Generally covariant quantum theory.

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Abstract

This paper represents a concise introduction to the quantum theory of point particles in a time orientable curved spacetime, part of which was presented in the DICE conference in Castiglioncello, Italy. Substantiated improvements in presentation and content have been made as well as some silly mistakes have been removed.

1 Introduction.

There is only one quantum theory on Minkowski and that is the one presented by Weinberg proceeding upon work started by Wigner and Von Neumann. It is axiomatic, starts from a clear definition of a particle and constrains the dynamics as such that the notion of a field becomes useful. Weinberg wrote his summary after all the Evil happened and the beast was babtised "quantum field theory" instead of relativistic particle dynamics. Often, it is useful to attribute the correct name to something as it must reflect its deepest inner workings.

The idea of this paper is to give two distinct proper introductions to RQT (relativistic quantum theory), a Weinbergian one - which we will end with and was not presented on the conference - and a divine one, starting from the most simple of considerations, having nothing to do a priori with probability theory and Hilbert bundles (instead of spaces). Both approaches provide one with a different view on classical and quantum mechanics; they are geometrical and entirely devoid of a coordinatised language as well as symplectic approaches due to globally hyperbolic foliations.

In order to properly understand my motivation for doing this, one must understand well the shortcomings of quantum theory and relativity. The latter, for example does not contain anything like a psychological now and leads to a block universe view wher you can communicate with one and another without the other person atually "being" there. It is predestind that he or she will be there and that is all that matters; there is no such thing as conscious perception in relativity. This, at least, is a good feature of quantum theory. A ramification of this viewpoint is that you can actually ask "localized" questions about the universe which reside in the actual psychological past and future! This is something quantum theory also forbids, all your personal questions are projected on the psychologiccal now which is reasonable. On the other hand, relativity allows you

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in principle to take any observer test line (which ideally has no gravitaional back reaction effects) and that well not change anything to the observations made in the rest of the universe. That is clearly a physical statement which is necessary to do science. If this were not true and the definition of our very being and observations would depend upon some global convention, then we would be totally lost. Similar comments apply to the idea that you neeed an atemporal (that means for all observer times) notion of stationarity to even just define what particles are. That is clearly nonsense in a universe which is not deterministic and moreover, again, you should know from the beginning the initial conditions of the entire universe in order to define a particle. Effects such as the Unruh an Hawking effect critically use this sort of unscientific ideas. We, on the other hand suggest that the very notion of a particle is locally determined and that everybody in the universe holds the same, meaning unitarily equivalent in the operational language, conventions. This is already the first indication for a global consciousness. Secondary, this does not exclude that you can ask localized (in space as well time) questions about "the vacuum" such that spontanious particle creation occurs. Moreover, I tink the priciple of Local Lorentz invariance is the universal guideline here; that means that the dynamical framework should be flexible enough to allow for well defined actions of the local Lorentz group. This does not imply, as said previously, that a locally boosted observer can ask localized questions about nature in his own extended (by means of the exponential map) personal reference space, since that would reintroduce the liberty that we can project our questions as well to the past and future of the actual now which will lead to causality vialotions. Indeed, as will turn out, the physical questions observers can ask which do not lead to causality violations in this extended framework all have to transform in the same wway under the local Lorentz boosts which means that somehow, we all refer to one and another, another indication of global awareness. Several other comments are in place here; first of all, we insist upon local lorentz covariance, but not a global one on Minkowki. This is the perennial distinction between an active and passive interpretation of the Lorentz group; in the former, you leave the points on spacetime invariant but you just change your view while in the latter you actually move in Minkowski. In relativity, the same comment applies to the local coordinate transformations and spacetime diffeomorphisms. What I am saying is that global Lorentz invariance for quantum particle on Minkowski is *not* a symmetry of nature. This dsoes not want to mean that I don't think hat there does not exit a proper implementation of the Lorentz group in quantum gravity, since that would just be diffeomorphism of some kind. The crux is that in the latter theory the gravitational backractions are taken into account whereas for our more limited endeavour of RQT on a fixed background this is not the case. I believe this is a meaningful point of view given that there is no known rigorous formulation of QFT on Minkowski with a well defined action of the nonperturbative Lorentz group. Even worse, Haag's theorem shows that such a realization, if it would exist (Haag doesn't say anything about that) is incompatible with the notion of a free theory. This is a pretty substantial result but here I take again the philosophical viewpoint in judging the real content of this theorem. Ultimately, we all reason and observe by splitting the world into a free world and interactions, this is how we ask questions about the world; renormalization has a natural place here in the sense that the full propagaor of a particle needs to coincide in some sense with the free propagator[1]. So the nice thing about the interaction picture, which is sometimes also wrongly called perturbative quantization, is that it offers a clear vision on physical questions to begin with. All those mathematicians who refer to nonpertubative quantization fall into the trap that they will not be able to formulate any decent question about the real world, their so called momentum and spin eigenstates, in case they exist, will not be realistic ones we see by decoupling particles from the rest of the universe. Moreover, we must reason a bit more generally here than just Minkowski, where you have at least still, in principle, a nonperturbative notion of spin and energy momentum; in a general spacetime such notions are completely absent! Still, we know this to be a very useful way to reason, so this promots my view that particles are locally defined with a fixed notion of energy momentum and spin. So, the lesson I draw from Haag's theorem is that the standard nonperturbative QFT formulations have to be given up, with a ramification that unitarity is lost in a general interacting theory. Before I proceed to further thoughts regarding these issues, let me mention upfront that the ambition threfore is to make the interaction picture well defined. This will require giving upon global or active Lorentz invariance and unitarity, but not local Lorentz invariance. Now, I am kicking sacred assumptions here and that i that there is no need for a unitary queantum theory. Indeed, there is as far as I know, no evidence of any kind that this should be like that; n most experiments made in ral laboratoris, lots on information gets lost and nobody takes into account that your source particles have a nonvanishing probability to tunnel through the walls of your laboratory. Moreover, unitarity does not make any sense in a cosmology where everything is space and time dependent since your statistics will be a nonlocal one in time and depend upon the intricate details of the universe! Also, in practise we do never measure probabilites, we always renormalize occurence of events. In reality, a precise notion cannot even be defined, since a measurement on a screen in principle also undergoes its own dynamics and an account of events is by no means stationary as is always silently assumed. There is another remark in place here which is that in all practical comutations, the rest of the universe does not matter even if, in principle, everything is nonlocal to the full extend of the universe. This indicates that quantum mechanics is in reality not that global as the theory suggests it is; that is, localized prdictions shouldn't depend much upon your surroundings, at least in the approximation of weak, localized, gravitational degrees of freedom; neither should it depend very much opun your definition of the actual now since nobody really knows what that is. Indeed, for localized experiments, there is not much freedom in the choice of the actual now since the speed of light is huge. This calls for a revision of the basic tenets of quantum theory where the propagator does not reach too far and dies out in the limit to space and time infinity. This is what we really observe in CERN everyday; there the theorist can impose without any effect classical boundary on the labaratory scale breaking global Lorentz invariance. Also, as the reader will see, the intraction picture has a nice relational interpretation

[1] Note that on Minkowski, by means of unrigorous, but formal nonperturbative arguments, there exists a clear guideline for the natural renormalization conditions. In curved spacetime, this is a widely open issue and we shall not treat it in this paper. Notice also that I reject the use of an S matrix; first of all, it does not exist, the so called in or out states have no mathematical meaning - a large frustration of mine while reading Weinberg's book on the matter who pretends as if everything is all right. Second, such an idea could, at best, only have some impact in a time independent cosmology which is not the case here.

where no reference towards global aspects has to be made. The only way in which our global "now" creeps in into the dynamics of the interacting theory is by means of imposing that no physical process can propagate beneath the actual now, and beyond the future "now". Since Einstein did not provide for any equations of this actual "now", this remains a freedom in our theory which we are willing to eliminate in future work. But again, the predictions for local laboratories here on earth, where the gravitational field is weak as measured in our units, this unknown factor should not matter in practice. This is the point of view defended in my work which is nowhere, even closely, adressed in the current literature. I will proceed, in contrast to the previous publication of this paper, in the most pedestrian way possible showing full equivalence with the standard formalism in Minkowski under some limiting assumptions.

2 Foundational arguments.

In this section, we don't care about operational formulations of quantum theory, but try to derive the known Wightman fuctions of QFT from first principles without relying upon any quantization procedure or issues regarding (unitary) representations of the Poincaré group. That is, I will derive clasical as well as quantum physics from the sam principle; to do this, I see nature as a communist reflects upon society, that is the foundational quantity of everything is contained in an action signifying "work" or "rabota". More precisely, consider $\phi(\gamma, p(\gamma) \in \mathbf{B}$ where $\gamma : [a,b] \to \mathcal{M}$ is a curve joining an event x to an event y in spacetime $\mathcal{M}$ in affine parametrization with respect to a Lorentzian or Riemannian metric where, moreover, p is a field on that line associated with the physical quantity of "momentum". We do not really know yet what momentum is but it represents a kind of weight or importance given to that motion. p must, a priori, not be proportional to $\dot{\gamma}$ as weight might sometimes be disfavourable to the current motion. Given that we all love calculus, $\mathbf{B}$ is a division algebra over the real numbers with standard operations $+, .$, that is $\mathbb{R}, \mathbb{C}$ or $\mathbb{Q}$ disregarding the non-associative octonions. A frictionless theory is a dreamworld as no waste is produced; mathematically, this translates as follows, there exists an involution $\dagger$ and operation $\star$ such that

$$\phi(\gamma(b-s), p(\gamma(b-s))) \star \phi(\gamma(s), p(\gamma(s)) = 1_\star$$

and

$$\phi(\gamma(b-s), p(\gamma(b-s))) = \phi(\gamma(s), p(\gamma(s)))^\dagger.$$

It is worthwile to comment upon those; the first one says that that reversing the process is arithmetically equivalent to taking the inverse, an expression that nothing gets lost, whereas the second says that the inverse has an arithmetical significance. This last stance is useful as inverting two processes must preserve the "distance" between them. No discussion about this viewpoint is allowed for.

As a consequence, the constant curve $\gamma_e(s) = x$ satisfies

$$\phi(\gamma_e(s), p(\gamma_e(s)))^2 = 1_\star$$

which for $\star = +$, $x^\dagger = -x$ and $\mathbf{B} = \mathbb{R}$ gives $\phi(\gamma(b-s), p(\gamma(b-s)) = -\phi(\gamma(s), p(\gamma(s))$ and $\phi(\gamma_e(s), p(\gamma_e(s))) = 0$. These simple observations give rise to the notion of work *and* classical physics. On the other hand, taking $\mathbf{B} = \mathbb{C}$, $\star = .$, $x^\dagger = \overline{x}$, we have that

$$\phi(\gamma(b-s), p(\gamma(b-s)) = \overline{\phi(\gamma(s), p(\gamma(s))}$$

and $|\phi(\gamma(s), p(\gamma(s))|^2 = 1$ what leads to the $U(1)$ Fourier waves in quantum theory. We shall first argue how classical physics arises.

2.1 The classical theory.

The idea is to write down a first order differential equation for the quantity of labour performed along a path up to some parameter value. Reparametrization invariance forces

$$\frac{d}{ds}\phi(\gamma(s), p(\gamma(s))$$

where the latter is, with a slight abuse of notation, the same as $\phi(\tilde{\gamma}_s, p(\tilde{\gamma}_s)$ for $\tilde{\gamma}_s$ the restriction of γ to the interval $[a, s]$. We demand that it is proportional to $\frac{d}{ds}\gamma(s)$; hence, the reversion property implies that

$$\frac{d}{ds}\phi(\gamma(s), p(\gamma(s)) = \frac{d}{ds}\gamma(s) \cdot \mathbf{F}(\gamma(s), p(\gamma(s)))$$

which is the old Newtonian expression with $\mathbf{F}$ having the meaning of force. Indeed, $\mathbf{F}(\gamma(s), p(\gamma(s)))$ cannot depend upon the history between a and s as otherwise the reversion condition does not hold in general; that is, it needs to be an ultralocal quantity. To complete the dynamics, Newton supposed that $p(\gamma(s))$ must maximally *stimulate* the direction in which the particle is moving implying that

$$p(\gamma(s)) = m\dot{\gamma}(s)$$

where $m > 0$ expresses the weight attached to persistance of the motion, called the *physical mass*. Another observation was of an Einsteinian nature, namely that the change of work should be equal to the change in an inherent physical property of the particle. Such invariant is to the lowest order given by the momentum squared

$$h(p(\gamma(s)), p(\gamma(s)))$$

where h is the spatial metric which leads to

$$\frac{d}{ds}\left(\frac{m}{2}\left(\frac{d}{ds}\gamma(s)\right)^2\right) = \frac{d}{ds}\phi(\gamma(s), m\frac{d}{ds}\gamma(s))$$

and bestowes $\phi(\gamma(s), m\frac{d}{ds}\gamma(s))$ with the dimension of $\frac{\mathrm{kg.m}^2}{\mathrm{s}^2}$ which it should be, given that the notion of force must be associated to something intrinsic which is, in this case, the change of momentum

$$\mathbf{F}(\gamma(s), p(\gamma(s))) := \frac{d}{ds}p(\gamma(s)).$$

This is the simplest idea possible, given that the kinetic term is the lowest order invariant and m can be thought of as some material based constant. This leads to

$$\frac{m}{2}\left(\frac{d}{ds}\gamma(b)\right)^2 - \frac{m}{2}\left(\frac{d}{ds}\gamma(a)\right)^2 = \phi(\gamma(b), p(\gamma(b))) - \phi(\gamma(a), p(\gamma(a)))$$

and in a way generalizes a conserved quantity given that ϕ depends upon the entire path and not just the endpoints in general.

One could make higher derivative theories also in this way and allow for Newtonian laws with third order derivatives. These naturally appear in the context of backreactions in electromagnetism for example and allow for "unphysical" solutions with causality going backwards in time. For example, an electron would accelerate prior to turning on a lightbulb. Note also that the interpretation of γ as the physical path of the particle natually emerges given that Newtons law fixes it entirely given two "initial data".

2.2 Quantum theory.

Now, we derive quantum theory, as well as the probability interpretation, in the same vein. One notices that the *obvious*, but not only, candidate for an equation of motion is given by

$$\hbar \frac{d}{ds}\phi(\gamma(s), p(\gamma(s))) = -ig(p(\gamma(s)), \dot{\gamma}(s))\phi(\gamma(s), p(\gamma(s)))$$

where p is the so called energy momentum vector, g the Lorentzian spacetime metric and $\dot{\gamma}(s)$ the dimensionless velocity in units where the velocity of light is. Notice that $\hbar$ is needed for dimensional reasons to get a non-trivial theory given that ϕ must in this case, contrary to the previous one, be dimensionless number as any *physical* quantity is a real and not complex unitary number. On flat spacetime $\phi(\gamma(s), p(\gamma(s)))$ is, assuming the *law* that energy-momentum is conserved along γ meaning $\frac{D}{ds}p(\gamma(s)) = 0$, topological as it just depends upon the homotopy class or winding number. For Minkowski, such winding number is zero and the solution is given by

$$\phi(\gamma(b), p(\gamma(b))) = e^{-ip.(y-x)}$$

where $p = p(x)$ and $x = \gamma(a)$, $y = \gamma(b)$ which is the standard Fourier wave in y with base point x. Given that $e^{-ip.(y-x)}$ provides for a trivial unitary mapping between $e^{-ip.(z-y)}$ and $e^{-ip.(z-x)}$, the waves are identical up to a momentum dependent constant multiplicative $U(1)$ factor. In traditional RQT, this is precisely the impact of the translation symmetry in Minkowski. Now, unlike the previous case, there is no constraint on p in terms of γ and therefore, in order to come to a meaningful, Lorentz invariant, theory where everything is determined by the curve γ, we should integrate over a minimal Lorentz invariant shell in momentum space. The latter is given by $p^2 = \pm m^2$, where we have made the convention that the signature of the spacetime metric is given by $+ - --$. Since we further impose energy to be positive, only $p^2 = m^2$ remains and we arrive at the quantity

$$D(x, y) = \alpha \int_{\mathbb{R}^4} d^4p\theta(p^0)\delta(p^2 - m^2)\phi(\gamma, p)$$

which is, up to a constant, precisely the expression for the standard QFT propagator for scalar particles with "mass" $m > 0$. So, a consistent view upon a frictionless theory with our second choice of algebra, leads to free quantum field theory for scalar particles where the integration over the on shell momenta utters nothing but the Heisenberg uncertainty principle that if the positions x, y are known sharply, then the momentum is totally uncertain apart from the fact that it needs to be forwards pointing in time and on shell. Indeed, the Wightman function is all there is to free QFT on

Minkowski; going over to interactions, it is desirable to define the Feynman propagator which expresses the idea that you must travel from the past to the future and there is no ambiguity regarding spacelike separated event since there $D(x, y) = D(y, x)$ at least in Minkowski. We have of course that $D(x, y) = \overline{D(y, x)}$ since that was the very requirement of a frictionless theory. Usually, $D(x, y)$ is interpreted as an amplitude for a particle to be born, or created, in x and annihilated in y. Since for spacelik separate events, the creation and annihilation processes at x and y can be swapped without altering the "propagator", we arrive at an expression for Bose-Einstein statistics, a desirable property in the general theory. In a general curved spacetime, ϕ depends upon the curve and not just the homotopy class due to the existence of local gravitational degrees of freedom. In light of Bose statistics, only *geodesics* give rise tp $D(x, y) = D(y, x)$ for x, y spatially separated, given that the scalar product is preserved under evolution. In the next section, we shall deduce the correct propagators for spin $0, \frac{1}{2}$ partices as well as massless spin 1 particles, as well as their statistial properties, in a general time oriented curved spacetime from first principles. This maaterial has already been published in a non peer reviewed book of mine [1] but here we shall repeat it again, make some small corrections as well as add new material to deepen our understanding.

3 General Propagators.

Before we proceed with the calculations, let me repeat that I insist upon particles to be (ultra)locally *defined*, which of course does not imply that they are local objects. But in general, they all have an on shell momentum with a mass which, as we agreed before, is the same for all observers in the universe. This rules out interesting propsals in the literature where the mass of a particle is history dependent albeit it is rather difficult to speak of the history of a particle in wave mechanics. Since momentum is attached to a vierbein, the reader will understand that we shall integrate over flat tangent space at a fixed spacetime point. Indeed, there is a unique notion of Fourier transform attached to a spacetime point by means of propagation through geodesics. We shall explain this first in the following subsection.

3.1 Fourier transform and generalized Heisenberg operators.

In what follows, we shall adopt an Einsteinian view on spacetime and consider the latter to be eternally given and fixed; therefore, take a generic, time-orientable spacetime $(\mathcal{M}, g)$ and select a base point x, k^a a Lorentz vector at x defined with respect to the local vierbein $e_a(x)$ and y any other point in $\mathcal{M}$. Let $\gamma(s)$ be a curve from x to y and denote by $k^\mu(s)$ the parallel transport of $k^\mu(x) = k^a e^\mu_a(x)$ along γ. Then, we can define a potential $\phi_\gamma(x, k^a, y)$ by means of the differential equation

$$\frac{d}{ds}\phi_\gamma(x, k^a, \gamma(s)) = -i\dot{\gamma}^\mu(s)k_\mu(s)\phi_\gamma(x, k^a, \gamma(s))$$

with boundary condition $\phi_\gamma(x, k^a, x) = 1$. Then, one easily calculates, as mentioned in the previous section, that in Minkowski spacetime the potential is independent from the choice of γ and is given by the following group representation

$$\phi(x, k^a, y) = e^{-ik_a(y^a - x^a)}$$

where the formula is with respect to the global inertial coordinates defined by the vierbein $e_a(x)$. Minkowski is special in many ways: (a) every two events are connected by a unique geodesic (b) the ϕ_γ are path independent and define a group representation. Neither (a) nor (b) are true in a general curved spacetime which means we have to select for a preferred class of paths: the natural choice being that the *information* about the birth of a particle at x travels freely, meaning on geodesics which implies that we should sum over *all* distinct geodesics between x and y. This inspires one to consider the following mapping

$$\tilde{\phi} : T^\star\mathcal{M} \times T^\star\mathcal{M} \to U(1) : (x, k^a, w^a) \to \tilde{\phi}(x, k^a, w^a)$$

where $\tilde{\phi}(x, k^a, w^a)$ is defined as before by means of integrating the potential over the unique geodesic emanating from x with tangent vector w^a and affine parameter length one. One has then that

$$\phi(x, k^a, y) = \sum_{w:\exp_x(w)=y} \tilde{\phi}(x, k^a, w^a)$$

and although $\tilde{\phi}$ is more fundamental, we will sometimes switch between $\tilde{\phi}$ and ϕ by assuming that they are the same meaning that every two points in spacetime can be connected by a *unique* geodesic: this last assumption will be abbreviated to GS standing for "geodesic simplicity". In a general spacetime,

$$\tilde{\phi}(x, k^a, w^b) = e^{-ik^a w_a} = e^{ik^a e^\mu_a(x)\sigma_{,\mu}(x, \exp_x(w))}$$

where we assume in the last equality GS to hold and

$$\sigma(x, y) = \frac{1}{2}\epsilon L^2(x, y)$$

is Synge's function where $\epsilon = -1$ if x and y are connected by a spacelike geodesic and 1 if they are connected by a timelike geodesic and $L(x, y)$ denotes the geodesic length. Covariant derivatives of $\sigma(x, y)$ with respect to x will be denoted by unprimed indices μ, ν whereas their counterparts with respect to y are denoted with primed indices. It is clear that as usual the standard Fourier identities hold between the two tangent spaces at x, that is

$$\int_{T^\star\mathcal{M}_x} \frac{dk^a}{(2\pi)^4} e^{-ik^a w_a} e^{ik^a v_a} = \delta^4(w^a - v^a)$$

and

$$\int_{T^\star\mathcal{M}_x} \frac{dw^a}{(2\pi)^4} e^{-ik_a w^a} e^{il_a w^a} = \delta^4(k^a - l^a)$$

being the inverse Fourier transform. Under the hypothesis of GS, the first integral reduces to

$$\int_{T^\star\mathcal{M}_x} \frac{dk^a}{(2\pi)^4} e^{ik^a e^\mu_a(x)\sigma_{,\mu}(x,y)} e^{-ik^a e^\mu_a(x)\sigma_{,\mu}(x,z)} = \frac{\delta^4(y, z)}{\sqrt{-g(y)}\Delta(x, y)}$$

and the second one under the additional assumption of geodesic completeness (GC) becomes

$$\int_{\mathcal{M}} \frac{d^4y}{(2\pi)^4} \sqrt{-g(y)}\Delta(x, y) e^{ik^a e^\mu_a(x)\sigma_{,\mu}(x,y)} e^{-il^a e^\mu_a(x)\sigma_{,\mu}(x,y)} = \delta^4(k^a - l^a).$$

Here,

$$\Delta(x, y) = \frac{|det(\sigma_{,\mu\nu'}(x, y))|}{\sqrt{-g(x)}\sqrt{-g(y)}}$$

is the absolute value of the Van Vleck-Morette determinant. Still working under the GS assumption, one recognizes the presence of a global coordinate system given by $\sigma_{,\mu}(x, y)$ which transforms as a co-vector under coordinate transformations at x; contracting with $e^{a\mu}(x)$, one obtains local Lorentz vector $\sigma^a(x, y)$ and associated momentum operators $-i\frac{\partial}{\partial\sigma^b(x,y)}$ which transform as a local Lorentz co-vector such that

$$-i\frac{\partial}{\partial\sigma^b(x, y)}\phi(x, k^a, y) = k_b\phi(x, k^a, y)$$

meaning our generalized exponentials are eigenfunctions of the relative momentum operators. Also,

$$-\eta^{ab}\frac{\partial}{\partial\sigma^a(x, y)}\frac{\partial}{\partial\sigma^b(x, y)}\phi(x, k^a, y) = k^2\phi(x, k^a, y)$$

meaning that the above operator is to be preferred over the generalized d'Alembertian. In Minkowski spacetime, something special happens as

$$\sigma^b(x, y) = x^b - y^b$$

and one can substitute $-i\frac{\partial}{\partial\sigma^b(x,y)}$ by $-i\frac{\partial}{\partial x^b}$ or $i\frac{\partial}{\partial y^b}$. In other words, the x, y coordinates factorize and one can identify all Fourier pictures in this way and obtain *one* Heisenberg pair only. Indeed, I have stressed before that the philosophy of Minkowski is misleading due to its translational invariance and the reader should appreciate that the latter just falls out from our formalism. Also, it is now clear that a generalized Heisenberg picture demands the condition of geodesic simplicity whereas there is no good physical reason why this should be the case: our geometric framework is far more interesting than that.

3.2 An interesting example.

In thos section, we will motivate *why* you have to sum over all geodesics connecting x, y. For this purpose, consider a timelike cylinder $\mathbb{R} \times S^1$ with coordinates (t, θ) where θ has to be taken modulo $L > 0$ and see if only the discretized modes $k^1 = \frac{2\pi n}{L}$ for some $n \in \mathbb{Z}$ play a part in the propagator. Indeed, global space imposes such boundary conditions but the local observer is totally unaware of this. So he or she will uphold a continuous momentum spectrum, the only thing that matters is that only the discrete Fourier modes propagate. The reader has to be capable of figuring out that

$$\phi(x, k^a, y) = e^{-i(\sqrt{(k^1)^2+m^2}\delta t - k^1\delta\theta)}\left[\sum_{n\in\mathbb{Z}} e^{ik^1Ln}\right]$$

where

$$y - x = (\delta t, \delta\theta)$$

in the global flat coordinate system with periodic boundary conditions. This function is clearly invariant under the translation $\delta\theta \to \delta\theta \pm L$ and it is therefore well defined on the cylinder. Forming now a wave packet at x

$$\psi_x(y) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int dk^1\widehat{\psi}(k^1)\phi(x, k^a, y) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int dk^1\widehat{\psi}(k^1)e^{-i(\sqrt{(k^1)^2+m^2}\delta t - k^1\delta\theta)}\left[\sum_{n\in\mathbb{Z}} e^{ik^1Ln}\right]$$

and taking the Fourier transform with

$$\frac{1}{\sqrt{L}}e^{i\frac{2\pi p\delta\theta}{L}}$$

gives

$$\psi_x(y) = \frac{1}{L}\sum_{p\in\mathbb{Z}}\left(\int_0^L \psi_x(y)e^{-i\frac{2\pi p\delta\theta}{L}}d(\delta\theta)\right)e^{i\frac{2\pi p\delta\theta}{L}}$$

and it is easy to calculate that

$$\frac{1}{L}\int_0^L \psi_x(y)e^{-i\frac{2\pi p\delta\theta}{L}}d(\delta\theta) = \frac{1}{(2\pi)^{\frac{1}{2}}L}\int_{-\infty}^{+\infty} dk^1 d(\delta\theta)e^{-i\left(\sqrt{(k^1)^2+m^2}\delta t+\left(\frac{2\pi p}{L}-k^1\right)\delta\theta\right)}\widehat{\psi}(k^1)$$

where the reader has noticed that the integral between $0, L$ has become an integral between $-\infty, +\infty$ due to translational invariance of the measure and taking into account all terms in the sum originating from the winding of geodesics. The latter now equals

$$\frac{(2\pi)^{\frac{1}{2}}}{L}e^{-i\sqrt{\left(\frac{2\pi p}{L}\right)^2+m^2}\delta t}\widehat{\psi}\left(\frac{2\pi p}{L}\right)$$

which results in the ordinary Fourier transform

$$\psi_x(y) = \frac{(2\pi)^{\frac{1}{2}}}{L}\sum_{p\in\mathbb{Z}}\widehat{\psi}\left(\frac{2\pi p}{L}\right)e^{i\left(\sqrt{\left(\frac{2\pi p}{L}\right)^2+m^2}\delta t-\frac{2\pi p\delta\theta}{L}\right)}.$$

So, the winding of geodesics kills off all modes which do not satisfy the global boundary conditions. A similar result of course holds for the propagator and the reader may enjoy making that exercise. This example obviously generalizes to higher dimensional cylinders over the spatial d-dimensional torus $\mathbb{T}^d$. The reader should see this as the ultimate victory of our approach; as a local observer, you don't have to *know* that the universe is spatially compact in order to get the correct propagator.

3.3 Spin, two point functions and particle statistics.

Now that we have the correct Fourier transform in our hands, we come to the point where we can generalize propagators. Notice, as I will show later on, that these propagators do not satisfy the naive generalizations of the Klein Gordon or Dirac equation towards curved spacetime. The salient feature about our approach is that it is much more general: it holds for any time oriented spacetime whereas the traditional approach requires global hyperbolicity. Moreover, we have a direct particle interpretation wheras the latter has not; some authors, including Wald have become so desperate with their theory that they suggest to entirely give up the particle notion and speak only in terms of detector clicks where a detector is defined as an external quantum system coupled to the quantum field with no backreaction effects on the quantum field itself. I wish them good luck with getting any physical prediction out. We now define the two point function for a scalar particle in a general time-orientable curved spacetime by means of

$$W_\gamma(x,y) = \int_{T^\star\mathcal{M}_x}\frac{d^4k}{(2\pi)^3}\delta(k^2-m^2)\theta(k^0)\phi_\gamma(x,k^a,y)$$

where I have kept γ as a general curve, but we already know it should be geodesics. I already said that the geodesics stand out if you want that $W_\gamma(x,y) = W_{\gamma^{-1}}(y,x)$ but I will make this explicit soon so that I prove that geodesics are really the only sensible curves. This definition is clearly locally Lorentz invariant, as it should, and from the equality

$$\overline{\phi_\gamma(x,k^a,y)} = \phi_{\gamma^{-1}}(y,k_\star^{a'},x)$$

and the fact that the mapping $\star(x,y); T^\star\mathcal{M}_x \to T^\star\mathcal{M}_y : k^a \to k_\star^{a'}$ defined by dragging along γ is an orthochronous Lorentz transformation, it follows that

$$\overline{W(x,y)} = W(y,x)$$

as it should. This result does not depend upon the path γ joining x to y; the following demand however leaves in general just one option open:

$$W(x,y) = W(y,x)$$

for all $x \sim y$ where $\sim$ stands for being spacelike related, by which we mean related through spacelike geodesics only. This is our demand of quantum causality, it says that the amplitude for propagation of a particle between two spacelike separated points x and y does not depend upon the order of the points. We now show that if γ is a geodesic between x and y, then this demand is automatically satisfied. By definition, this geodesic must be a spacelike geodesic (it may be possible for timelike separated points to be joined by a spacelike geodesic such as occurs on the timelike cylinder); hence

$$\phi(x,k^a,y) = e^{-ik_a w^a}$$

where $w^a w_a = 2\sigma(x,y)$, w^a is tangent to the geodesic at x and $\sigma(x,y)$ is Synge's function as before. Equivalently,

$$\phi(x,k^a,y) = e^{i\sigma(x,y)_{,\mu} e_a^\mu(x) k^a}$$

as the reader may show or $w^a = -e^{a\mu}(x)\sigma_{,\mu}(x,y)$. To prove that the associated two point function satisfies indeed quantum causality, consider the reflection around w^a, the latter is a Lorentz transformation, preserving the sign of k^0 if k^a is a causal vector and maps $k^a w_a$ to $-k^a w_a$; hence, $W(x,y) = \overline{W(x,y)}$ which proves our assertion. In the case of general paths, the reader may easily see that this reflection of k^a does not need to flip the sign of $w^\mu(s)k_\mu(s)$ as this quantity is *not* preserved under general transport; the very preservation requires the geodesic equation to be fullfilled. One can now wonder to what extend the Klein-Gordon equation still plays a roll; we calculate that $W(x,y) \equiv W(\sigma_{,\mu}(x,y))$ satisfies

$$\left(\Box' + m^2\right) W(x,y) = i g^{\alpha'\beta'}\sigma_{,\mu\beta'\alpha'}\frac{\partial}{\partial\sigma_{,\mu}}W(x,y) + m^2 W(x,y) - g^{\alpha'\beta'}\sigma_{,\mu\alpha'}\sigma_{,\nu\beta'}\frac{\partial^2}{\partial\sigma_{,\mu}\partial\sigma_{,\nu}}W(x,y)$$

where primed indices refer to y and unprimed to x and all derivatives of σ are covariant derivatives. The reader now notices that in the coincidence limit $y \to x$, we have that the right hand side reduces to zero where we use Synge's rule $[\sigma_{,\mu\beta'}] = -g_{\mu\beta}$ and $[\sigma_{,\mu\alpha'\beta'}] = 0$ where the square brackets indicate that the limit $y \to x$ is taken. Before we proceed, let us stress that our point of view is relational in the sense that it is the way we have build the two point function, the point of view of field operators was absolute in the sense that propagation is a derived concept of composite entities whereas here, the bifunction is fundamental. Notice also that the above formula gives our covariantization of the flat space time equation and as anticipated in the previous chapter, the right hand side is in general not zero; we will come to other, more substantial deviations later on. Our two point function is natural in the sense that it only depends upon the geodesics joining the two points which is as "local" as one may get whereas in the standard approach there is no natural spacetime replacement for the flat Klein Gordon equation. There is a useful information interpretation of our formula which is that the information of the creation of a particle travels on geodesics possibly exceeding the local speed of light: therefore, the interacting theory will be constructed as a theory of interacting information currents.

3.3.1 Spin-0 extended.

So, we have now uncovered why our paths, along which information travels, have to be geodesics and why we have to sum over all of them. Given the somewhat more general character of our setup, we will introduce some extra notation needed for future reference. In particular, we need to change $k_\star^{a'}$, being a Lorentz vector at y, to $k_{\star\,w}^{a'} = \Lambda(x,w)_a^{a'} k^a$ being a Lorentz vector at $y = \exp_x(w)$ determined by the parallel transporter $\Lambda(x,w)$ which is defined by dragging a generic vector over the geodesic connecting x with y with tangent vector at x given by w. Wed shall now give the full definition of the Feynman propagator. These remarks *naturally* lead to the following definition of the Feynman propagator

$$\Delta_F(x,y) = \sum_{w:\exp_x(w)=y \text{ and w is in the future lightcone of x}} W(x,w) + \sum_{w':\exp_y(w')=x \text{ and w' is in the future lightcone of y}} W(y,w') + \sum_{w:\exp_x(w)=y \text{ and w is spacelike at x}} W(x,w)$$

which shall be used in defining the interaction theory. The reader should appreciate its full generality where we allow for points to be connected by as well spacelike as causal geodesics. This allows one to study our theory without any problem on closed spacelike cylinders with closed timelike curves whereas standard QFT does not make any chance here; for example, the Pauli Jordan function is only locally well defined. It is obvious that the singularity structure of our two point function is of Hadamard type and therefore identical to the one of the standard Minkowski vacuum; this leads to *infinite* renormalizations in the interaction picture which one would preferably avoid if one insists upon a sensible mathematical theory and I have motivated several regularization schemes in order to yield finite calculations [1]. The reader interested in that might consult that reference; this article is meant to be an introduction to these ideas and to show, in absence of these regulations, equivalence to the standard picture on Minkowski. The interaction theory is alo treated in [1] but until so far I have not suggested a general *physical* renormalization scheme. The reader should forgive me this inconvenience but the interacting theory, as it stands there, is perfectly sensible when one restricts to diagrams without loops, that is without radiative self energy corrections. We now come to the treatment of particles of higher spin.

3.3.2 General theory of spin.

The theory of spin has been developed from different points of view. Historically, Dirac rediscovered the gamma matrices and simply noticed that the generator of rotations around the j axis is of the form

$$-\frac{1}{2}\begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}$$

where σ_j is the j'th Pauli matrix which gives a double copy of the irreducible spin $\frac{1}{2}$ representation of $SU(2)$. The boosts however come with a relative minus sign and the Dirac representation is of type $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ of the homogeneous Lorentz group, see [2] for more details. So, in this section, we shall take an approach which is closer related to the "field" point of view constructed by Weinberg, rather than what he calls the particle point of view which is based upon abstract reasoning regarding transformation laws of the eigenstates of the energy momentum operators and some rotation operator in an irreducible unitary representation of the Poincaré group. In the latter view, he first derives transformation properties of those states, which upon an investigation of statistics, induce a transformation on particle creation an annihilation operators which in turn induce transformation laws on the particle-antiparticle vectors associated to a certain momentum and spin. This is all very universal and nice and we shall shamelessly copy his results in the next section when I discuss our novel insights from the operatioanl point of view. In this section, we directly copy his transformation laws for vectors associated to a certain momentum and spin and take that as the basis for our understanding. This requires no fields or anything like it, albeit it boils down to the same mathematics, but we simply ask for a *natural* definition of vectors associated with a state of definite momentum and spin. We shall only treat the theory for particles with a mass here and make some comments for massless spin one particles. Notice that we have not said anything yet about statistics in general, those properties have to be "read off" from symmetry properties of the propagator which we did already for spin zero particles. They have to be bosons, since you can switch the locations of particle creation and annihilation without modifying the Wightman function for events connected exclusively by spacelike geodesics. Thi approach has certain advantages because in the operational approach, it is for example unknown why massless particles do not have a continuous spectrum of internal degrees of freedom whereas in the "vector" or "field" representation, this is utterly clear. So beware, I am going to use some results in the book of Weinberg, but I shall *directly* motivate why these transformation laws should hold without undergoing the entire analysis from particles to fields.

So, this section contains no new material, albeit I do provide for a different point of view than Weinberg does. To make the presentation as clear as possible, we organize the discussion as follows: first, we introduce general properties of massive particles with spin and then impose a constraint on the Lorentz tranformations connection particles with different momenta and the same spin. This is also something Weinberg does and he does not fully motivate why this constraint has to hold apart from the fact that the relativistic notion of spin should coincide with the Euclidean one. I provide here no new input and I will use the same *convention* as he does. Nothing physical depends upon that convention as we shall show explicitely for Dirac particles given that the propagators are insensitve to such recalibrations. Next, we go over to Dirac particles and flesh out that theory; you will see that there are naturally two kinds of particles in this representation corresponding to vectors and covectors, the former which we call particles and the latter anti-particles. So, the transformation laws for those vectors and covectors have to be the same regarding the representation of the spin group $SU(2)$ but of course a covector transforms covariantly under the (nonunitary) irreducible representation of the Lorentz group whereas a vector transforms contravariantly. Now, in the Dirac representation, there is a natural correspondence between vectors and covectors so that the transformation law for the anti-particle covector can be turned into vector form. These are the general transformation laws we are interested in and they are precisely the ones obtained by Weinberg in the general case of arbitrary spin. Consider a free particle with mass $m > 0$ such that its wave vector k is timelike, $k^{\perp}$ is therefore a three dimensional Euclidean space, with inner product defined by $-\eta_{ab}$, and carrying the defining representation of the little group[2] of k, which is $SO(3)$. To start with, we can consider the case $k = (m, 0, 0, 0)$ and one notices that, for example, that the rotation around the z axis belongs to the little group. In fact, any rotation belongs to the little group, but we are picking one generator because you caqn only diagonalize one generator in any irreducible representation of the Lie algebra of $SO(3)$; the latter being given by

$$[L_j, L_k] = i\epsilon_{jkl} L_l$$

where the L_j are hermitean matrices, the square brackets denote the commutator and ϵ_{jkl} is the usual total antisymmetric tensor. A little bit of algebra reveal that

$$(L_1)^2 + (L_2)^2 + (L_3)^2 = L^2$$

commutes with every generator; therefore in an irreducible representation, it should be a multiple of the identity operator. As is well known, all finite dimensional irreducible representations are characterized by a half integer number j, and has dimension $2j+1$. Therefore, let σ, σ' be indices running from $-j \ldots j$, then the generators take the following standard form

$$(L_3)_{\sigma\sigma'} = \sigma\delta_{\sigma\sigma'} \quad (1)$$

$$(L_1 \pm iL_2)_{\sigma'\sigma} = \delta_{\sigma'\sigma\pm 1}\sqrt{(j \mp \sigma)(j \pm \sigma + 1)} \quad (2)$$

and as is well known

$$[L_1 + iL_2, L_1 - iL_2] = 2L_3$$

and therefore serve as lowering and raising operators. We are also interested in tensors mixing different representations of this Lie algebra, since those serva as natural intertwiners in a rotationally invariant theory. Such tensors are conqtructed from taking a tensor prouct representation

$$A \otimes B$$

with A, B half integers and noticing that this can be decomposed into a direct sum of spin j representations where j varies between $|A - B|$ an $A + B$ with multiplicity one. The Clebsch Gordan coeffiecients are the the natuaral projectors on those representations, meaning that a vector Ψ_{ab} where $a : -A \ldots A$ and $b : -B \ldots B$ gets projected to

$$\Psi_j(m) := \sum_{ab} C(jm, ab)\Psi_{ab}$$

[2]The little group of k^a is defined as the subgroup of the Lorentz group leaving k^a invariant.

where $m : -j \ldots j$ and $\Psi_j(m)$ transforms irreducible undr the spin j transformation. These coeifficients have several symmetries and you can find a more general analysis of what followq in Weinberg [2]. I am not going to repeat all this material here, since it has been explained before. Now, so far for spin; what we are interested in now are representations of the homogeneous Lorentz group which intertwine properly with spin j representations of $SO(3)$. We know this already to be the case for the Dirac representation, since the rotation around the three axis is a double copy of the spin $\frac{1}{2}$ rotation matrix. Now, I shall give you the right answe straight away and then show that this fully coincides with my view on vectors and covectors in the Dirac representation to be related to particle and anti-particle creation. Before we proceed, let me explain one further thing: we start out with the canonical four vector k and we attach thereon vectors $u(k, \sigma), v(k, \sigma)$ where the $u's$ are the particle vectors and the $v's$ the raised covectors (by means of the appropriate map); then we are going to look for Lorentz transformations $\Lambda(p)$ such that $D(L(p))$ brings $u(k, \sigma), v(k, \sigma)$ into $u(p, \sigma), v(p, \sigma)$ where D is our irreducible representation of the Lorentz group. Notice also that the litte group $D(R)$ where R is a spatial representation has to induce a spin j transformation $S^j(R)$ on these base vectors, that is

$$D(R)u(k, \sigma) = S^j(R)_{\sigma' \sigma} u(k, \sigma') \quad (3)$$
$$D(R)v(k, \sigma) = \overline{S^j(R)_{\sigma' \sigma}} v(k, \sigma') \quad (4)$$

and the only mysterious thing here is the complex conjugation in the second formula. We shall explain where that comes from if we go to the Dirac theory. Consider now $\Lambda(p)$ to be chosen and take an arbitrary Lorentz transformation Γ then the action of $D(\Gamma)$ on, say, $u(p, \sigma)$ reads

$$D(\Gamma)u(p, \sigma) = D(\Lambda(\Gamma(p))D(\Lambda(\Gamma(p))^{-1}\Gamma\Lambda(p))u(k, \sigma)$$

Obviously, $\Gamma(p))^{-1}\Gamma\Lambda(p)$ belongs to the little group of k and hherefore we denote it by $W(\Gamma, p)$. Thus,

$$D(\Gamma)u(p, \sigma) = S^j(W(\Gamma, p))_{\sigma' \sigma} u(\Gamma(p), \sigma').$$

Now, what we demand is that the $\Lambda(p)$ are chosen as such that for any spatial roation R holds that $W(R, p) = R$. Weinberg defines $\Lambda(p) = R(\hat{p})B(|p|)R^{-1}(\hat{p})$ where $B(|p|)$ is given by the standard boost around the z axis bringing k into $(\sqrt{|\vec{p}|^2 + m^2}, 0, 0, |\vec{p}|)$:

$$B(|p|) = \begin{pmatrix} \gamma & 0 & 0 & \sqrt{\gamma^2 - 1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sqrt{\gamma^2 - 1} & 0 & 0| & \gamma \end{pmatrix}$$

where $\gamma = \frac{\sqrt{|\vec{p}|^2 + m^2}}{m}$. Upon writing $\frac{\vec{p}}{|\vec{p}|} = \hat{p} = (0, \sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$ we can write

$$R(\hat{p}) = e^{i\phi L_3} e^{i\theta L_2}.$$

With those conventions one arrives at

$$\Lambda(p) = \begin{pmatrix} \gamma & (\gamma^2 - 1)\hat{p}_1 & (\gamma^2 - 1)\hat{p}_2 & (\gamma^2 - 1)\hat{p}_3 \\ (\gamma^2 - 1)\hat{p}_1 & 1 & (\gamma - 1)\hat{p}_1\hat{p}_2 & (\gamma - 1)\hat{p}_1\hat{p}_3 \\ (\gamma^2 - 1)\hat{p}_2 & (\gamma - 1)\hat{p}_1\hat{p}_2 & 1 & (\gamma - 1)\hat{p}_2\hat{p}_3 \\ (\gamma^2 - 1)\hat{p}_3 & (\gamma - 1)\hat{p}_1\hat{p}_3 & (\gamma - 1)\hat{p}_2\hat{p}_3 & 1 \end{pmatrix}.$$

Now, Weinberg checks that this boost satisfies our convention; that is, take any rotation R then

$$W(R,p) = R(R\hat{(p)})B^{-1}(|p|)R^{-1}(R\hat{(p)})RR(\hat{p})B(|p|)R^{-1}(\hat{p}$$

and notice that $R^{-1}(R\hat{(p)})RR(\hat{p})$ brings the w axis into itself. Therefore, it commutes with $B^{-1}(|p|)$ and the entire expression reduces to R as it should. The reader understands from this computation that the only ambiguity consists of a redefintion of $R(\hat{p})$ by $T(\hat{p})R(\hat{p})$ where $T(\hat{p})$ is a rotation around the $\hat{p}$ axis. Now, the general transformation law is given by

$$u(p,\sigma) = \sqrt{\frac{m}{p^0}}D(\Lambda(p))u(k,\sigma),\; v(p,\sigma) = \sqrt{\frac{m}{p^0}}D(\Lambda(p))v(k,\sigma).$$

The factor $\sqrt{\frac{m}{p^0}}$ stems from the fact that you still have to multiply $u(p,\sigma)$ with a wave $e^{ip.w}$ and that the latter has a norm squared p^0 with regard to the Klein Gordon inner product for waves. Hence, $\sqrt{\frac{m}{p^0}}$ serves as a normalization factor. The reader who is interested into the kind of irreducible representations (A,B) of the homogeneous Lorentz group which allow for vectors transforming as above under the spin j representation of the rotation group, is advised to consult [2]. There, a full classification and explicit formulae in terms of the Clebsch Gordan coefficients is givven. We now see how this realizes into the Dirac picture and show there where the "strange" transformation properties under the spin $\frac{1}{2}$ representation comes from.

The Dirac representation is defined by means of the γ^a matrices satisfying

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}1$$

and $(\gamma^a)^\dagger = \eta^{aa}\gamma^a$ with a special role for γ^0 since

$$\gamma^0(\gamma^a)^\dagger\gamma^0 = \gamma^a.$$

Note that we take the opposite convention to Weinberg who took the spacetime signature to be $-+++$. In matrix form

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^j = \gamma^0 = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix}.$$

As is well known, the generators of the Lorentz group are given by six tensor valued operators J^{ab} where $a,b : 0\ldots 3$ and $J^{ab} = -J^{ba}$ which obey

$$\left[J^{ab}, J^{cd}\right] = -i\left(\eta^{bc}J^{ad} - \eta^{ac}J^{bd} + \eta^{ad}J^{bc} - \eta^{bd}J^{ac}\right)$$

and the reader verifies that

$$\mathcal{J}^{ab} = \frac{-i}{4}\gamma^{[a}\gamma^{b]}$$

, where the brackets denote anti-symmetrization, satisfy this commutator algebra. Usually, we denote the rotations by

$$J^i := \epsilon_{ijk}J^{jk}$$

and the reader verifies that they obey

$$\left[J^j, J^k\right] = -i\epsilon_{jkl}J_l$$

which is the "rotation"valgebra in a space[3] signature $---$. In matrix notation, they read

$$J^j = \frac{1}{2}\begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}$$

which shows that the Dirac representation contains two irreducible spin-$\frac{1}{2}$ unitary representations of $SU(2)$. Note that the third Pauli matrix is real and self adjoint, so it equals its comolex conjgate. We will now find the correct vectors $u(k,\sigma), v(k,\sigma)$ from a *different* point of view than Weinberg, he neeeds the parity transformation as well as relativistic causality to fix those vectors given that their transformation proprties leaves for three ambiguities. Unlike in the operational approach of the next section, we do not dispose of a parity transformation or a time reversal and neither do I know what relativistic causality is since we are not working with operator fields. We shall eliminate all three of these ambiguities at once without using these supplemenary assumptions. The split we are looking for arises naturally if one makes the following observations: notice that

$$\left[\gamma^0, (J^{ab})^\dagger\right] = J^{ab}$$

and that therefore $D^{\frac{1}{2}}(\Gamma) := \Gamma^{\frac{1}{2}}$ obeys

$$(\Gamma^{\frac{1}{2}})^\dagger \gamma^0 = \gamma^0 \Gamma^{-\frac{1}{2}}$$

where $\Gamma^{-\frac{1}{2}}$ is the inverse of $\gamma^{\frac{1}{2}}$. The reader may want to check the covariance property

$$\Gamma^a_b \Gamma^{\frac{1}{2}} \gamma^b \Gamma^{-\frac{1}{2}} = \gamma^a.$$

We are interested in finding canonical projection operators $P(k)$ such that $\Gamma^{\frac{1}{2}} P(k) \Gamma^{-\frac{1}{2}} = P(\Gamma(k))$. The reader verifies that

$$P^{\pm}(k) = \frac{1}{2m}(k_a\gamma^a \pm m).$$

Trivially,

$$\Lambda^{\frac{1}{2}} P_{\pm}(k) \Lambda^{-\frac{1}{2}} = P_{\pm}(k)$$

for Λ in the little group of k and

$$P_+(k)P_-(k) = 0$$

as well as the "hermiticity" properties with respect to the indefinite scalar product

$$\langle v|w\rangle = \overline{v}^T\gamma^0 w.$$

Now, it remains to find a preferred basis for those two dimensional subspaces: for this purpose, we introduce commuting operators with $P_{\pm}(k)$ which are defined by means of an infinitesimal rotation in a two plane perpendicular to k; more in particular, let m, n denote two unit spacelike vectors perpendicular to k and one and another, then a generator of rotations in the n, m plane is given by by

$$R(n,m) = n_{[a}m_{b]}J^{ab}$$

which constitutes an hermitian operator with respect to the indefinite scalar product and defines two hermitian projection operators

$$P_{\pm}(n,m) = \frac{1}{2}\left(\mp 2R(n,m) + 1\right)$$

[3]Here, we differ a bit with Weinberg, who uses the convention $+++$ for space; hence, we have to reverse the sign of the generators of the spin $\frac{1}{2}$ representation too.

satisfying

$$P_+(n,m)P_-(n,m) = 0.$$

Therefore, we can *define* four canonical, normalized, wave vectors $u_{n,m,k;\pm}, v_{n,m,k;\pm}$ as solutions to

$$P_+(k)P_\pm(n,m)u_{n,m,k;\pm} = u_{n,m,k;\pm}$$

and

$$P_-(k)P_\pm(n,m)v_{n,m,k;\pm} = v_{n,m,k;\pm}.$$

We study these vectors now in somewhat more detail; under a general Lorentz transformation, we have that

$$u_{\Lambda n,\Lambda m,\Lambda k;\pm} = \Lambda^{\frac{1}{2}} u_{n,m,k;\pm}$$

and likewise for $v_{n,m,k;\pm}$. We now choose a Lorentz frame such that $k = me_0, n = e_1, m = e_2$; in that case $P_\pm(e_0)$ and $P_\pm(e_1,e_2)$ are *also* hermitian operators with respect to the standard Euclidean inner product so that the $u_{e_1,e_2,me_0;\pm}, v_{e_1,e_2,me_0;\pm}$ constitute both an orthonormal basis with respect to the Lorentzian as well as the Euclidean inner product. In particular, we have that

$$\frac{1}{4}\left(\gamma^0 + 1\right)\left(\pm i\gamma^1\gamma^2 + 1\right) u_{e_1,e_2,me_0;\pm} = u_{e_1,e_2,me_0;\pm}$$

which reduces to

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \pm\sigma_3 + 1 & 0 \\ 0 & \pm\sigma_3 + 1 \end{pmatrix} u_{e_1,e_2,me_0;\pm} = 4u_{e_1,e_2,me_0;\pm}$$

and therefore

$$u_{e_1,e_2,me_0;\pm} = \frac{\epsilon_\pm 1}{\sqrt{2}}\begin{pmatrix} \chi_\pm \\ \chi_\pm \end{pmatrix}$$

and likewise

$$v_{e_1,e_2,me_0;\pm} = \frac{\kappa_\pm 1}{\sqrt{2}}\begin{pmatrix} \chi_\pm \\ -\chi_\pm \end{pmatrix}$$

where $\sigma_3\chi_\pm = \pm\chi_\pm$ and $\chi_\pm^\dagger\chi_\pm = 1$; $\kappa_\pm, \epsilon_\pm$ are for now unknown unitary numbers. Now, we wish to identify the $u_{e_1,e_2,me_0;\pm}$, $v_{e_1,e_2,me_0;\pm}$ with the $u(k,\pm), v(k,\pm)$. The latter have to obey

$$\theta_i J^i u(k,\alpha) = \frac{1}{2}\sigma^i_{\beta\alpha}\theta_i u(k,\beta),\ \theta_i J^i v(k,\alpha) = -\frac{1}{2}\overline{\sigma^i}_{\beta\alpha}\theta_i v(k,\beta).$$

Taking only rotations around the z-axis, we have arrived at the natural candidates

$$u(k,\pm) = u_{e_1,e_2,me_0;\pm},\ v(k,\alpha) = v_{e_1,e_2,me_0;\mp}.$$

Insisting upon the full rotation conditions fixes all those vectors, see [2], up to an overall unitary number which has no influnce on the physics and can be set to one. In particular, we have in standard form

$$u(k,\pm) = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi_\pm \\ \chi_\pm \end{pmatrix},\ v(k,\pm) = \pm\frac{1}{\sqrt{2}}\begin{pmatrix} \chi_\mp \\ -\chi_\mp \end{pmatrix}.$$

This fixes our theory of spin-momentum vectors associated to particles; rmains to clarify the transformation laws for $v(p,\sigma)$. Heere, we start from our philosophy that vectors are associated to particles whereas covectors to anti-particles. As said previously, the anti-particle co-vectors should transform in the same way under spin rotations as particle vectors do.

It therefor suffices to say that the natural mapping between vectors and covectors is given by

$$v \to \overline{v}^T \gamma^0$$

which explains our last concern. We shall later on derive the propagator fro first principles and all formulae in Weinberg automatically fall out which suggests further generalizations towards particls of higher spin, but this is a concern for later.

Let me make some brief comments about massless spin 1 particles in the vector, or defining, representation of the Lorentz group. Here the little group of a null vector k is the Euclidean group in two dimensions $E(2)$, from which only the rotation part, with respect to any timelike vectorfield e_0 and spatial axis e_3, is unitarily represented. It is well known that the acction of the translations does not leave the helicity vectors $e(\pm)_\alpha$ invariant but the bilinear $k_{[\alpha}e(\pm)_{\beta]}$ is an invariant under the little group. This results in the well known stance that only antisymmetric tensors in two indices can carry a spin one particle and transform covariantly. Notice further that the represention of the rotation group is real so that the $u's$ and $v's$ all transform in the same way and hence no distinction between particle and anti-particle shows up. Indeed, the canonical raising or lowering operation of indices simply happens with the (inverse) spacetime metric and no complex conjugation is involved. Moreover, in traditional QFT on flat Minkowski, the Ward identities show that all ambiguities in the definition of the helicity states and propagators proportional to k vanish. This is not expected to hold on a general curved spacetime and one might indeed question the relevance of gauge invariance here.

3.4 Spin-$\frac{1}{2}$ particles.

Now that we hav st the preliminaries for the discussion, let us return now the definition of the straightforward generalization of our "Schroedinger" equation for particles with internal degrees of freedom living in an irreducible representation of the Lorentz group. In particular, we shall treat the case of spin $\frac{1}{2}$ particles first. We completely abandon the "quantum field" viewpoint here and derive the entire theory from a novel implementation of well known physical principles. That is, we aim to generalize the entire framework and derive all well known results of the free theory in flat Minkowski from novel principles without ever speaking about Hamiltonians, field operators, action principles and so on. So, what I propose is a "nouvelle cuisine" for quantum theory: a purely geometrical framework with a realist ontology. Since we work in a general curved space time, we need a Lorentz connection $\omega_{\mu\,b}^{a}$ and the reader may verify that the associated spin connection is given by

$$\omega_{\mu j}^{k} = i\omega_{\mu ab}(\mathcal{J}^{ab})_j^k$$

where the $k, j : 0 \ldots 3$ denote spinor indices and the generator of spin rotations J^{ab} has been introduced before. Therefore, the spin covariant derivative looks like

$$\nabla_\mu^s = \nabla_\mu + \omega_{\mu b}^{a} + i\omega_{\mu ab}(\mathcal{J}^{ab})_l^k$$

where $\omega_{\mu b}^{a}$ is given by

$$\omega_{\mu b}^{a} = -e_b^\nu \nabla_\mu e_\nu^a$$

and one may directly verify the antisymmetry property

$$\omega_{\mu ab} = -\omega_{\mu ba}.$$

Coming back to the main line of our story, we would like to introduce a function $\phi_m(x, k^a, y)^i_{j'}$ where primed indices again refer to y and m is the mass of the particle such that

$$W(x,y)^i_{j'} = \int_{T^\star \mathcal{M}_x} \frac{d^4k}{(2\pi)^3}\delta(k^2 - m^2)\theta(k^0)\phi_m(x, k^a, y)^i_{j'}$$

denotes some "propagator". Upper indices refer to spin properties of a vector while lower indices to those of a covector and moreover, annihilation and creation always go in a vector-covector pair. We maintain the convention that particle creation in the propagator corresponds to a covector (indeed, it has to contract with a vector) in the propagator while anti-particle creation corresponds to a vector index in the propagator. Note also that for simplicity of notation, we did not include for a sum over all geodesics but it is of course undeerstood that you should do that. So, the above propagator signifies the amplitude for an anti-particle to be created at x, with spin component i, and be annihilated at y with spin component j'. Likewise, we should have an amplitude $\psi_m(x, k^a, y)^{j'}_i$ to denote the "propagation" of a particle from x, with spin i towards y with spin j'. To fix the propagator, we will proceed in the same way as for the particle of zero spin, arguing what the coincidence limit $\phi_m(x, k^a, x)$ should look like and then solve for the entire spacetime by using the Schrodinger equation associated to (geodesic) paths γ:

$$\frac{D'^s}{dt}\phi(x, k^a, \gamma(t))^i_{j'} = -i\dot{\gamma}^\mu(t)k_\mu(t)\phi(x, k^a, \gamma(t))^i_{j'}.$$

Indeed, the latter is our replacement for the Dirac equation and we will study its solution later on. Let us start by the most straightforward principles of which the first does not necessarily need to be satisfied in a general curved space time but it is for sure true in Minkowski due to spatial homogeneity. That is, the coincidence limit $\phi_m(x, k^a, x)^i_j$ does not depend upon x and it transforms in the adjoint representation of $SL(2, \mathbb{C})$ meaning that

$$\phi_m(x, (\Lambda k)^a, x) = \Lambda^{\frac{1}{2}}\phi_m(x, k^a, x)\Lambda^{-\frac{1}{2}}.$$

The latter requirement, taken together with our generalized Schrodinger equation, ensures that the definition of the propagator shall be independent of the Lorentz frame chosen. Both conditions, taken together, imply that our only building blocks are $k_a\gamma^a$ and $m1$ and since we only work with on shell momenta, $\phi_m(x, k^a, x)$ may be chosen of the form $\alpha(k_a\gamma^a + \beta m1)$ where α and β are complex numbers: the mass dimension should be zero so that the limit of zero mass gives a nonvanishing result. Now, we arrive at our third and most important principle which says that the creation and annihilation of both a particle and antiparticle with the same four momentum should give a vanishing amplitude on shell when summing over all internal degrees of freedom, that is:

$$\phi_m(x, k^a, x)\psi_m(x, k^a, x) = \psi_m(x, k^a, x)\phi_m(x, k^a, x) \sim (k^2 - m^2).$$

This gives that $\phi_m(x, k^a, x) = \alpha(k_a\gamma^a \pm m1)$ and $\psi_m(x, k^a, x) = \alpha'(k_a\gamma^a \mp m1)$. Finally, we have our fourth condition which I call the positive energy condition, which says that

$$\frac{1}{4}\mathrm{Tr}(\gamma^0\phi_m(x,k^a,x)) = k^0 = \frac{1}{4}\mathrm{Tr}(\gamma^0\psi_m(x,k^a,x))$$

which states that the energy of a particle equals the zero'th component of its momentum vector. This further limits $\alpha = \alpha' = 1$; so we are left with

$$\phi_m(x,k^a,x) = (k_a\gamma^a \pm m1) = \pm 2mP_\pm(k), \quad \psi_m(x,k^a,x) = (k_a\gamma^a \mp m1) = \mp 2mP_\mp(k)$$

and given our previous analysis, it is clear that $\psi_m(x,k^a,x) = k_a\gamma^a + m1)$ and rhz other way around for $\phi_m(x,k^a,x)$. This ends our discussion of the coincidence limit; our novel principles have brought us to matrices which equal $\pm 2mP_\pm(k)$ giving the propagator a dimension of mass^3 in contrast to the propagator for a spin-0 particle.

Now, we come to the integration of the Schrodinger equation: the latter is easy and natural and before giving its solution, denote by $(\Lambda^{\frac{1}{2}}(x,w))_i^{j'}$ the *spin* holonomy attached to the geodesic from x to $y = \exp_x(w)$ determined by tangent vector w and similarly fo $(\Lambda(x,w))_a^{b'}$ the associated Lorentz holonomy. Thus given our initial conditions, the solutions to the "equation of motion" read

$$\tilde{\phi}_m(x,k^a,w)^i_{j'} = (k_a(\gamma^a)^i_r - m\delta^i_r)(\Lambda^{-\frac{1}{2}}(x,w))^r_{j'}\tilde{\phi}(x,k^a,w)$$

and

$$\tilde{\psi}_m(x,k^a,w)_i^{j'} = (\Lambda^{\frac{1}{2}}(x,w))_r^{j'}(k_a(\gamma^a)^r_i + m\delta^r_i)\tilde{\phi}(x,k^a,w).$$

We will now prove a remarkable property which shows that quantum causality, as it is usually understood, holds for this propagator. Indeed, the very structure of our formulae suggests that there may be a relationship between $\tilde{\psi}_m(x,k^a,w)$ and $\tilde{\phi}_m(y,k^{a'}_{\star w},-w_{\star w})$ where, as before, $k^{a'}_{\star w} = (\Lambda(x,w))_b^{a'}k^b$. Indeed, a small calculation reveals that

$$\begin{aligned}\tilde{\phi}_m(y,k^{a'}_{\star w},-w_{\star w})_i^{j'} &= (k_b((\Lambda(x,w))^{-1})^b_{a'}(\gamma^{a'})^{j'}_{k'} - m\delta^{j'}_{k'})(\Lambda(x,w)^{\frac{1}{2}})^{k'}_i\tilde{\phi}(y,k^{a'}_\star,-w_{\star w}) \\ &= (\Lambda^{\frac{1}{2}}(x,w))_l^{j'}\left(k_b(\gamma^b)^l_i - m\delta^l_i\right)\overline{\tilde{\phi}(x,k^a,w)}\end{aligned}$$

where we have used on the first line that $\Lambda^{\frac{1}{2}}(x,w) = (\Lambda^{\frac{1}{2}}(y,-w_{\star w}))^{-1}$; in the second line, we used covariance of the gamma matrices under joint spin and Lorentz transformations as well as the previous established formula for $\tilde{\phi}(x,k^a,w)$. Now, the way in which this formula becomes useful is by means of the particle and antiparticle propagators:

$$W_p(x,y)_i^{j'} = \int_{T^\star\mathcal{M}_x}\frac{d^4k}{(2\pi)^3}\delta(k^2-m^2)\theta(k^0)\psi_m(x,k^a,y)_i^{j'}$$

and

$$W_a(x,y)^i_{j'} = \int_{T^\star\mathcal{M}_x}\frac{d^4k}{(2\pi)^3}\delta(k^2-m^2)\theta(k^0)\phi_m(x,k^a,y)^i_{j'}$$

where, as before,

$$\psi_m(x,k^a,y)_i^{j'} = \sum_{w:\exp_x(w)=y}\tilde{\psi}_m(x,k^a,w)_i^{j'}$$

and likewise for $\phi_m(x, k^a, y)$. Indeed,

$$
\begin{aligned}
W_a(y,x)_i^{j'} &= \sum_{w:\exp_x(w)=y} \int_{T^\star \mathcal{M}_y} \frac{d^4 k_{\star w}}{(2\pi)^3} \delta(k_{\star w}^2 - m^2)\theta(k_{\star w}^0)\tilde{\phi}_m(y, k_{\star w}^{a'}, -w_{\star w})_i^{j'} \\
&= \sum_{w:\exp_x(w)=y} (\Lambda^{\frac{1}{2}}(x,w))_l^{j'} \int_{T^\star \mathcal{M}_x} \frac{d^4 k}{(2\pi)^3} \delta(k^2 - m^2)\theta(k^0) \left(k_b(\gamma^b)_i^l - m\delta_i^l\right) \overline{\tilde{\phi}(x, k^a, w)}
\end{aligned}
$$

and we concentrate now on points $x \sim y$ which are exclusively connected by spacelike geodesics. In that case, we could write

$$\tilde{\phi}(x, k^a, w) = e^{-ik_a w^a}$$

where w^a is the spacelike tangent at x to the geodesic connecting x with y. Choosing now for each term a different Lorentz frame at x such that the vector w is parallel to the three axis e_3; we perform, as before, a reflection around w given by $k^3 \to -k^3$ to obtain

$$(\Lambda^{\frac{1}{2}}(x,w))_l^{j'} \int_{T^\star \mathcal{M}_x} \frac{d^4 k}{(2\pi)^3} \delta(k^2 - m^2)\theta(k^0) \left(k_b(\gamma^b)_i^l - 2k_3(\gamma^3)_i^l - m\delta_i^l\right) e^{-ik_3 w^3}$$

where $e^{ik_3 w^3} = \tilde{\phi}(x, k^a, w)$. Summing this formula with the corresponding part of $W_p(x,y)_i^{j'}$ in the same frame gives

$$(\Lambda^{\frac{1}{2}}(x,w))_l^{j'} \int_{T^\star \mathcal{M}_x} \frac{d^4 k}{(2\pi)^3} \delta(k^2 - m^2)\theta(k^0) \left(2 \sum_{j=0\ldots 2} k_j(\gamma^j)_i^l\right) e^{-ik_3 w^3}$$

which is immediately seen, due to the antisymmetry of some part of the integrand under $k_1, k_2 \to -k_1, -k_2$, to reduce to

$$(\Lambda^{\frac{1}{2}}(x,w))_l^{j'} (\gamma^0)_i^l i \int_{T^\star \mathcal{M}_x} \frac{d^3 k}{(2\pi)^3} e^{-ik_3 w^3}$$

where the last integral equals $\delta^3(w^a)$ which proves that

$$W_p(x,y)_i^{j'} + W_a(y,x)_i^{j'} = 0$$

in any local Lorentz frame. This constitutes a proof of the well known statement that the amplitude for a particle with spin i to travel form x to y and be annihilated with spin j' equals the amplitude for an antiparticle with spin j' to travel from y to x where it is annihilated with spin i. The very minus sign reveals that spin-$\frac{1}{2}$ particles are fermions, meaning that exchanging two particles comes with a minus sign; this constitutes the proof of the spin statistics theorem in our setting at least for spin-0 and spin-$\frac{1}{2}$ particles. I should really mention that the standard approach towards fermions on a general curved spacetime did not even come close in obtaining such a general result. As before, we can now define the Feynman propagator for particle propagation $\Delta_{F,p}(x,y)_i^{j'}$ as we did for for scalar particles by summing over all geodesics between x and y and insisting upon propagation twards the future possibly replacing a particle propagatot by minus the anti-particle propagator. We also could define a Feynman propagator for anti-particle propagation as $\Delta_{F,a}(x,y)_{j'}^i$ as before by replacing p with a and the reader immediately notices that $\Delta_{F,a}(x,y)_{j'}^i = -\Delta_{F,p}(y,x)_{j'}^i$. This concludes our discussion of the free Fermi theory and the reader notices that all salient features of the standard Minkowski theory have been saved. We can now, as in the previous case suggest gravitational modifications of the two point function for causally related points such that causality remains valid but the singularity structure of the propagator changes. The way to do this is exactly identical to the one suggested before for the scalar two point function and therefore, we do not have to discuss this further on here. Evidently, our propagator does not satisfy the Dirac equation anymore and the reader is invited to investigate if the latter would still hold in the coincidence limit $y \to x$ just as the Klein Gordon equation did for the scalar two point function.

3.5 Spin 1 "gauge" particles.

In contrast to what one may expect, the two point function for massless spin-1 particles is extremely easy to guess, even when they carry another charge such as is the case for non-abelian gauge theories. We do not speak anymore in terms of gauge transformations which were necessitated by the quantum field viewpoint but we derive the main formula for the two point function and the Feynman propagator from two simple demands. The reader should appreciate the plain simplicity of the construction as the computation of the two point function for non-abelian gauge fields in standard quantum field theory is a matter of laborious work, the proof that gauge particles satisfy bosonic statistics being evident. Hence, we are interested in computing a quantitity

$$W^{\alpha\beta'}_{\mu\nu'}(x,y) = \int_{T^\star\mathcal{M}_x} \frac{d^4k}{(2\pi)^3}\delta(k^2)\theta(k^0)\psi(x,k^a,y)^{\alpha\beta'}_{\mu\nu'}$$

and again, we derive the correct form of the two point function. Note here that our group transformations are global transformations and therefore do *not* depend upon the space time point; so, the indices α, β' stands for the adjoint representation of the compact simple Lie group whose algebra is defined by

$$[t_\alpha, t_\beta] = i f^\gamma_{\alpha\beta} t_\gamma$$

where $f_{\alpha\beta\gamma} = f^\delta_{\alpha\beta} g_{\delta\gamma}$ is totally antisymmetric and the positive definite invariant Cartan metric is given by $g_{\alpha\beta}$. The fact that we do not make any distinction between covariant and contravariant vectors is due to the possibility to raise and lower indices with both metrics $g_{\mu\nu}$ and $g_{\alpha\beta}$. Let us study the coincidence limit $y \to x$ of $\psi(x,k^a,y)^{\alpha\beta'}_{\mu\nu'}$ first. Since there is no mass parameter, the only object of mass dimension zero which we can write down is a multiple of $g_{\mu\nu}g^{\alpha\beta}$, the only other term one can write down on shell has mass dimension squared and is given by a multiple of $k_\mu k_\nu g^{\alpha\beta}$. So here, we make our first law, $\psi(x,k^a,y)^{\alpha\beta'}_{\mu\nu'}$ has mass dimension zero and we can absorb any positive, real constant in the definition of the Cartan metric; so we obtain that

$$\psi(x,k^a,x)^{\alpha\beta}_{\mu\nu} = -g_{\mu\nu}g^{\alpha\beta}$$

where the minus sign originates from the fact that the vectors of helicity ± 1 should come with a plus sign. Writing out our Schrodinger equation is extremely easy

$$\frac{D'}{dt}\psi(x,k^a,\gamma(t))^{\alpha\beta'}_{\mu\nu'} = -i\left[\dot{\gamma}(k)\right](t)\psi(x,k^a,\gamma(t))^{\alpha\beta'}_{\mu\nu'}$$

and when $\gamma(t)$ is a geodesic, the solution is given by

$$\psi(x,k^a,y)^{\alpha\beta'}_{\mu\nu'} = -g_{\mu\nu'}(x,y)\phi(x,k^a,y)g^{\alpha\beta'}$$

where $g_{\mu\nu'}(x,y)$ denotes the parallel transport of the metric along the geodesic. The latter can be written as a composition of the Van Vleck matrix with Synge's function and since the metric is covariantly constant one has that $g_{\mu\nu'}(x,y) = g_{\nu'\mu}(y,x)$. In case multiple geodesics join x and y, we obtain that

$$W^{\alpha\beta'}_{\mu\nu'}(x,y) = -\sum_{w:\exp_x(w)=y} g_{\mu\nu'}(x,w)g^{\alpha\beta'}W(x,w)$$

where $W(x,w) = \int \frac{d^3k}{(2\pi)^3}\delta(k^2 - m^2)\theta(k^0)e^{-ik_a w^a}$, which shows that the two point[4] function for spin-1 particles transforming under a global, compact symmetry group is determined by the two point function of the scalar theory, a transporter and the Cartan metric. From our previous results and the symmetry of the transporter as well as the Cartan metric it follows that

$$W^{\alpha\beta'}_{\mu\nu'}(x,y) = W^{\beta'\alpha}_{\nu'\mu}(y,x)$$

for $x \sim y$ so that our theory satifies quantum causality and has bosonic exchange properties. Clearly, massless spin-1 particles are their own antiparticles as there exists only one two point function and not two. Let us better understand the magic which happened here: instead of following the quantization procedure of a theory with a local gauge symmetry and impose a gauge, we simply took the transformation group of the quantum numbers to be a *global* one. This is a meaningful point of view since those numbers themselves do not correspond to any force field, they are attributes of particles which is something different. It is possible to introduce classical gauge fields and introduce a dynamical gauge bundle so that we have to use the holonomies associated to this gauge field. This would be new physics and I hold it entirely possible that the future may lead us there; for now, we obtain on one sheet of paper a result which can be found in every textbook and which requires a long introduction to derive. As mentioned in the previous section, the structure constants $f_{\alpha\beta\gamma}$ and Cartan metric $g_{\alpha\beta}$ will be used to build interactions, everything is perfectly consistent with quantum chromo dynamics and quantum electro dynamics. The Feynman propagator $\Delta^{\alpha\beta'}_{F\,\mu\nu'}(x,y)$ has precisely the same prescription as is the case for spin-0 particles, which concludes the discussion for spin-1 particles. We now come to the discussion of Faddeev-Popov ghosts; first, let us ask ourselves why we insist upon spin-1 particles to transform in the adjoint representation and spin-$\frac{1}{2}$ in the defining one. The general reason is that it allows us to write down intertwiners of the kind

$$(\gamma^a)^i_j e^\mu_a(x)(t_\alpha)^m_n$$

and as the reader may verify, this is the only way to couple spin-1 and spin-$\frac{1}{2}$ particles. This leaves us with the question of coupling spin-0 particles to spin-1, the relevant intertwiner is given by

$$f_{\alpha\beta\gamma}\nabla^\mu$$

where the derivative acts on the ghost fermi propagator and therefore these spin-0 particles should transform as a vector in the adjoint representation; moreover they should have fermionic exchange properties since $f_{\alpha\beta\gamma}$ is totally anti-symmetric. and it is very easy to derive the correct propagator

$$W_p(x,y) = \theta(x)\overline{\theta(y)}g^{\alpha\beta}W(x,y)$$

[4]The fact that we need the Cartan metric for the construction of the two point function is precisely the reason why the Lie group had to be compact and simple in the first place.

where we have used Grassman numbers $\theta(x), \theta(y)$.

4 Haute Weinbergian cuisine.

In this final section, we look at relativistic quantum theory again from the operational point of view in a way which is fully equivalent to the one in the previous section. Here, we use without too much restriction the results of Weinberg [2] and I refer the reader to that book if anything looks mysterious to him or her. I feel it is not my duty to rewrite an analysis wich takes around eighty pages to motivate what I do. Sometimes, I shall explicitely state all assumptions I am making as I deem appropriate and mandatory for the discussion. This story goes way back in time as the ideas expresed in this paper were already present some 15 years ago. In 2011, I wrote a book [3] about an operational approach to quantum theory with local vacua delineating a Fock-Hilbert bundle $\otimes_{x \in \mathcal{M}} \mathcal{H}_x$ over the space-time manifold $\mathcal{M}$. However, the approach was troublesome and muddled with two "fundamental errors" of mine, not due to a lack of mathematical precision, but being the consequence of a poor understanding of what curved spacetime really means, something most authors don't reeally understand. This error found a natural solution in [1] written on generally covariant quantum theory from the point of view of the Dyson-Feynman "perturbative" exxpansion.
Concretely, we assumed $\mathcal{H}_x$ to be constructed by means of a cyclic quasi-free vacuum state $|0\rangle_x$ and multiparticle states showing Bose or Fermi statistics constructed in the Fock way. The dynamical object was a unitary bi-field $U(x, y)$ mapping $\mathcal{H}_y \to \mathcal{H}_x$ and obeying a Schroedinger like differential equation

$$\frac{d}{dt}U(t, s) = iHU(t, s)$$

but then with the times t, s replaced by x, y. The two errors in the book originated from the mathematical implementation of this idea I conceived; first of all $U(t, s) = U(t)U^{\dagger}(s)$ and moreover the only covariant first order differential operator *homogeneous* in the spacetime coordinates is given by the covariant Dirac operator D. The first condition is equivalent to a "cohomology" condition

$$U(x, y)U(y, z)U(z, x) = 1$$

which turns out to hold in Minkowski or any maximally symmetric spacetime only and reflects the absence of local gravitational degrees of freedom. Consequently, the only solution I was able to find of my field equations was free quantum field theory on Minkowski. The Dirac operator gives all sorts of trouble meaning we have to replace the complex numbers by an appropriate Clifford algebra of signature $(1, 3)$ or $(3, 1)$. This gives rise to negative probabilities and huge problems with the spectral theorem even for finite dimensional Clifford bi-modules. The approach was clearly dead as it stood which I realised later on. The crucial realization was that you just cannot relate particle notions like that, such relation is path dependent and as we explained before, the natural paths are the geodesics. So the idea of a Hilbert bundle *is* adequate, but the correct differential equation for $U(y, x)$ needs to run over geodesics connecting x with y in a fully reparametrization invariant way. Before we proceed, we make again the convention that at any spacetime point any obqserver sees the sme particles with identical massses and so on. Furthermore all unitary representations of the Poincaré algebra are te same; a Lorentz transformation merely relating one local vielbein to another. I will assume the point of view that the translations are only infinitesimally represented as our particles live on tangent space and it makes little sense to walk a finite distance away from the origin, albeit you can perfectly imagine this to be the case. Hence, if we have a vierbein $e_a(x)$ which we *passively* boost meaning $e_a(x) \to (\Lambda(x)^{-1})^b_{\ a} e_b(x)$ and $k^a \to \Lambda(x)^a_b k^b$ then the effect of this change on the states of the theory iz given by $\Psi \to U(\Lambda(x))\Psi$. We denote the generators of the translations by P^a and J^{ab} for the boosts. The Lorentz algebra yields [2] that

$$U(\Lambda)P^a U(\Lambda)^\dagger = (\Lambda^{-1})^a_b P^b$$

and this is the only property we really need. The obvious candidate for a dragging law is being given by

$$\frac{d}{ds}U(w^a, x, e_a(x)) = -iw_a P^a U(w^a, x, e_a(x))$$

where $w^a e_a(x)$ determines a unique geodesic connecting x with y and P_a equals the free four momentum generator, given by the expression

$$P_a = \sum_{\text{particles j, internal degrees } \sigma_j} \int \frac{d^3k}{\sqrt{k_0}} k_a a^\dagger_{k;j,\sigma_j} a_{k,j,\sigma_j}$$

and the final result has to be taken with respect to the *dragged* vierbein $e_a(x)$ in $y = \exp_x(w)$. This invites one to define quantities

$$U(w^a, x, e_b(y), e_b(x)) = U(\Lambda(y))U(w^a, x, e_a(x))$$

where $\Lambda(y)$ is the unique Lorentz transformation relating $(\exp_x(w)_\star)e_a(x)$ to $e_b(y)$. Therefore, if you change from reference frame in y you simply have to perform $U(\Lambda(y))U(w^a, x, e_b(y), e_b(x))$. Likewise, suppose you change of reference frame at x, meaning $e_a(x) \to (\Lambda^{-1})^b_{\ a} e_b(x)$ then $w_a \to (\Lambda^{-1})^b_{\ a} w_b$ and therefore our differential equation

$$\frac{d}{ds}U(\Lambda^a_b w^b, x, (\Lambda^{-1})^b_{\ a} e_b(x)) = -i(\Lambda^{-1})^b_{\ a} w_b P^a U(\Lambda^a_b w^b, x, (\Lambda^{-1})^b_{\ a} e_b(x))$$

$$= -iU(\Lambda)U(w_a P^a U(\Lambda^a_b w^b, x, (\Lambda^{-1})^b_{\ a} e_b(x))U(\Lambda)^\dagger$$

which is up to an equivaqlence preciely the same equation as for $U(w^a, x, e_b(y), e_b(x))$ which shows that

$$U(\Lambda^a_b w^b, x, (\Lambda^{-1})^b_{\ a} e_b(x), (\exp_x(w))_\star((\Lambda^{-1})^b_{\ a} e_b(x))) = U(\Lambda)U(w^a, x, e_b(x), (\exp_x(w))_\star(e_b(x)))U(\Lambda)^\dagger.$$

Therefore, in order to go from $(\exp_x(w))_\star((\Lambda^{-1})^b_{\ a} e_b(x))$ to $(\exp_x(w))_\star(e_b(x))$ we have to perform an inverse Lorentz transformation $U(\Lambda^{-1}) = U(\Lambda)^\dagger$ from the left resulting in

$$U(\Lambda^a_b w^b, x, (\Lambda^{-1})^b_{\ a} e_b(x), (\exp_x(w))_\star(e_b(x))) = U(w^a, x, e_b(x), (\exp_x(w))_\star(e_b(x)))U(\Lambda)^\dagger$$

which gives our covariance properties of $U(w^a, x, e_b(y), e_b(x))$. The coincidence limit is of course fixed by $U(0, x, e_n(x)) = 1$. So, as before, we have a canonical transport equation relation particles with different spin and momenta to one and another. In order to get an equivqlent viewpoint on the propagator, we have to introduce the notion of a bi-field. I shall only comment here for real scalar fields associated to bosonic particles of spin zero; considr the observer $x, e_a(x)$ then his fieldat the originof tangent space is given by

$$\Phi(x, x, e_b(x)) = \int \frac{d^3k}{(2\pi)^3 2\sqrt{k^2 + m^2}} \left(a_k + a_k^\dagger\right)$$

which is a locally Lorentz invariant expression so I could drop reference towards $e_b(x)$ in its definition. Now, we propagate the field by means of

$$\Phi(w, x, e_b(x), e_c(\exp_x(w))) = U(w^a, x, e_b(x), e_c(\exp_x(w)))\Phi(x, x)U(w^a, x, e_b(x), (\exp_x(v))_\star e_c(\exp_x(w)))^\dagger.$$

which transorms covariantly under local Lorentz transfromations at ay and is invariant rgarding local Lorentz transformations at x. This leads one to define

$$\Phi(y, x, e_b(x), e_c(y)) = \sum_{w \in T^\star \mathcal{M}_x : \exp_x(w) = y} \Phi(w, x, e_b(x), e_c(y))$$

Now, the reader should understand that the Wigthman function of the previous section in this case equals

$$W(x, y) = \langle 0 | \Phi(y, y, e_b(y)) \Phi(y, x, e_b(y)) | 0 \rangle$$

which is independent of the local Lorentz frame at y given dat the vacuum is Lorentz invariant and we have dropped any reference towards $e_b(x)$ since it does not matter at all. Again, on Minkowski, one simply chooses y to be the origin 0 and one defines a *field*

$$\Phi(x) = \Phi(0, x, e_b(0))$$

whose transformation law has become a "global" one since it only depends upon the Lorentz frame at the origin. One can, in this framework always define commuting observables in case x, y are spatially separated, but those all need to be relatipnal; they cannot depend freely upon the reference frames at x and y. It is not difficult, by using our definition, to see that for x and y spatially separated, there exist open neighborhoods $O(x), O(y)$ such that the smeared bi-field operators

$$V(y) = \int_{O(y)} d^z y \sqrt{g(z)} h(z) \Phi(y, z, e_b(y))$$

an d

$$\int_{O(x)} d^4 z \sqrt{g(z)} s(z) \Phi(y, z, e_b(y))$$

commute with one and another for arbitrary smeearing functions h, s. This raises profound questions about the "psychic" interconnectedness of observers, something which was already implicitly present in the standard formulation but becomes here, due to the rich local quantum symmetry group very explicit. Regarding interactions, it is clear that that one *cannot* write down expressions of the kind

$$\lambda \int_{\mathcal{M}} \sqrt{g(y)} \Phi(y, x) \Phi(y, z) \Phi(y, p) \Phi(y, q)$$

since it is impossible to make such vertex locally Lorentz invariant; the only thing one can do is to take propagators, which are locally Lorentz invariant and *define* more general operators using these special matrix elements. There is no way to write down directly a product of Bi-field operators, that is simply meaningless. This corrects a few errors made in [4].

5 Conclusions.

Quantum gravity is a long standing problem and according to this author, our first task is to close the gap between both obtaining a picture of the world which is commensurable with the way we do science. I have argued that locally defined particle notions is mandatory here and I explaind why on Minkowski for parrallel reference frames and inertial hypersufaces, our view coincides with the old one. However, radical departures emerge when you go over to a general curved spacetime; not only is our theory much more general here than is usually the case, it is (up to a choice of "now") also uniquely defined. This is not so for curved spacetime generalizations of standard Minkowskian laws, since for example, you can add any higher curvature corrections to the Klein Gordon equation you like. Moreover, even if you would select such choice of dynamics, the issue of which representation to choose and which question to ask remains fully open. Here, we have a direct, albeit nonunitary, interpretation with a broad claas of observables whose ontology is very clear. I hope, in this way to contribute a valuable new point of view to the way quantum theory should be done.

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