A BOUND FOR THE ISOTROPIC CONSTANT IN THE SYMMETRIC CASE

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ABSTRACT. In this preprint we will prove an explicit bound for the isotropic constant in the symmetric case.

1. INTRODUCTION

We say that a convex body $K$ is centralized, if

$$0 = \int_K \langle x, \phi \rangle,$$

for all $\phi \in S^{n-1}$. The entries of the covariance matrix of a convex body $K$ are defined as

$$(a_{ij}) = \frac{\int_K x_i x_j}{|K|} - \frac{\int_K x_i \int_K x_j}{|K|^2}.$$ We define the isotropic constant of any convex body $K$ via

$$L_K^2 := \frac{\text{Det}(\text{Cov}K)}{|K|^2}$$

[2]. We define the polar of $K$ as

$$K^\circ := \{ x \in \mathbb{R}^n | \langle x, y \rangle \leq 1 \ \text{for all} \ y \in K \}.$$ The Mahler volume $s(K)$ of $K$ is defined as

$$s(K) := \frac{|K|}{|K^\circ|}.$$ We say that the convex body is in isotropic position if it is centralized and the covariance matrix is a constant times the unit matrix. This kind of position exists [4]. The reverse Santaló inequality says that there is an universal constant $c$ such that

$$c^n |B_n|^2 \leq |K||K^\circ|,$$

where $B_n$ is the $n$-dimensional euclidean unit ball [1]. The isotropic constant for a ball $L_{B_n}$ is well know to be bounded. We will prove

**Theorem 1.** For all isotropic symmetric convex bodies $K$ it holds that

$$L_K \leq L_{B_n}^1(s(K))^{-1/n} \leq C,$$

where $C$ is an universal constant.

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2. The proof of the main theorem

Let $K$ be a unit ball. Let $C \in LG(n, \mathbb{R}^n)$ that takes $K$ to the John’s position [1]. So $B_n \subset C(K)$. It means that $C(K)$ contains the unit ball. And moreover $C(K) \subset \sqrt{n}B_n$ [1]. Because $C(K)$ contains the unit ball it follows that $(C(K))^o \subset C(K)$. Now,

$$\{x \in \mathbb{R}^n \mid \langle x, Cy \rangle \leq 1, \ y \in K \} = \{C^{-1}x \in \mathbb{R}^n \mid \langle x, y \rangle \leq 1, \ y \in K \}.$$ 

Thus, $C^{-1}(K^o) \subset C(K) \subset \sqrt{n}B_n$. So we have also $C^{-1}(K^o) \subset \sqrt{n}B_n$. Now using that there exists $C \in SLG(n, \mathbb{R}^n)$ such that $C(K) \subset \sqrt{n}B_2$ we can calculate

$$(2) \quad L^2_{A(C(K))} \leq \frac{\int_{C(K)}|x|^2}{n|C(K)|} |C(K)|^{-2/n} \leq \frac{|C(K)|n}{|C(K)|n} * |C(K)|^{-2/n} = |C(K)|^{-2/n}. $$

Moreover using that we have $C^{-1}(K^o) \subset \sqrt{n}B_n$ we can calculate

$$(3) \quad L^2_{B(C^{-1}(K))} \leq \frac{\int_{C^{-1}(K^o)}|x|^2}{n|C^{-1}(K^o)|} |C^{-1}(K^o)|^{-2/n} \leq \frac{|C^{-1}(K^o)|n}{|C^{-1}(K^o)|n} * |C^{-1}(K^o)|^{-2/n} = |C^{-1}(K^o)|^{-2/n}, $$

where $A, B \in SLG(n, \mathbb{R}^n)$ and $A(C(K))$ and $B(C^{-1}(K))$ are isotropic modulo scaling. Thus from (2) and (3) we obtain

$$(4) \quad L_{B'}L_{A(C(K))} \leq L_{A(C(K))}L_{B(C^{-1}(K))} \leq \frac{1}{s(K)^{1/n}}, $$

where we use the fact that $L_{B'} \leq L_K$ for all convex bodies $K$ and that

$$|C(K)||C^{-1}(K^o)| = |K||K^o|$$

Combining (4) with Milman-Bourgain (or reverse Santaló) inequality (1) and $L_{B'} > cL_{B_2}$

we obtain

$$L_{T(K)} \leq \frac{1}{L(B_2)s(K)^{1/n}} \leq C,$$

which implies the main theorem 1.

References

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