An optimization approach to Fermat’s last theorem

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Introduction to Fermat’s last theorem
The so-called Fermat’s last theorem is actually a conjecture that was formulated by Pierre de Fermat in 1637 where he stated that the Diophantine equation \( x^n + y^n = z^n \), with \( x, y, z \) and \( n \) positive integers, has no nonzero solution for \( n > 2 \). This conjecture was one of the most famous unsolved problems of mathematics for over three and a half centuries. Early on, the following few specific cases were proved [1]: Fermat, for \( n = 4 \), Euler for \( n = 3 \), Dirichlet and Lagrange for \( n = 5 \), Lamé for \( n = 7 \), and Dirichlet for \( n = 14 \). The proof was eventually extended by other mathematicians to cover all prime exponents up to four millions [2]. This conjecture was finally proven by Andrew Wiles in late 1994 [2] using very long and complex analyses. In this note, we present a direct, short and easy to grasp solution based on the following optimization approach.

Problem formulation and solution
Let’s consider Fermat’s Diophantine equation \( D(x,y,z;n) \) (1) and the associated function \( F(x,y,z;n) \) (2)

\[
D(x,y,z;n) = x^n + y^n = z^n \\
F(x,y,z;n) = x^n + y^n - z^n
\]

The Diophantine equation \( D(x,y,z;n) \) has a solution if and only if \( F(x,y,z;n) \) achieves minimum values of zero for some positive integer values \( x, y, \) and \( z \). Hence, the task of finding out whether \( F(x,y,z;n) \) has a solution, or not entails identifying its optimality and feasibility conditions, given that \( x^{n-1} + y^{n-1} = z^{n-1} \) (e.g. \( n-1 = 3 \)) has no solution, i.e. given that:

\[
F(x,y,z;n-1) = x^{n-1} + y^{n-1} - z^{n-1} \neq 0
\]

Optimality conditions can be derived from the following unconstrained optimization problem which is to:

\[
\text{Minimize } F(x,y,z;n) = x^n + y^n - z^n
\]

The necessary optimality condition is that the gradient of \( F \) be equal to zero\(^1\) [3], that is:

\[
\frac{\partial F}{\partial x} = n x^{n-1} = 0; \quad \frac{\partial F}{\partial y} = n y^{n-1} = 0; \quad \frac{\partial F}{\partial z} = -n z^{n-1} = 0
\]

(5) From (4) we get: \( n (x^{n-1} + y^{n-1} - z^{n-1}) = 0 \), hence \( x^{n-1} + y^{n-1} - z^{n-1} = 0 \), since \( n > 0 \),

Thus, in order for \( F(x,y,z;n) \) to vanish, it is necessary that \( D(x,y,z;n-1) \) has a solution, which is not feasible since it is in contradiction with condition (3). \( D(x,y,z;n-1) \) having no solution. Hence, this condition implies that the Diophantine \( D(x,y,z;n) = x^n + y^n = z^n \), also has no solution. Since condition (3) is true for \( n = 3 \), by induction, this result is valid for all exponents \( n > 3 \), which proves the conjecture in Fermat’s last theorem.

References

1. Fermat’s last theorem: https://mathworld.wolfram.com/FermatsLastTheorem.html
2. Fermat’s last theorem: https://en.wikipedia.org/wiki/Fermat%27s_Last_Theorem

\(^1\) Originally formulated by Fermat in 1637