Fractional Calculus, Irreversible Time, and Hubble's Law

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There is no remembrance of former things; neither shall there be any remembrance of things that are to come with those that shall come after. (Ecclesiastes 1:11)

Abstract

In this essay we suggest that description of processes in classical physics based on differential calculus of integer order is an approximation. We propose use of fractional calculus with orders very close to integers to replace classical equations. We define time axiomatically in terms of set theory: time is partially ordered, can be continuous or discrete, homogeneous, and, generally, nonuniform. Use of fractional calculus predicts irreversible time, temporal indefinitism, and evolution of the Universe, as described by the Hubble's Law of expansion. Dimensionless Hubble and Cosmological constants are numerically equated to first and second fractional derivatives deviations from corresponding integers. The deviations are extremely small and their ratio corresponds to the Universe expansion deceleration parameter.

1. Introduction

Understanding and description of Nature relies heavily on Differential and Integral Calculus, based, in turn, on the concept of the derivative. This is the reason why in classical physics, the universe appears predictable: when initial state of a physical system is known, it should be possible to track its evolution indefinitely into the future and restore the past. This notion is held true whether using Newton's laws, Maxwell's equations, or Einstein's theory of general relativity to predict the evolution of the shape of space-time. So, the evolution process can be determined from initial data. However, use of calculus of integer orders is limited in its description ability of some phenomena, sometimes generating questionable entities or properties not observable or detectable in the Nature. To overcome this insufficiency to explain observable or not observable phenomena additional assumptions are required like mysterious dark matter and dark energy (clammed 95.5% of mater-energy balance composing of the Universe) just to be able to explain discrepancies of dominating theory with observations. The intent of this essay is to explore an alternative approach on formulation laws of physics by turning to Fractional Calculus. Fractional calculus is extensively utilized in all branched of science including physics, chemistry, astrophysics, cosmology etc. In most cases the fractional calculus is used to generalize, extend, interpolate, or extrapolate classical equations (1), (2). In this paper, we suggest to *reformulate* classical equations in fractional calculus terms.

2. Concepts of Time (3)

Was there time when there was no time and what was before that time?

Interest in time and its nature is based on expectations in terms of use and handling. Philosophers want to know what time is and how real it is; the scientists want to know the structure, properties of time, and relation to space and physical phenomena; science fiction promoters are looking for possibility of time travel. However, in reality, time seems irreversible, the past is gone, and time travel is not feasible.

i. The Wheel of Time

In ancient Indian philosophy (2nd millennia BC), a concept of time, the so-called "**wheel of time**", presents repeating cycles over the infinite life of the Universe – the "**hypertime**". The Universe endlessly goes through cycles of creation and destruction with each cycle lasting 4.320 billion years (compare with establishes Earth age of 4.54 billion years)

ii. Infinite Universe and Time

The early Greek philosophers (5th century BC) generally believed that the universe (and therefore time itself) was infinite with no beginning and no end. Many philosophers asserted that time is not a reality, but a concept or mental construction, They saw time (as well as motion) as an illusion because, Some notions were based on believes that reality was limited to what exists in the here and now, and the past and future are unreal and imaginary. Zeno's logical constructions - **Zeno's Paradoxes** - were devised to support doctrine of time as an illusion, ironically lead to absurdity. However, other contemporary philosophers, like Heraclitus, firmly believed that the flow of time is real and the very essence of reality. Solving Zeno's paradoxes contributed to development of differential and integral calculus.

iii. Creation of Time

Starting with Plato, in the 4th Century BC, believes were evolving that time was created by the Creator at the same instant as the heavens. But, be more scientific, Plato identified time with the periods of motion of the heavenly bodies. Plato was also aware of the so-called "**Great Year**", a complete cycle of the equinoxes around the ecliptic (effectively the return of the planets and the "fixed stars" to their "original relative" positions, the Earth precession, a process that takes about 25,800 years). Pythagorean and some Stoic philosophers saw the end of this cycle as the end of time itself, after which history would start to repeat itself all over again in an endless repetition (see **The Wheel of Time**).

iv. Back to Infinite Universe and Time

Aristotle saw time as something that does not exist on its own but is relative to the motions of things. He called time "the numeration of continuous movement" or "the number of change in respect of before and after". He saw time as the measure of change and believed that, although space was finite, time was infinite, and that the Universe has always existed and will always exist. Aristotle's view that time is **continuous** (not discrete) was adopted by Newton and incorporated and in his mechanics.

v. Creationism again: Modern Scientific and Creationist Point of View on Time and Universe

Influenced by Judaeo-Christian faith the medieval philosophers and theologians developed the concept of Universe with **finite past** and definite **beginning**: an omnipotent **God** created world and time, therefore, they are finite. The concepts of **creation** exist today, though not all versions are as literal as those of medieval philosophers. In the modern world both, **evolutionists** (scientists) and **creationists** view time as linear and finite: a doctrine of **temporal finitism -** timeline for Universe includes a beginning and an end. The creationist views timeline in thousands of years with creation and following destruction by **God's** will. The evolutionist evaluate the age of the Universe in billions of years with the beginning (creation) in a **cataclysmic** event (Big Bang) and destruction by another **cataclysm** (Big Crunch, Big Freeze, Big Bounce, etc,). Some creationists accept the "scientifically proven" age of the Universe.

3. Reversibility of Time

The time reversibility is usually seen as Process Reversibility and Time Travel, include travel to the past and into the future. In science, time reversibility is associated with reversed processes for negative time values. Indeed, practically, all physical laws (including thermodynamics and quantum physics) are time direction invariant **(4)**. This is so the called **T-symmetry (5)** or **time reversal symmetry**. Even the Second Law of thermodynamics **(6)** – Entropy Increase – is a statistical law – it does not exclude probability (as small as it could be) of rearranging system into its original state. In quantum mechanics existence of antiparticles sometimes is interpreted as particles moving backward in time (7).

4. Axiomatic Approach: basic properties of Time

What is Time?! It seems something almost tangible, but elusive and even allusive, escaping mental grip and deceiving intuition about it. We will not define what is time, but postulate its properties.

- one-dimensional
- omnipresence
- order
- continuity
- homogeneity
- uniformity

4.1 Omnipresence of Time

Time is everywhere, but where? In space, supposedly! How to formalize this property?

4.2 Time as Order

Intuitively, time is an ordered entity. Thus, it can be considered as an ordered set. The order on a set is defined by four specific binary relations. In general, an "element" of time before most "recent element" is identified as preceding, and the term "following" is excluded from binary relation presented below.

Definition of Order (8)

O1. ***reflexivity*** $\boldsymbol{a \le a}$ meaning: element ***a*** precedes or coincides with itself
O2. ***anti-symmetry*** **if** $\boldsymbol{a \le b}$ **and** $\boldsymbol{b \le a}$**, then** $\boldsymbol{a=b}$; meaning: if element ***a*** precedes or coincides with ***b*** and element ***b*** precedes or coincides with ***a***, then ***a*** coincides with ***b***
O3. ***transitivity*** **if** $\boldsymbol{a \le b}$ **and** $\boldsymbol{b \le c}$**, then** $\boldsymbol{a \le c}$; meaning: if element ***a*** precedes or coincides with ***b*** and ***b*** precedes or coincides with ***c***, then ***a*** precedes or coincides with ***c***
O4. ***comparability***: **for all elements either** $\boldsymbol{a \le b}$ **or** $\boldsymbol{b \le a}$; or in terms ***trichotomy***: **for** $\boldsymbol{a<b}$**,** $\boldsymbol{a=b}$**, or** $\boldsymbol{b<a}$**;** meaning: or ***a*** preceding ***b***, or ***b*** proceeding ***a***, or ***a*** coinciding with ***b***

Conditions 1) and 2) define ***pre-order*** or ***quasi-order***
Conditions 1), 2), and 3) define ***partial order*** and a partial order set is called ***poset***
Conditions 1) through 4) define ***strict*** or ***total order***

The universal and absolute time, implied in Newton's equations, obeys all four order conditions and ***Newton's time is a strict order.*** On the other hand, in general (and special) relativity theory in different inertial and not inertial reference frames different time flow not the same due to relativity effects like dilation. In these situations the trichotomy (**O4.**) condition will not hold, the time follows the first 3 relation, thus, it is partially ordered, and the ***Einstein time should be qualified as partial order.***

Time Topology

The last three properties (continuity, homogeneity, and uniformity) are related to topology: The concept of topology is based on a notion of ***neighborhood***. Intuitively, time appears to be one dimensional entity. It should be evident, that as in an ordered set any point in the past (exuding the most recent “now”) should have two closest neighbors: preceding and following. The “now” has only one neighbor – the preceding moment, Newtonian and Einstein times are expected to have all the point from $t = -\infty$ to $t = +\infty$ with 0 at an arbitrary point.

Definition of Neighborhood (9)

A set $U \subset R$ is a ***neighborhood*** of a point $x \in R$ if $U \supset (x - \varepsilon, x + \varepsilon)$ for ($\varepsilon \to 0$ & $\varepsilon > 0$). The ***open interval*** $(x - \varepsilon, x + \varepsilon)$ or equivalently $(x - \varepsilon < c < x + \varepsilon)$ is called a ***ε-neighborhood*** of x.

In words, a subset U of the set R is a ***neighborhood*** of a point x if the subset U includes vicinity of x such as x is between $x - \varepsilon$ and $x + \varepsilon$ when $\varepsilon > 0$ and $\varepsilon \to 0$.

4.3 Continuous or Discrete Time

Two types of sets satisfy this neighborhood condition: real numbers presenting continuum and discrete system based on integer numbers. In general, a neighborhood is required for differentiation and integration. In view of Riemann–Stieltjes integral (10),

Eq.1. $$RS \stackrel{\text{def}}{=} \int_a^b f(x)\,du(x) \quad \text{vs} \quad R \stackrel{\text{def}}{=} \int_a^b f(x)\,u'(x)\,dx$$

based on differentials vs derivatives of the classical Riemann integral, the continuity of time is not necessary nor essential: time ***can be continuous or discrete***

4.4 Homogeneity of Time

Fundamentally, time structure or topology can be expected to be the same everywhere and at all times (looks like tautology): ***time should be homogeneous***

4.5 Uniformity of Time

In **Newton’s physics** time is absolute and uniform on all scales and frames of reference – local and global – independent on on forces or movements. But, according to **General Relativity**, only two types of frames of reference can be expected to have uniform time: ***inertial frames*** and those ***containing constant force fields***, like gravitation. An ***inertial frame*** is a frame of reference in which a body is at rest or is moving at a constant speed in a straight line. In inertial frames time dilation depends on relative velocities. In force field time dilation is determined by its strength. When state of a frame of reference is changing time is not uniform until an inertial or constant force field states are established.

But what is **uniformity**? Being seemingly a simple concept, the uniformity definition in mathematics is quite involved and based on concept of a topological space. In Set theory (11), there are three equivalent definitions based on concept of ***entourage*** (surrounding). Use of those definitions is not trivial.

Definition of uniformity in terms of differentials

A simple and intuitive definition of uniformity is formulated in this essay in terms of first order differentials. Introducing a local self-derivative to be called 1st order ***s-derivative*** as a ratio of two sequential differentials of time makes explicit use of a topological neighborhood.

Eq.2. $t^{(1)} \overset{\text{def}}{\equiv} \lim\limits_{\Delta t \to \partial t} \dfrac{\partial t}{\partial t(\Delta t)} = \dfrac{dt}{dt(\partial t)} \equiv t^{(1)}_{\partial t}$

Generalization of **s-differentials** to arbitrary orders can be presented as:

Eq.3. $dt(\partial t) \cdot t^{(1)}_{\partial t} = dt$ and $dt(\partial t) \cdot t^{(\beta)}_{\partial t} = dt^{\beta}; \ \beta > 0$

Now, the time is uniform when 1st s-derivative equals 1 and all other derivatives are 0

Eq.4. $t^{(1)} \equiv 1$, then $dt(\partial t) \equiv dt;$ and $t^{(\beta)} \equiv 0$ for all $\beta \neq 1$.

As it was mentioned before, generally, ***time is not uniform***. For mathematical description of physical processes related to time we intend to use **differential equations,** (slightly) **deviating** from **integer orders**, or, in other words, **Fractional Calculus**.

6. Fractional Calculus (12), (13) (14),

To proceed with analysis we turn to fractional calculus using Riemann-Louisville integral transform ${}_cI^{\alpha}_t f(t)$ for *left fractional integral of order **α** (real number)* from a *function **f (t).*** Using standard unified notation as derivative of a negative order (-***α***) this integral is defined in a following way

Eq.5. ${}_cI^{\alpha}_t f(t) = {}_cD^{-\alpha}_t f(t) = \dfrac{1}{\Gamma(\alpha)} \int\limits_c^t (t-\tau)^{(\alpha-1)} f(\tau)\, d\tau,\ \alpha \geq 0,\ c < t$

To illustrate relation of time uniformity with fractional calculus this equation is presented in Riemann–Stieltjes form:

Eq.6. $(t-\tau)^{(\alpha-1)} d\tau = d\theta^{\alpha-1} \rightarrow {}_cI^{\alpha}_t f(t) = {}_cD^{-\alpha}_t f(t) = \dfrac{1}{\Gamma(\alpha)} \int\limits_c^t f(\theta - t)\, d\theta^{\alpha-1},\ \alpha \geq 0$

The *left fractional derivative of order **α*** from a function ***f (t)*** is defined as first integer derivative of a left fractional integral as following:

Eq.7. ${}_cD^{\alpha}_t f(t) = \dfrac{d^n}{dt^n} {}_cI^{n-\alpha}_t f(t) = \dfrac{d^n}{dt^n} {}_cD^{-n+\alpha}_t f(t) = \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dt^n} \int\limits_c (t-\tau)^{(n-\alpha-1)} f(\tau)\, d\tau,\ \alpha \geq 0,\ n \in \mathbb{N},\ n-\alpha \subset (0,1)$

As it follows from equation above, the fractional derivative is not local, but for integer orders it becomes local, as it supposed to be. At any value of the lower limit (the past) including -∞, this integral is a constant, and integer derivative of a constant is 0. Consequently, at the lower (left) limit the fractional derivative Eq.7. , “disappears” or becomes irrelevant. Applied to time this could mean, that the notion of finite (had beginning) or infinite (always existed) time is, actually, irrelevant. This makes it impossible to trace time back to its “beginning”. Generally, the lower limit can be different for integrals in different situation and for multiple integration attention should be paid how the lower limit are handled. But, wherever it is unambiguous, the subscripts will be omitted

Eq.8. $$D^{\alpha} f(t) \stackrel{\text{def}}{=} \frac{d^{\alpha}(f)}{dt^{\alpha}} \equiv \frac{1}{\Gamma(-\alpha)} \int_{c}^{t} (t-\tau)^{(-\alpha-1)} f(\tau) d\tau , \quad \alpha \geq 0$$

It should be noted, that unlike derivatives of integer orders, fractional derivatives are not local. Sequential application of differentiation and/or integration is commutative

Eq.9. $$D^{\pm\beta} D^{\pm\alpha} f(t) = D^{\pm\alpha} D^{\pm\beta} f(t) = D^{\pm\alpha\pm\beta} f(t), \ \alpha \geq 0, \ \beta \geq 0, t \geq 0 \quad ,$$

and exhibits “identity” property

Eq.10. $$D^{\pm\alpha} D^{\mp\alpha} f(t) = D^{0} f(t) = f(t), \ \alpha \geq 0, t \geq 0 \quad .$$

For integer values of $\alpha, \beta, \alpha \pm \beta = \pm 1,2, \ldots N$ fractional calculus converges to classical integer. For power functions such as

Eq.11. $$f(t) = C \cdot t^{\mu}, \ \mu > -1, \ C \in \mathbb{R}_{<0}, \ \mathbb{R}_{<0} = \{ x \in \mathbb{R} \mid x > 0 \}$$

where C is a constant with positive real value, the fractional integral and derivative *Power Rule* can be respectively defined as recursive formula with commutative property for sequential application differentiation and integration (12):

Eq.12. $$D^{\pm\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-(\pm\alpha)+1)} t^{\mu-(\pm\alpha)}, \ \alpha > 0, \ \mu > -1, \ t \geq 0$$

Few recursive properties of Gamma function to be used in this work.

Eq.13. $$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) \quad , \quad \Gamma(\alpha+1) = \alpha\,\Gamma(\alpha) \quad , \text{or} \quad \Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha}$$

Using Taylor series $\varepsilon \rightarrow 0$ we can get values of Gamma function for small arguments

Eq.14. $$\Gamma(\varepsilon) = \frac{\Gamma(\varepsilon+1)}{\varepsilon} = \frac{1}{\varepsilon} - \gamma + \varepsilon \frac{6\gamma^{2}+\pi^{2}}{12} + O(\varepsilon^{2}), \ \varepsilon \rightarrow 0$$

where $\gamma = 0.57721\ldots$ is the Euler–Mascheroni constant (also called Euler's constant). Further approximations yield

Eq.15. $\Gamma(\varepsilon)\approx\frac{1}{\varepsilon}-\gamma+\varepsilon\cdot 0.9890\ldots\approx\frac{1}{\varepsilon}+\varepsilon-\gamma\approx\frac{1}{\varepsilon},\ \varepsilon\rightarrow 0$

I. Classic Newton's Law and Reversible Time

Mathematically, time reversal means replacing the time ***t*** by **−*t***. This way one can verify whether the physical law is unchanged under time reversal. Almost all physical laws are time reversible.
As an example let us consider Newton's second law which governs the motion of bodies in classical mechanics (4). The classical equation of the second Newton's law is

Eq.16. $F=m\frac{d^2x}{dt^2},\ \frac{d^2x}{dt^2}=\frac{F}{m}=a,\ \frac{F}{m}=\frac{dv}{dt},\ F=const$

where ***m*** is a mass, ***F*** → force, ***a*** → acceleration, and ***v*** → speed. Integrating, we've got

Eq.17. $v=\int_0^t\frac{F}{m}dt=\frac{F}{m}t+v_0$

with $\mathbf{v_0}$ as the starting speed (**t=0**). Recalling that $v=\frac{dx}{dt}$ **,** and integrating again, we have

Eq.18. $x=\int_0^t(\frac{F}{m}t+v_0)dt=v_0t+\frac{1}{2}\frac{F}{m}t^2+x_0$ and finallly $x=v_0t+\frac{1}{2}\frac{F}{m}t^2+x_0$

The Newton's law does not change because it is insensitive to the substitution of $\mathbf{(+t)^2}$ by $\mathbf{(-t)^2}$. Mathematically this means that when ***x*(*t*)** is a solution, then so is ***x*(−*t*)**. Physically, this means that an object following a certain trajectory for positive times will retrace its original trajectory backward when time is reversed, thus following reversed process.

II. Fractional Force in Newton's Law and Irreversible Time

The Newton's equation above is based on assumption that the force is proportional **precisely** to second derivative of displacement $a=d^2x/dt^2$. This is seemingly very good description, but in this work we suggest that the force could be proportional to fractal derivative sightly deviating from 2 by λ→±0, and the first derivative slightly deviating from 1 by η→±0.

Eq.19. $F=m\frac{d^{2+\lambda}x}{dt^{2+\lambda}}=mD^{2+\lambda}x(t),\frac{d^{2+\lambda}x}{dt^{2+\lambda}}=\frac{F}{m}=a,\frac{F}{m}=\frac{d^{1+\eta}v}{dt^{1+\eta}},\ F=const,$

Clearly, when λ=0 and η=0 the equation becomes classical Newton's law. Applying fractional integral Eq.12. to modified Newton's second law once and using values of μ=0 and α=−(1+η), and recalling lower limit, we obtain:

Eq.20. $v=\frac{F}{m}\frac{\Gamma(1)}{\Gamma(1+\eta+1)}t^{1+\eta}+v_{c1}=\frac{F}{m}\frac{t^{1+\eta}}{\Gamma(2+\eta)}+v_{c1}\approx\frac{F}{m}t^{1+\eta}+v_{c1},\ \eta\rightarrow\pm 0,$

For positive time the speed $v=(F/m)t^{1+\eta}$ has **one** real value, while for negative values expression $v=(F/m)(-t)^{1+\eta}$ **does not have real values**, but only infinite number of complex roots.
This effect is evident by presenting numbers in complex exponential form. For a positive number *k* and a negative number -*k* it can be written respectively

Eq.21. $k=|k|(+1)^{\alpha}=|k|e^{i(0+2n\pi)\alpha}$ and $-k=|k|(-1)^{\alpha}=|k|e^{i(\pi+2n\pi)\alpha}$

where α is an arbitrary real number and *n* are natural numbers including zero: *n=0,1,2...*
Thus, for positive time values we have real value solution for *n*=0

Eq.22. $t^{1+\eta}=|t|(1)^{1+\eta}=|t|e^{i(0+2n\pi)(1+\eta)}$, or $|t|e^{i(0)(1+\eta)}=t$,

and infinite number of complex solutions with no physical meaning while for all other values *n=1,2...*:

Eq.23. $t^{(1+\eta)}=|t|(1)^{(1+\eta)}=|t|e^{i(2n\pi)(1+\eta)}$

For negative time values no real solutions exists, but only complex ones for all values of *n=0,1,2…*:

Eq.24. $-t^{(1+\eta)}=|t|(-1)^{1+\eta}=|t|e^{i(\pi+2n\pi)(1+\eta)}$

Applying fractional integral Eq.12. second time with μ=0 and α=−(1+δ), and considering that generally, deviations from integer powers don't have to be the same, we obtain for displacement

Eq.25. $x=\frac{F}{m}\frac{\Gamma(1+\eta+1)}{\Gamma(1+\eta+1+\delta)}t^{2+\eta+\delta}+v_{c1}\frac{\Gamma(1)}{\Gamma(2+\delta)}t^{1+\delta}+x_{c2}\approx\frac{F}{2m}t^{2+\lambda}+vt^{1+\delta}+x_{c2},\quad \lambda=\eta+\delta$

As demonstrated above, we have real solution t > 0 only, but not for t < 0, thus for negative times and fractional derivatives the Newton's law doesn't hold:

Eq.26. $m\frac{d^{2+\lambda}x}{dt^{2+\lambda}}=F(x)\rightarrow m\frac{d^{2+\lambda}x}{d(-t)^{2+\lambda}}\neq F(x)$

III. "Fixed" Distance in the Universe

Moving in space requires speed which is the first derivative of displacement. In classical mechanics two bodies separated by some fixed distance $S=constant$ will remain at this distance as long no forces are applied. Mathematically this means: first derivative by time of displacement (relative speed) is zero. Indeed, presenting distance $S=S(t^0)$ with explicit (in)dependence on time *t* and using Power Rule for derivative, we can express this statement by the following equation

Eq.27. $D^1S(t^0)=S\frac{\Gamma(0+1)}{\Gamma(0-1+1)}t^{0-1}=S\frac{\Gamma(1)}{\Gamma(0)}\frac{1}{t^1}=\frac{S}{t^1\cdot\infty}=0,$ or $\frac{dS(t^0)}{dt}=0,$

As it was expected the 1st order integer derivative from a constant equals zero and seems consistent with everyday observation. To whatever extend it could appear strange, it will be shown that using fractional calculus unexpectedly reveals phenomena observed on cosmological scale. Applying fractional derivative Eq.12. by time from a fixed distance S_0 we obtain (recall Eq.15.)

Eq.28. $$\frac{d^{1+\eta}(S)}{dt^{1+\eta}}=\dot{S}=D^{1+\eta}S(t^0)=S\frac{\Gamma(0+1)}{\Gamma(0-(1+\eta)+1)}t^{0-(1+\eta)}=S\frac{\Gamma(1)}{\Gamma(-\eta)}\frac{1}{t^{(1+\eta)}}\approx\frac{S\cdot(-\eta)}{t^{(1+\eta)}}$$

Obtained formula has dimension of velocity and reflects relative positional change of objects or, as it is called in cosmology, recessional velocity v_r .

Eq.29. $$v_0\approx\frac{S\cdot\eta}{t^{(1-\eta)}}=\frac{\eta}{t^{(1-\eta)}}\cdot S\sim v_r$$

The order of fractional derivative has been chosen $(1-\eta)<1$ to comply with observed expansion of the Universe when the objects are moving away from the observer, thus defining recessional velocity as positive. In cosmology the recessional velocity is defined by the Hubble’s Law as a product of Hubble parameter H and separation between objects

Eq.30. $v_r=H\cdot S$, then $\frac{\eta}{t^{(1-\eta)}}\sim H$ and $\eta=H\,t^{(1-\eta)}\approx H\,t$

This equation indicates, that the farther is the astronomical object from the observer the higher is the velocity. For very large distances, comparable with the radius of the observable Universe, increasing velocity will reach the speed of light. Indeed, this is how the observable radius of the Universe is defined. An object positioned far enough that emitted light, traveling during time equal to the age of the Universe, cannot reach the observer yet. The age of the universe is estimated to be around *T*=13.80±0.02 billion years, and beyond distance $R_O=13.8\,Gly=4.25\,Gpc$ (Gig-parsec) the Universe is expanding at speed exceeding speed of light *c*. The distance $R_0=c\cdot A_U$ is defined by the age of the universe $A_U\approx H^{-1}$ as inverse of Hubble Parameter H . The actual radius of the observable Universe is calculated to be lager $R_U=14.25\,Gpc=46.5\,Gly$, because we see the light emitted when Universe had radius R_O .

The value of η can be estimated by using known value of Hubble constant. The measurements from the Hubble Space Telescope and the Gaia space telescope together showed that the rate of expansion is ~73.5 *km/s/Mpc.* But the more distant background universe, according to previous measurements from the Planck telescope, is moving somewhat slower at ~67.8 *km/s/Mpc*. Though, this discrepancy is of serious concerns and debates among astrophysicists, for the purpose of this paper, the most acceptable value of $H=67.8\,km/s/Mpc$ will be used. This means that a galaxy, far away from Earth, moves away with speed 67.8 km/s faster for every *Mpc* = 3.26 *Mly* = $3.086\cdot10^{19}$ km. Presenting Hubble's constant in time units we have $H=67.8\,km/s/3.09\cdot10^{19}=2.195\cdot10^{-18}s^{-1}$. To evaluate η from Eq.30. , a “characteristic or fundamental” time is required. The Hubble's law is expected to be universal and to be applicable on all spatial scales, so the Plank’s time $t_p=\sqrt{\hbar G c^{-5}}=5.39\cdot10^{-44}s$ can be used. Thus, we find the value for of the deviation η of the first derivative from one is numerically equal to the Hubble parameter expressed in Plank’s time units.

Eq.31. $$\eta=\frac{v_r}{S}\cdot t_p^{(1-\eta)}=H\cdot t_p^{(1-\eta)}=2.195\cdot10^{-18}\cdot5.39\cdot10^{-44}\approx1.18\cdot10^{-61}t_p^{-1}=H_P$$

The second fractional derivative Eq.12. from a “fixed” distance using Eq.13. and Eq.15. yields:

Eq.32. $$D^{2+\lambda}S=\ddot{S}=\frac{S\cdot\Gamma(1)}{\Gamma(-(2+\lambda)+1)t^{2+\lambda}}=\frac{S}{\Gamma(-(1+\lambda))t^{2+\lambda}}\approx\frac{S\cdot(-\lambda)\cdot(-(1+\lambda))}{t^{2+\lambda}}\approx\frac{S\cdot\lambda}{t^{2+\lambda}}$$

The fractional acceleration is proportional to the distance and has real value for positive time. In general, the second fractional derivative with deviation from the integer ($\lambda\neq 2\eta$).

IV. Hubble Cosmological and Deceleration Parameters

The Hubble parameter describes expansion (evolution) of the Universe

Eq.33. $$H\equiv\frac{\dot{S}}{S}\qquad H=D^{1-\eta}S=\dot{S}=\frac{S}{\Gamma(-\eta)\cdot t^{1-\eta}}$$

The evolution of Hubble parameter, associated with the second derivative $\ddot{S}$ is (Eq.32.)

Eq.34. $$\dot{H}=\frac{\ddot{S}}{S}-\frac{\dot{S}\cdot\dot{S}}{S^2}=\frac{\ddot{S}}{S}-H^2$$

The evolution of Hubble parameter $\dot{H}$ is described by a deceleration parameter *q* defined as

Eq.35. $$\dot{H}=-H^2\cdot(1-\frac{\ddot{S}}{S\cdot H^2})=-H^2\cdot(1+q)\ \text{ and }\ q\overset{\text{def}}{=}-\frac{\ddot{S}\cdot S}{(\dot{S})^2}=\frac{-\ddot{S}}{S\cdot H^2}$$

The definition of “accelerating” expansion is that the second time derivative of the cosmic scale factor $\dot{S}$ is positive, which implies that the deceleration parameter is negative. Finally, the deceleration parameter can be expressed as a ratio second derivative deviation to square of the first derivative deviation from integer values:

Eq.36. $$q=-\frac{\ddot{S}\cdot S}{(\dot{S})^2}=-\frac{\Gamma^2(-\eta)}{\Gamma(-(1+\lambda))}\approx-\frac{-\lambda}{(-\eta)^2(-1-\lambda)}\approx-\frac{\lambda}{\eta^2}$$

As it follows from this equation the deceleration parameter is positive when the order of second derivative is $2-\lambda<2$ indicating that the Universe should expand with decreasing rate, and it is negative when $2+\lambda>2$ indicating accelerating expansion. And, indeed, until recently deceleration of Universe expansion was considered well established. However, according to now-days cosmological models, the Universe went through phases of acceleration and deceleration. The **accelerating** or **decelerating expansion of the Universe** is characterized by continuously **increasing** or **decreasing** with time speed at which a distant galaxy is receding from the observer. The latest observations suggest that the expansion evolution during age of the Universe should be classified as follows: **accelerating** in the very early universe, then **decelerating** for most of its age (about 8 billion years) until “recently” (6 billion years ago), and **accelerating** again since then until present (see Figure 1). The accelerated expansion was discovered in 1998 and is accepted by astronomers and astrophysicists. To explain this intermittent behavior of the Universe expansion the ideas of “dark matter” and “dark energy” were introduced. But, as of now, no other properties or manifestations of these mysterious entities have been detected or observed.

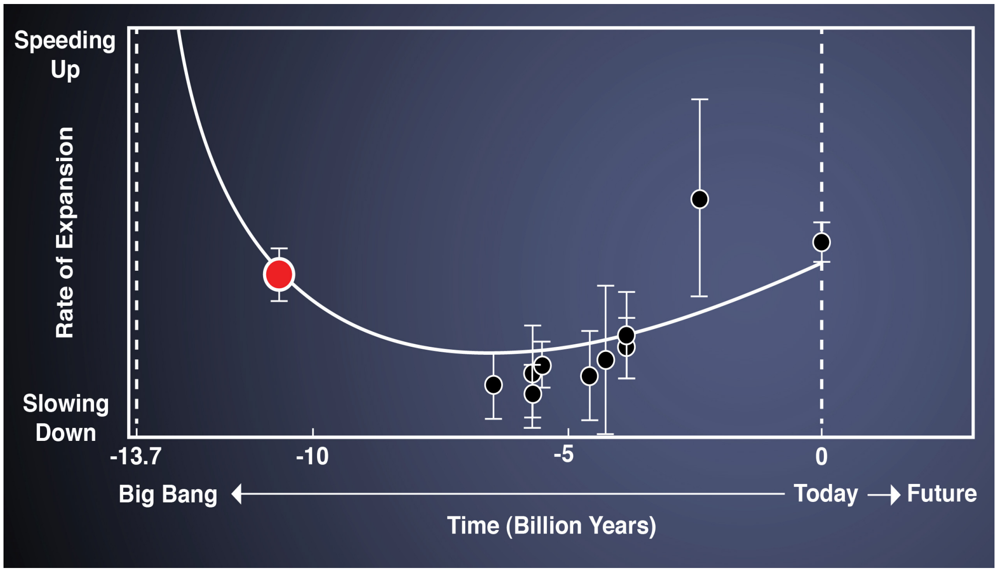

Figure 1, Universe expansion last 13.8 Billion Years (*Illustration by Z. Rostomian (LBNL)/N. Ross & BOSS Lyman-alpha team (LBNL)* ((15))

Data presented in this plot (excluding outlier at ~2.5 billion years ago) have been used to evaluate evolution of the Universe during last 11 billion years by fitting a second order polynomials. These equations describe Hubble parameter $H(t)$, its change $\dot{H}(t)$, and deceleration parameter q respectively:

Eq.37. $H(t)=4.28\cdot 10^{-53}t^2-2.13\cdot 10^{-35}t+3.39\cdot 10^{-18}\,[s^{-1}],\ R^2=0.94$

Eq.38. $\dot{H}(t)=8.56\cdot 10^{-53}t-2.13\cdot 10^{-35},\,[s^{-2}]$

Eq.39. $q=-\dfrac{\ddot{S}\cdot S}{(\dot{S})^2}\approx-\dfrac{\lambda}{\eta^2}=\dfrac{\Lambda}{\mathrm{H}^2}$ and $\lambda=q\cdot\eta^2=4.0$

Now-days (using Planck’s time units) with $\mathrm{H}=\eta=1.18\cdot 10^{-61}t_p^{-1}$ and deceleration parameter $q=4.0$ (calculated using these equations) the cosmological parameter found to be (16) $\Lambda=\lambda=5.6\cdot 10^{-122}t_p^{-2}$.

V. Cosmology Discussion

The Universe evolution, as considered in modern cosmology, is based on the Big Bang theory, so the Universe is finite in time and space and its expansion is controlled by inertia, gravitational forces (mostly mysterious dark matter) and energy field (mostly mysterious dark energy). It was shown in this work, that use of fractional calculus predicts evolution of the Universe without Big Bang, inertia, forces or energy fields. In view of a fact that left fractional derivative integral is 0 at the lower limit (Eq.7.) “beginning of time disappears” or becomes irrelevant. This observation is of a significant importance when analyzing evolution of processes in time. Specifically, absolute 0 (origin) on time axis and current events on this axis cannot be determined nor defined, thus, notion of **temporal finitism** becomes nonsensical in Wittgenstein's terms (17): the concept cannot be defined in its own terms without creating a logical paradox. In all fairness, “absence” of lower limit of fractional derivative, applied to cosmology, does not necessarily indicates infinite time in the past either. In this respect, the choices should not be limited to classical dichotomy: are time and age of Universe finite or infinite. The third option could be ***irrelevance***. Following the approach discussed in this essay, there is no way to determine if the beginning was in the infinity (no beginning) or at some finite point in the past. Mathematically, this is a consequence of fractional derivative being independent on its lower limit. What, if indeed, the Nature cannot be described by concepts based on of temporal finitism-infinitism dichotomy, but rather as indefintism? If so, the future cant' be predicted precisely and the past never would be possible to restore uniquely.

VI. Summary:

- Today, the laws of physics are formulated in terms of differential and integral calculus with integer orders leading to time in-variance or symmetry
- In this essay we suggest that integer calculus in physics is an approximation and propose to use fractional calculus with orders close to integers to reformulate classical equations
- This approach yielded following results
  - Time is irreversible and can flow only in positive direction
  - Global Time is not necessarily uniform and this non-uniformity can be observed on cosmological scale as evolution of the Universe
  - Temporal finitism/infinitism concept could be irrelevant, and indefintism seems to be more appropriate: beginning of time cannot be determined or defined
  - Obtained equations based on fractional first and second derivatives suggest that Universe can't be stationary, but should expand or contract (without any forces or inertia) with a rate proportional to the distance from the observer predicting Hubble's cosmology law
  - Behavior of the Universe, as experimentally established by the Hubble's Law of expansion and acceleration, corresponds to first derivative being slightly less than **1** (one)and second derivative slightly more than **2** (two)
  - The deviation of the first derivative from the integer η is <u>negative</u> and is numerically equal to the Hubble Parameter in Planck’s time units $\mathbf{H_P}$**:** $\boldsymbol{\eta = H_P = -1.18 \cdot 10^{-61}}$
  - The deviation of the second derivative from the integerλ is <u>positive</u> and is numerically equal to the Cosmological Parameter in Planck’s time units $\boldsymbol{\Lambda_P}$**:** $\boldsymbol{\lambda = \Lambda_P = +5.6 \cdot 10^{-122}}$
  - The deceleration parameter is expressed as a ratio of the second derivative deviation to square of the first derivative deviation from their integer values, $\boldsymbol{q = \lambda/\eta^2}$ or equivalently, as a ratio of Cosmological Parameter to squared Hubble Parameter**:** $\boldsymbol{q = \Lambda/H^2}$
  - Hubble and Cosmological Parameters slowly vary with time, what can be associated with change of fractal derivatives deviation from integers

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