Riemann's Hypothesis and Stieltjes' Conjecture

Dr. Clemens Kroll

Abstract

It is shown that Riemann's hypothesis is true by showing that an equivalent statement is true. Even more, it is shown that Stieltjes' conjecture is true.

Key words

Riemann hypothesis, Stieltjes conjecture, Möbius function, Mertens function

1) Introduction

Riemann stated his hypothesis in 1859 [1]: the non-trivial zeroes of his zeta-function in the complex plane are all on the line with real part *1/2*.

Closely related is the Möbius function $\mu(n)$ [2], [9, page 234] which indicates if there are even or odd numbers of distinct primes and which can be used for an equivalent formulation of the Riemann hypothesis (5), precisely:

- $\mu(n) = 0$, if n has one or more repeated prime factors (is not square-free).
- $\mu(n) = (-1)^k$, if n is a product of k distinct primes. This is: $\mu(n) = 1$ if there is an even number (including zero) of distinct primes and $\mu(n) = -1$ if there is an odd number of distinct primes.

Furthermore, the Mertens function $M(x)$ [5, page 370] is summing up the Möbius function:

(1) $M(x) = \sum_{k=1}^{n \leq x} \mu(k).$

Here is a sequence of equations, analysis and theorems around the Möbius function that will be used in this paper:

- (2) Equation: $\lim_{n\to\infty} \frac{\left(\sum_{k=1}^{n} |\mu(k)|\right)}{n} = 6/\pi^2$; asymptotic density of square-free numbers $q(n) = |\mu(n)|$, [9, page 270].
- (3) Equations: Similarly [3], [10, page 606], and using Iverson's notation, the asymptotic densities of $\mu(k)=1$ or $\mu(k)=-1$ are:

  $\lim_{n\to\infty} \frac{\left(\sum_{k=1}^{n}[\mu(k)=1]\right)}{n} = 3/\pi^2$; and $\lim_{n\to\infty} \frac{\left(\sum_{k=1}^{n}[\mu(k)=-1]\right)}{n} = 3/\pi^2$.
- (4) Equation: $\lim_{n\to\infty} \frac{\left(\sum_{k=1}^{n} \mu(k)\right)}{n} = 0$; average order of μ, which is equivalent to the prime number theorem [11, page 64].
- (5) Theorem: $M(x) = O(x^{0.5+\varepsilon}),\ \varepsilon>0$, is equivalent to Riemann's hypothesis, using big O notation [4], [5, page 370], [6, page 47], [7, page 251]. We will make use of this theorem to verify the Riemann hypothesis.
- (6) Theorem: $M(x) = \Omega(x^{0.5})$; this shows a lower bound, using Ω notation [5, page 371].

- (7) Conjecture: $M(x) = O(x^{0.5})$; Stieltjes' conjecture, implying Riemann's hypothesis [4], [7, page 250].

Denjoy's probabilistic interpretation of Riemann's hypothesis [8, pages 268f] is not used for the proof in chapter 3).

2) Outline of the Proof

Step 1: It follows from (2) and (9) a linear term and a remainder term $O(\sqrt{x})$ for the summatory function of square-free numbers (big O notation).

Step 2: The same remainder term for the summatory function of the set of numbers with $\mu(k)=1$ and the set of numbers with $\mu(k)=-1$ can be concluded.

Step 3: Finally, the equation for $M(x)$ is created, using the result of step 2, which shows Riemann's hypothesis. Supporting evidence is presented in chapter 4).

3) Riemann's Hypothesis

Looking at the square-free numbers (2), there is a remainder term in [9, page 270]:

(8) $\sum_{k=1}^{n} |\mu(k)| = (6/\pi^2) * n + O(\sqrt{n}),$

where the sum - using Titchmarsh's notation [5, page 370] - is noted as:

(9) $Q(x) = \sum_{k=1}^{n \leq x} |\mu(k)| = (6/\pi^2) * x + O(\sqrt{x}).$

The remainder term (big O notation) takes care of details not described by the linear term. This does not imply randomness, it shows that additional terms are of order $O(\sqrt{x})$.

Let us define with Iverson's notation:

(10) $Q_{+1}(x) = \sum_{k=1}^{n \leq x} [\mu(k) = 1]$, the summatory function over all numbers with $\mu(k)=1$.

(11) $Q_{-1}(x) = \sum_{k=1}^{n \leq x} [\mu(k) = -1]$, same for all numbers with $\mu(k)=-1$.

From (10) and (11) together with (9) and (1) there is [10, page 606]:

(12) $Q(x) = Q_{+1}(x) + Q_{-1}(x)$, and:

(13) $M(x) = Q_{+1}(x) - Q_{-1}(x)$.

From (3) it is known that $Q_{+1}(x)$ and $Q_{-1}(x)$ both have linear terms, but I am not aware of any publication regarding the remainder terms. So there is:

(14) $Q_{+1}(x) = (3/\pi^2) * x + O(f_{+1}(x))$, and:

(15) $Q_{-1}(x) = (3/\pi^2) * x + O(f_{-1}(x))$, where both $f(x)$ limit the order of additional terms.

From (9) it is known that the remainder term of (12) is $O(\sqrt{n})$.

Hence $O(f_{+1}(x))$ and $O(f_{-1}(x))$ both are of maximal order $O(\sqrt{n})$, otherwise there would be a contradiction. From this, (3), (13) and the calculation rules of big O we conclude:

(16) $M(x) = O(\sqrt{x})$, complying with (6).

This is Stieltjes' conjecture and (16) implies Riemann's hypothesis by (5).

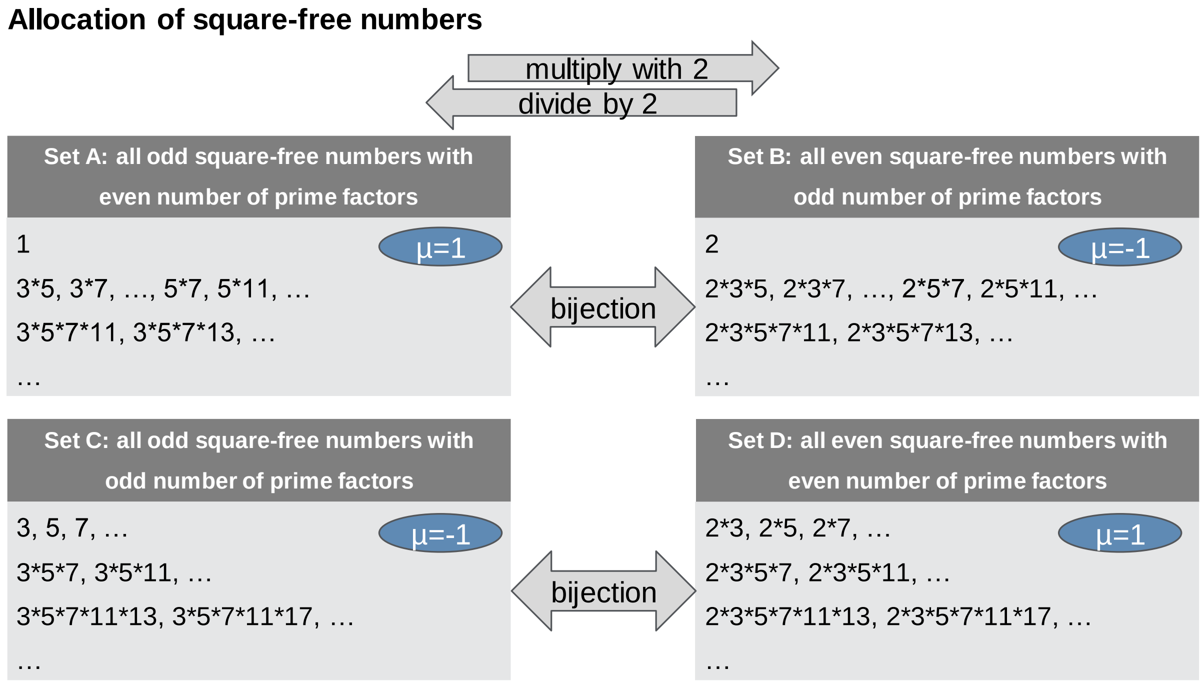
Hence Riemann's hypothesis is true.

(4) Supporting Evidence – Part 1

In [7, page 323] – while following Denjoy's probabilistic interpretation of Riemann's hypothesis – it is argued that from a strict 1:1 correlation between numbers with $\mu(k)=1$ and numbers with $\mu(k)=-1$ the Riemann hypothesis follows. Let us have a look on equations (3) and (4). They suggest that $\mu(k)=1$ is as frequent as $\mu(k)=-1$. But still there might be a huge deviation from a strict 1:1 correlation.

(17) Theorem: There is a bijection between the set of odd square-free numbers and the set of even square-free numbers.

(18) Theorem: There is a bijection between the set of numbers with $\mu(k)=1$ and the set of numbers with $\mu(k)=-1$.



(19) Picture

Proof: In picture (19) all square-free numbers are allocated into four sets *A, B, C, D*. By construction these four sets do not share any common numbers and together these four sets cover all square-free numbers. There is a bijection between sets *A* and *B*, and a bijection between sets *C* and *D*, both implemented as a multiplication with *2* or division by *2* respectively. From this there is a bijection between $\{A \cup C\}$ and $\{B \cup D\}$ which delivers theorem (17). There also is a bijection between $\{A \cup D\}$ and $\{B \cup C\}$ which delivers theorem (18) and is more strict than (3) which states same asymptotic density.

(4) Supporting Evidence – Part 2

(20) Theorem: There is a bijection between the set of odd square-free numbers with an odd number of prime factors and the set of odd square-free numbers with an even number of prime factors (sets *A* and *C* in picture (19)).

(21) Theorem: There is a bijection between the set of even square-free numbers with an odd number of prime factors and the set of even square-free numbers with an even number of prime factors (sets *B* and *D* in picture (19)).

Proof: picture (22) shows the square-free numbers generated with a set of four odd prime numbers. The number of elements with a certain number of prime factors are counted.

For 0, 1, 2, 3, 4 prime factors there are 1, 4, 6, 4, 1 elements respectively, which is determined by the binomial coefficients. Numbers of elements in Set 1 and Set 2 are the same.

Example with primes 3, 5, 7, 11

Set 1: all odd square-free numbers with even number of prime factors	Set 2: all odd square-free numbers with odd number of prime factors
0 prime factors: |{ 1 }| = 1	1 prime factor: |{ 3, 5, 7, 11 }| = 4
2 prime factors:	3 prime factors:
|{ 3*5, 3*7, 3*11, 5*7, 5*11, 7*11 }| = 6	|{ 3*5*7, 3*5*11, 3*7*11, 5*7*11 }| = 4
4 prime factors: |{ 3*5*7*11 }| = 1	

(22) Picture

If Set 1 and Set 2 are created with more and more odd prime factors, the result stays the same: |Set 1| = |Set 2|.

The reason is the formula for the sum of alternating binomial coefficients:

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k = 0.$$

This formula holds for arbitrary many prime factors (alternatively |Set 1| = |Set 2| could be shown by induction when adding the next odd prime factor) and thus delivers theorem (20). Taking "*2*" as one of the prime factors in the sets delivers theorem (21).

From theorems (17), (18), (20) and (21) it follows: $|A| = |B| = |C| = |D|$ with the four sets defined in picture (19).

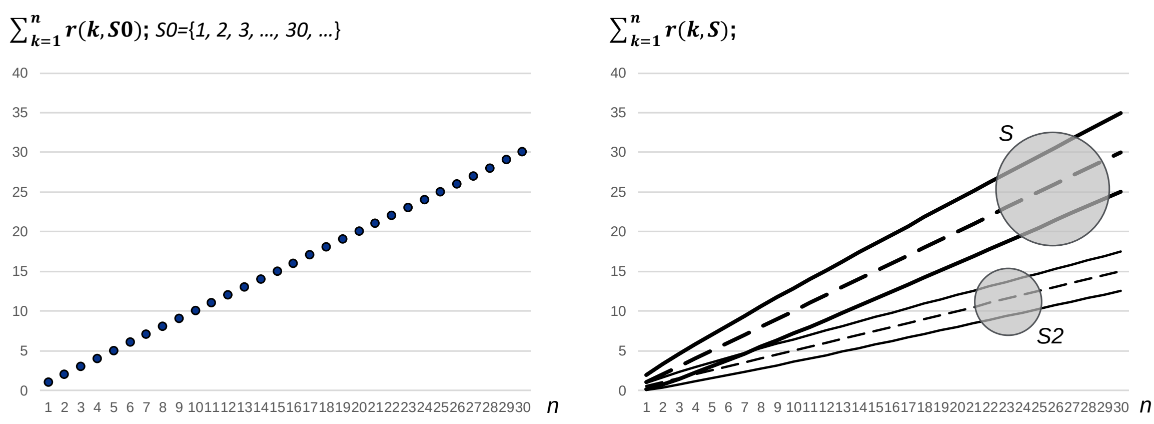
(4) Supporting Evidence – Part 3

Let S be an infinite set of numbers: $S=\{a, b, c, \ldots\}$; where $a, b, c, \ldots$ are arbitrary numbers. For ease of notation let us define a function r that returns k if t is k times included in S:

(23) $r(t, S)=1$, if $\{t\}\subseteq S$; $r(t, S)=2$, if $\{t, t\}\subseteq S$; $r(t, S)=k$, if $\{t, t, \ldots, t\}\subseteq S$, with $|\{t, t, \ldots, t\}|=k$; otherwise $r(t, S)=0$. (Returns maximum possible k).

If all elements in S are different, r is an indicator-function, showing if a number t is in S.

Summatory function of *r*



(24) Picture

The summatory function of *r* is shown in picture (24), on the left hand side there is a simple example with a set *S0* containing natural numbers starting with *1*. On the right hand side there is a more complex *S* resulting in $\sum_{k=1}^{n} r(k,S)$ = *const*n* + *O*($\sqrt{n}$). The dotted line represents the linear term, the range between the continuous lines shows the effect of the big *O* (illustrative). Summatory *r* of another set *S2* is shown additionally, described below.

Now let us check what happens to $\sum_{k=1}^{n} r(k,S)$ = *const*n* + *O*($\sqrt{n}$) when all elements of *S* are multiplied with some number *c*, for example *2* like in picture (19). Let us call this set *S2*, with *S2*={*a*2, b*2, c*2, …*}. Then there is $\sum_{k=1}^{n} r(k,S2)$ = *const*n/2* + *O*($\sqrt{1/2} * \sqrt{n}$). Recalling the invariance of big *O* regarding constant factors results in:

(25) $\sum_{k=1}^{n} r(k,S2)$ = *const*n/2* + *O*($\sqrt{n}$).

Similarly, having a set *S* with even elements only – like sets *B* and *D* in picture (19) – all elements in *S* can be divided by *2* and *S3*={*a/2, b/2, c/2, …*}. Like above there is:

(26) $\sum_{k=1}^{n} r(k,S3)$ = *const*n*2* + *O*($\sqrt{n}$).

From (25) and (26) it is seen that for linear scaling operations on elements of a set *S* with $\sum_{k=1}^{n} r(k,S)$ = *const*n* + *O*($\sqrt{n}$), the remainder term stays in the same order *O*($\sqrt{n}$), and the linear term is re-scaled inversely to the multiplicative factor but still is a linear term. This applies to sets *A*, *B* and to sets *C*, *D* in picture (19).

This is shown for integer multipliers or divisors. There are multiple more general settings than used here regarding the multipliers or the functions within the big *O*.

Let us come back to the summatory function of *r*: $\sum_{k=1}^{n} r(k,S)$ and have two infinite sets *S1* and *S2*. Without loss of generality, it is assumed that the elements in the sets show up in ascending order. From this ordering there directly is:

(27) $\sum_{k=1}^{n} r(k,\{S1 \cup S2\}) = \sum_{k=1}^{n} r(k,S1) + \sum_{k=1}^{n} r(k,S2)$.

This supports the conclusion regarding the big *O* terms in (14) and (15).

Finally – taking the four sets in picture (19) – we have:

$\sum_{k=1}^{n} r(k,A) = (1.5/\pi^2) * n + O(\sqrt{n})$;

$\sum_{k=1}^{n} r(k,B) = (1.5/\pi^2) * n + O(\sqrt{n})$;

$\sum_{k=1}^{n} r(k,C) = (1.5/\pi^2) * n + O(\sqrt{n})$;

$\sum_{k=1}^{n} r(k,D) = (1.5/\pi^2) * n + O(\sqrt{n})$.

(5) References

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