Special value of Riemann zeta function and L function, Approximate calculation formula of ζ (N), L(N)

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Explanation and calculation result of the created formula.

1 introduction

First of all, this sentence is created by machine translation.[1] I apologize first because I think it is difficult to read. In the formula, when N is small, the accuracy is very bad, but as N increases, the accuracy also improves. I hope this formula will help you discover new things.

2 function description

2.1 Tn

The tangent number is a number sequence whose generation function is a tangent function or an individual number belonging to it. That is, it is defined as an integer string T_k defined by the following Taylor expansion.[2] The Euler number is defined as the expansion coefficient in the Taylor expansion of the hyperbolic secant function. Formally, Taylor series Formally,E_k in the Taylor series is the Euler number.[3]

The secant number is a number sequence obtained by correcting the Euler number.

secant number tangent number Euler number

$\sec z = \sum_{k=0}^{\infty} \frac{\hat{E}_k}{k!} z^k \quad \tan z = \sum_{k=0}^{\infty} \frac{T_k}{k!} z^k \quad \operatorname{sech} z = \sum_{k=0}^{\infty} \frac{E_k}{k!} z^k \quad \hat{E}_{2k} = (-1)^k E_{2k}$

Consider the expansion series $\hat{T}_k$ of the Taylor expansion of the sum of the tangent function and the secant function as follows. This expansion coefficient is the sum of the secant number and the tangent number, that is, $\hat{T}_k = E_k + T_k$

$T_n = \hat{T}_{(n-1)} = E_{(n-1)} + T_{(n-1)}$

$\tan z + \sec z = \sum_{k=0}^{\infty} \frac{\hat{T}_k}{k!} z^k$

T_n	1	2	3	4	5	6	7	8	9	10	···
tangent number	-	1	-	2	-	16	-	272	-	7936	···
secant number	1	-	1	-	5	-	61	-	1385	-	···

2.2 function

$$f(x) = \frac{T_n\ \pi^n}{\Gamma(n)\ 2^{n+1}} \times f(x)_\alpha$$

3 zeta function

(Riemann zeta function)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (s > 1) \tag{1}$$

3.1 ζ (2n)

In the case of $\zeta(2n)$, the correct value is obtained.

A formula using Euler numbers has already been found.[4]

$$f(x)_\alpha = \frac{2^n}{2^n - 1}$$

$$(n \geqq 2) \quad n \equiv 0 \pmod 2 \quad f(n) = \frac{T_n\ \pi^n}{\Gamma(n)\ 2^{n+1}} \times \frac{2^n}{2^n - 1} \tag{2}$$

$$\zeta(2) = \frac{\pi^2}{\Gamma(2)\ 2^3} \times \frac{2^2}{2^2 - 1} = \frac{\pi^2}{6}$$

n	2	4	6	8	10	⋯
(1) $\zeta(n)$	$\frac{\pi^2}{6}$	$\frac{\pi^4}{90}$	$\frac{\pi^6}{945}$	$\frac{\pi^8}{9450}$	$\frac{\pi^{10}}{93555}$	⋯
(2) $f(n)$	$\frac{\pi^2}{6}$	$\frac{\pi^4}{90}$	$\frac{\pi^6}{945}$	$\frac{\pi^8}{9450}$	$\frac{\pi^{10}}{93555}$	⋯

3.2 ζ (2n+1)

I could not put it together precisely and concisely.

$$\zeta(3) = \frac{\pi^3}{\Gamma(3)\ 2^4} \times \frac{2^3}{2^3 - 1} \times \frac{3^3 + 1}{3^3 - 1} \times \frac{7^3 + 1}{7^3 - 1} \times \frac{11^3 + 1}{11^3 - 1} \times \frac{19^3 + 1}{19^3 - 1} \times \frac{23^3 + 1}{23^3 - 1} \cdots$$

$$(n \geqq 2) \quad n \equiv 1 \pmod 2 \quad f(n) = \frac{T_n\ \pi^n}{\Gamma(n)\ 2^{n+1}} \times f(x)_\alpha \tag{3}$$

$$f(x)_{\alpha 1} = \frac{2^n}{2^n - 1} \times \frac{3^n + 1}{3^n - 1} \times \frac{7^n + 1}{7^n - 1} \times \frac{11^n + 1}{11^n - 1} \tag{4}$$

$$f(x)_{\alpha 2} = \frac{2^n}{2^n - 1} + \left(\frac{2^n}{2^n - 1} \times 2 \times \frac{T_n\ \pi^n}{\Gamma(n)\ 2^{n+1}} \times 3^{-n} \times \frac{e^n}{e^n - 1} \right) \tag{5}$$

$$f(x)_{\alpha 3} = \frac{2^n}{2^n-1} \times \frac{\alpha_n \times T_n \times f_{tn}(n)\sqrt{\Gamma(2n)}}{\sqrt{\alpha_{2n} \times T_{2n}} \times \Gamma(n)\sqrt{2}} \qquad \alpha_n = \frac{3^n}{3^n - \left(\frac{2^n}{2^n+\left(\frac{6}{5}\right)^n}\right)} \tag{6}$$

$$f_{tn}(n) = \frac{3^n+1}{3^n-1} \times \frac{7^n+1}{7^n-1} \times \frac{11^n+1}{11^n-1} \times \frac{e^{2n}}{e^{2n} - \left(\frac{19}{18}\right)^n}$$

n	(1) $\zeta(n)$	$(1)-(3)(4)$	$(1)-(3)(5)$	$(1)-(3)(6)$
3	$1.202056903\cdots$	7.30035×10^{-4}	8.36482×10^{-3}	-3.83030×10^{-4}
5	$1.036927775\cdots$	1.26273×10^{-6}	1.15083×10^{-4}	-1.80318×10^{-5}
7	$1.008349227\cdots$	2.93473×10^{-9}	2.13580×10^{-6}	-3.08151×10^{-7}
9	$1.002008392\cdots$	7.40497×10^{-12}	4.31260×10^{-8}	-4.75492×10^{-9}
- -				
19	$1.000001908212\cdots$	1.03781×10^{-24}	1.67328×10^{-16}	-3.14119×10^{-18}
- -				
39	$1.0\cdots004656629\cdots$	2.69086×10^{-50}	2.19333×10^{-33}	-2.54884×10^{-37}

4 L-function

(Leibniz series)

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots = \frac{\pi}{4}$$

(Dirichlet L-function)

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \qquad \chi(n) = \begin{cases} 0 & (n \equiv 0,2 \bmod 4) \\ 1 & (n \equiv 1 \bmod 4) \\ -1 & (n \equiv 3 \bmod 4) \end{cases}$$

(Euler L-function)

$$L(s) = \sum_{n \geqq 1:(Odd)}^{\infty} (-1)^{\frac{n-1}{2}} n^{-s} = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \frac{1}{9^s} - \cdots \quad [5] \tag{7}$$

$$L(s) = \prod_{p;(odd\ \ prime)} (1 - (-1)^{\frac{p-1}{2}} p^{-s})^{-1} = \frac{1}{1+3^{-s}} \times \frac{1}{1-5^{-s}} \times \frac{1}{1+7^{-s}} \times \cdots \quad [5]$$

$$L(2n+1) = \frac{E_{2n}}{(2n)!\ 2^{2n+2}} \pi^{2n+1} \qquad (n = 0,1,2\cdots) \qquad [6]$$

E_{2n} : (Euler number)

4.1 L(2n+1)

In the case of L (2n + 1), the correct value is obtained.

$$f(x)_\alpha = 1$$

$$(n \geqq 2) \quad n \equiv 1 \pmod 2 \quad f(n) = \frac{T_n\ \pi^n}{\Gamma(n)\ 2^{n+1}} \tag{8}$$

n	3	5	7	9	11	⋯
(7) $L(n)$	$\frac{\pi^3}{32}$	$\frac{5\pi^5}{1536}$	$\frac{61\pi^7}{184320}$	$\frac{277\pi^9}{8257536}$	$\frac{50521\pi^{11}}{14863564800}$	⋯
(8) $f(n)$	$\frac{\pi^3}{32}$	$\frac{5\pi^5}{1536}$	$\frac{61\pi^7}{184320}$	$\frac{277\pi^9}{8257536}$	$\frac{50521\pi^{11}}{14863564800}$	⋯

4.2 L(2n)

$$(n \geqq 2) \quad n \equiv 0 \pmod 2 \quad f(n) = \frac{T_n\ \pi^n}{\Gamma(n)\ 2^{n+1}}\ f(x)_\alpha \tag{9}$$

$$f(x)_{\alpha 1} = \times \frac{3^n - 1}{3^n + 1} \times \frac{7^n - 1}{7^n + 1} \times \frac{11^n - 1}{11^n + 1} \tag{10}$$

$$f(x)_{\alpha 2} = -2 \times \frac{T_n\ \pi^n}{\Gamma(n)\ 2^{n+1}} \times 3^{-n} \times \frac{e^n}{e^n - 1} \tag{11}$$

$$f(x)_{\alpha 3} = \times \frac{\sqrt{\alpha_{2n} T_{2n}} \times \Gamma(n)\sqrt{2}}{\alpha_n T_n \sqrt{\Gamma(2n)}} \qquad \alpha_n = \frac{3^n}{3^n - \left(\frac{2^n}{2^n + \left(\frac{6}{5}\right)^n}\right)} \tag{12}$$

n	(7) $L(n)$	(7) − (9)(10)	(7) − (9)(11)	(7) − (9)(12)
2	0.915965594 ⋯	-0.0159839	-6.69078×10^{-4}	-1.41466×10^{-2}
4	0.988944551 ⋯	-2.58753×10^{-5}	-2.12264×10^{-4}	-2.77082×10^{-4}
6	0.998685222 ⋯	-5.88042×10^{-8}	-7.57298×10^{-6}	-6.06204×10^{-6}
8	0.999849990 ⋯	-1.45902×10^{-10}	-2.07585×10^{-7}	-1.34902×10^{-7}
10	0.999983164 ⋯	-3.77056×10^{-13}	-5.04624×10^{-9}	-2.95733×10^{-9}
- -				
20	0.9⋯9971321⋯	-5.43741×10^{-26}	-2.37213×10^{-17}	-1.21506×10^{-17}
- -				
40	0.9⋯9917747⋯	-1.41610×10^{-51}	-3.13417×10^{-34}	-1.56918×10^{-34}

5 function summary

n $\equiv 0 \pmod 2 \quad (n \geqq 2)$

$$L(n) = \frac{\overset{(T)}{T_n}\ \pi^n}{\Gamma(n)\ 2^{n+1}} \begin{cases} -2\alpha \\ \times\alpha(3\beta)^n \\ \times\gamma^{(-1)^n} \end{cases} \qquad \zeta(n) = \frac{\overset{(T)}{T}_n\ \pi^n}{\Gamma(n)\ 2^{n+1}} \times \frac{2^n}{2^n - 1}$$

n $\equiv 1 \pmod 2 \quad (n \geqq 2)$

$$L(n) = \frac{\overset{(E)}{T}_n\ \pi^n}{\Gamma(n)\ 2^{n+1}} \qquad \zeta(n) = \frac{\overset{(E)}{T}_n\ \pi^n}{\Gamma(n)\ 2^{n+1}} \times \frac{2^n}{2^n - 1} \begin{cases} +\frac{2^n}{2^n - 1} \times 2\alpha \\ \times\alpha(3\beta^{-1})^n \\ \times\gamma^{(-1)^n} \end{cases}$$

$$(n \geqq 2) \quad \zeta(n) = \prod_{\substack{p;odd \\ prime}} \frac{p^n}{p^n - 1} \times \frac{2^n}{2^n - 1}$$

$$(n \geqq 2) \qquad L(n) = \prod_{\substack{p;odd\\prime}} \frac{p^n}{p^n-1} \times \gamma \quad = \quad \prod_{\substack{p;odd\\prime}} \frac{p^n}{p^n-1} \quad \times \quad \prod_{\substack{p;odd\\prime\\(p\equiv 3 \bmod 4)}} \frac{p^n-1}{p^n+1}$$

6 memo

$$\zeta(n) = \prod_{p;prime} (1-p^{-n})^{-1} = \prod_{p;prime} \frac{p^n}{p^n-1} \qquad [7]$$

$$\frac{\zeta(2n)}{\zeta(n)} = \prod_{p;prime} \frac{1-p^{-n}}{1-p^{-2n}} = \prod_{p;prime} \frac{p^n}{p^n+1} \qquad [7]$$

$$\frac{(\zeta(n))^2}{\zeta(2n)} = \prod_{p;prime} \frac{p^n}{p^n-1} \times \prod_{p;prime} \frac{p^n+1}{p^n} = \prod_{p;prime} \frac{p^n+1}{p^n-1} \qquad [7]$$

$$\tanh\left(\frac{x}{2}\right) = \frac{e^x-1}{e^x+1} \qquad \frac{e^n - \left(\frac{e}{3}\right)^n}{e^n + \left(\frac{e}{3}\right)^n} = \frac{3^n-1}{3^n+1} = \tanh\left(\frac{n}{2} \times \log(3)\right)$$

$$\text{Tangent number(n)} = 2^{n+1} \times (2^{n+1}-1) \times |B_{n+1}| \times (n+1)^{(-1)}$$

B_n = Bernolli number

$$\alpha = \sum_{k=0}^{\infty} \frac{1}{(4k+3)^n} \qquad \alpha = \frac{T_n\ \pi^n}{\Gamma(n)\ 2^{n+1}} \times f(x)_1 \qquad f(x)_1 = 3^{-n} \times \frac{e^n-1}{e^n}$$

$$\beta = (\tanh(n) \times f(x)_2)^{(-1)^n}$$

$$\gamma = \prod_{\substack{p;odd\ prime\\(p\equiv 3 \bmod 4)}} \frac{p^n-1}{p^n+1}$$

$$(a) \quad \prod_{\substack{p;odd\\prime}} \frac{p^{2n}}{p^{2n}-1} \times \prod_{\substack{p;odd\\prime}} \frac{p^n-1}{p^n} \quad \times \prod_{\substack{p;odd\\prime\\(p\equiv 3 \bmod 4)}} \frac{p^n+1}{p^n-1}$$

$$= \prod_{\substack{p;odd\\prime}} \frac{p^{2n}}{(p^n+1)(p^n-1)} \times \prod_{\substack{p;odd\\prime}} \frac{p^n-1}{p^n} \quad \times \prod_{\substack{p;odd\\prime\\(p\equiv 3 \bmod 4)}} \frac{p^n+1}{p^n-1}$$

$$= \prod_{\substack{p;odd\\prime}} \frac{p^n}{p^n+1} \times \prod_{\substack{p;odd\\prime\\(p\equiv 3 \bmod 4)}} \frac{p^n+1}{p^n-1}$$

$$(b) \quad \prod_{\substack{p;\\prime}} \frac{p^n-1}{p^n} \times \prod_{\substack{p;\\prime}} \frac{p^n-1}{p^n} \times \prod_{\substack{p;\\prime}} \frac{p^{2n}}{p^{2n}-1}$$

$$= \prod_{\substack{p;\\prime}} \frac{(p^n-1)^2}{p^{2n}} \times \prod_{\substack{p;\\prime}} \frac{p^{2n}}{(p^n+1)(p^n-1)} = \prod_{\substack{p;\\prime}} \frac{p^n-1}{p^n+1}$$

$$(c)\quad \prod_{\substack{p;odd\\prime}} \frac{p^n-1}{p^n} \times \prod_{\substack{p;odd\\prime\\(p\equiv 3\\ \text{mod } 4)}} \frac{p^n+1}{p^n-1} \times \prod_{\substack{p;odd\\prime}} \frac{p^n-1}{p^n} \times \prod_{\substack{p;odd\\prime\\(p\equiv 3\\ \text{mod } 4)}} \frac{p^n+1}{p^n-1} \times \prod_{\substack{p;odd\\prime}} \frac{p^{2n}}{p^{2n}-1}$$

$$= \prod_{\substack{p;odd\\prime}} \frac{(p^n-1)^2}{p^{2n}} \times \prod_{\substack{p;odd\\prime\\(p\equiv 3 \text{ mod } 4)}} \frac{(p^n+1)^2}{(p^n-1)^2} \times \prod_{\substack{p;odd\\prime}} \frac{p^{2n}}{(p^n+1)(p^n-1)}$$

$$= \prod_{\substack{p;odd\\prime}} \frac{p^n-1}{p^n+1} \times \prod_{\substack{p;odd\\prime\\(p\equiv 3\\ \text{mod } 4)}} \frac{(p^n+1)^2}{(p^n-1)^2} = \prod_{\substack{p;odd\\prime\\(p\equiv 1\\ \text{mod } 4)}} \frac{(p^n-1)}{(p^n+1)} \times \prod_{\substack{p;odd\\prime\\(p\equiv 3\\ \text{mod } 4)}} \frac{(p^n+1)}{(p^n-1)}$$

$$\prod_{\substack{p;odd\\prime\\(p\equiv 1 \text{ mod } 4)}} \frac{(p^3-1)}{(p^3+1)} \times \prod_{\substack{p;odd\\prime\\(p\equiv 3 \text{ mod } 4)}} \frac{(p^3+1)}{(p^3-1)} = \frac{32}{\pi^3} \times \frac{32}{\pi^3} \times \frac{16\pi^6}{\Gamma(6)\times 2^7} = \frac{16}{15}$$

$$(d)\quad \prod_{\substack{p;odd\\prime}} \frac{p^n}{p^n-1} = \frac{T_{odd(n)}\pi^n}{\Gamma(n)\times 2^{n+1}}$$

$$\prod_{\substack{p;odd\ prime\\(p\equiv 3 \text{ mod } 4)}} \frac{p^n}{p^n-1} = \sqrt{\frac{\alpha_n T_{odd(n)}\pi^n}{\Gamma(n)\times 2^{n+1}}} \qquad \prod_{\substack{p;odd\ prime\\(p\equiv 1 \text{ mod } 4)}} \frac{p^n}{p^n-1} = \sqrt{\frac{\beta_n T_{odd(n)}\pi^n}{\Gamma(n)\times 2^{n+1}}}$$

$$\prod_{\substack{p;odd\\prime}} \frac{p^n}{p^n-1} = \sqrt{\frac{\alpha_n T_{odd(n)}\pi^n}{\Gamma(n)\times 2^{n+1}}} \times \sqrt{\frac{\beta_n T_{odd(n)}\pi^n}{\Gamma(n)\times 2^{n+1}}} \qquad \sqrt{\alpha_n} \times \sqrt{\beta_n} = 1$$

$$\prod_{\substack{p;odd\\prime\\(p\equiv 3\\ \text{mod } 4)}} \frac{p^n-1}{p^n+1} = \left(\sqrt{\frac{\Gamma(n)\times 2^{n+1}}{\alpha_n T_{odd(n)}\pi^n}}\right)^2 \times \sqrt{\frac{\alpha_{2n} T_{odd(2n)}\pi^{2n}}{\Gamma(2n)\times 2^{2n+1}}} = \frac{\sqrt{\alpha_{2n} T_{odd(2n)}}\,\Gamma(n)\sqrt{2}}{\alpha_n T_{odd(n)}\sqrt{\Gamma(2n)}}$$

$$L(2n) \qquad \gamma = \frac{\sqrt{\alpha_{2n} T_{2n}} \times \Gamma(n)\sqrt{2}}{\alpha_n T_n \sqrt{\Gamma(2n)}} \qquad \alpha_n = \frac{3^n}{3^n - \left(\frac{2^n}{2^n + \left(\frac{6}{5}\right)^n}\right)}$$

$$\zeta(2n+1)$$

$$T_{odd(2n+1)} = T_{(2n+1)} \times f(x)$$

$$\prod_{\substack{p;odd\\prime}} \frac{p^{2n+1}}{p^{2n+1}-1} = \sqrt{\frac{\alpha_{(2n+1)} T_{(2n+1)} f(2n+1)\pi^{2n+1}}{\Gamma(2n+1)\times 2^{2n+2}}} \quad \times$$

$$\sqrt{\frac{\beta_{(2n+1)}T_{(2n+1)}f(2n+1)\pi^{2n+1}}{\Gamma(2n+1)\times 2^{2n+2}}} \qquad \sqrt{\alpha_{(2n+1)}}\times\sqrt{\beta_{(2n+1)}}=1$$

$$\prod_{\substack{p;odd\\ prime\\ (p\equiv 3\\ \mathrm{mod}\ 4)}} \frac{p^n+1}{p^n-1} = \left(\sqrt{\frac{\alpha_n T_n f(x)\pi^n}{\Gamma(n)\times 2^{2n+1}}}\right)^2 \times \sqrt{\frac{\Gamma(2n)\times 2^{2n+1}}{\alpha_{2n}T_{2n}\pi^{2n}}} = \frac{\alpha_n T_n f(x)\sqrt{\Gamma(2n)}}{\sqrt{\alpha_{2n}T_{2n}}\times\Gamma(n)\sqrt{2}}$$

$$\zeta(2n+1) \qquad \gamma^{(-1)} = \frac{\alpha_n T_n f(x)\sqrt{\Gamma(2n)}}{\sqrt{\alpha_{2n}T_{2n}}\times\Gamma(n)\sqrt{2}}$$

$$\alpha_n = \frac{3^n}{3^n - \left(\frac{2^n}{2^n+\left(\frac{6}{5}\right)^n}\right)} \qquad f(x) = \frac{3^n+1}{3^n-1}\times\frac{7^n+1}{7^n-1}\times\frac{11^n+1}{11^n-1}\times\frac{e^{2n}}{e^{2n}-\left(\frac{19}{18}\right)^n}$$

$$\gamma = \frac{\sqrt{\alpha_{2n}T_{2n}}\times\Gamma(n)\sqrt{2}}{\alpha_n T_n\sqrt{\Gamma(2n)}} \qquad \gamma^{(-1)} = \frac{\alpha_n T_n f(x)\sqrt{\Gamma(2n)}}{\sqrt{\alpha_{2n}T_{2n}}\times\Gamma(n)\sqrt{2}}$$

$$\alpha_n = \frac{3^n}{3^n - \left(\frac{2^n}{2^n+\left(\frac{6}{5}\right)^n}\right)} \qquad f(x) = \frac{3^n+1}{3^n-1}\times\frac{7^n+1}{7^n-1}\times\frac{11^n+1}{11^n-1}\times\frac{e^{2n}}{e^{2n}-\left(\frac{19}{18}\right)^n}$$

The formula is unlikely to contain π^{-n},so we expect that $L(n)$ can be represented by $L(n) = \pi^x \times f(n) \qquad (n = 2, 3, 4, \cdots)$

If $f(x)$ does not contain π^{-n}, $\zeta(2n+1)$ can be expressd as $\pi^x \times f(n)$.

I think the accuracy can be improved by improving the formula, but I don't know if I can find the exact value.

References

[1] https://translate.google.com google translation

[2] https://ja.wikipedia.org/wiki/Tangent number

[3] https://ja.wikipedia.org/wiki/Euler number

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