On the Normalizer of a Proper Subgroup of a Group of Prime Power Order

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Abstract

In this paper we lay out the proof of this result in group theory using only elementary facts in group theory.

Definition Let $G$ be a group, $p$ be a prime number, and $Z(G)$ be the center of $G$.

Theorem 1 If $|G| = p^n$ and $H \neq G$ is a subgroup of $G$, then $G$ has a subgroup of order $p|H|$ that contains $H$.

Proof.
Use induction on $n$. Suppose that the result is correct for $n - 1$. Let $G$ be a group of order $p^n$ and $H \neq G$ be a subgroup of $G$. By Lagrange’s theorem, $|Z(G)| = p^k$ for some integer $0 \leq k \leq n$. Since $Z(G) \neq (e)$, $p$ divides $|Z(G)|$ and so $Z(G)$ has an element $a$ of order $p$. Let $N$ be the subgroup of $G$ generated by $a$. Then $N$ is of order $p$. Since $a \in Z(G)$, $N$ must be normal in $G$. Moreover, $|N \cap H|$ divides $|N|$. So $|N \cap H|$ divides $p$. Thus $|N \cap H| = 1$ or $p$. Suppose $|N \cap H| = 1$. Then

$$|NH| = \frac{|N||H|}{|N \cap H|} = p|H|.$$ 

So $NH$ is a subgroup of $G$ of order $p|H|$ that contains $H$. Now suppose $|N \cap H| = p$. Since $N \cap H$ and $|N \cap H| = |N|$, it follows that $N \cap H = N$ and hence $N \subseteq H$. Since $H \neq G$, there is an $x \in G$, $x \notin H$. Clearly $xN \subseteq G/N$. Suppose $xN \in H/N$. Then $xN = hN$ for some $h \in H$. Since $x \in xN \text{ and } xN \subseteq hN$, so $x \in hN$. Hence $x = hN$ for some $n \in N$. Since $h \in H$ and $N \subseteq H$, so $x \in H$, a contradiction. To conclude $xN \notin H/N$ and thus $H/N \neq G/N$. Since $G/N$ is a group of order $p^{n-1}$ and $H/N \neq G/N$ is a subgroup of $G/N$, by the induction hypothesis, $G/N$ has a subgroup $\bar{P}$ of order $p|H/N|$ that contains $H/N$. Let $P = \{x \in G \mid xN \in \bar{P}\}$. Thus $P$ is a subgroup of $G$ and $\bar{P} \cong P/N$. As the result of

$$|H| = |\bar{P}| = \frac{|P|}{|N|} = \frac{|P|}{p},$$

so $|P| = p|H|$. Let $h \in H$. Then $hN \in H/N$. Moreover, $H/N \subseteq \bar{P}$. To conclude $hN \subseteq \bar{P}$ and thus $h \in P$. As the result, $H \subseteq P$.

Theorem 2 Any subgroup of order $p^{n-1}$ in a group $G$ of order $p^n$ is normal in $G$.

Theorem 3 If $|G| = p^n$ and $H \neq G$ is a subgroup of $G$, then there exists an $x \in G$, $x \notin H$ such that $x^{-1}Hx = H$.

Proof.
By Theorem 1, $G$ has a subgroup $K$ of order $p|H|$ that contains $H$. By Lagrange’s theorem, $|H| = p^i$ for some integer $0 \leq i \leq n - 1$. So $|K| = p^{i+1}$ and hence $H$ is normal in $K$ by Theorem 2. Since $|K - H| = |K| - |H| = p^{i+1} - p^i > 0$, there is an $x \in K$, $x \notin H$. Since $H$ is normal in $K$,

$$x^{-1}Hx = x^{-1}H(x^{-1})^{-1} = H.$$ 

Finally $x \in K \subseteq G$ as required.

References