of a *base*, or a representation. A representation in quantum mechanics basically establishes a line of sight between the observer and the object or – if you don't like the idea of an observer (consciousness has nothing to do with it) – it establishes the geometric relation between the electron (or whatever other thing we're measuring) and the measurement apparatus (this is a Stern-Gerlach apparatus in this case).

electron. It's just like reducing the cat in our box to the mentioned fuzzy state of being dead and alive at the same time: that doesn't feel very real either!

What we are doing here, is to reduce a three-dimensional object – our electron – to some *flat* mathematical object.[47] Can't we come up with something more exciting? I think we can. I think we should.

Jean Louis Van Belle, 18 October 2018

References

This paper discusses general principles in physics only. Hence, references can be limited to references to physics textbooks only. For ease of reading, any reference to additional material has been limited to a more popular undergrad textbook that can be consulted online: Feynman's Lectures on Physics (http://www.feynmanlectures.caltech.edu). References are per volume, per chapter and per section. For example, Feynman III-19-3 refers to Volume III, Chapter 19, Section 3.

Most of the illustrations in this paper are open source or have been created by the author. References and credits have been added.

As mentioned in the text, *How to understand quantum mechanics* (2017) from John P. Ralston, Professor of Physics and Astronomy at the University of Texas, is one of a very rare number of exceptional books that address the honest questions of amateur physicists and philosophers upfront. I love the self-criticism: "Quantum mechanics is the only subject in physics where teachers traditionally present haywire axioms they don't really believe, and regularly violate in research." (p. 1-10)

[47] We call it flat here because it has two (mathematical) dimensions only.