New formula to generate all Pythagorean triples with proof and geometrical interpretation

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Abstract. This way has the convenience to find easily all Pythagorean triples x, y, z ∈ N, where x is a predetermined integer, which means finding all right triangles whose sides have integer measures and one cathetus is predetermined.

RESULTS

Let x be an integer with $x \geq 1$. We define

$$D(x) = \{d \in \mathbb{N} \text{ such that } d \leq x \text{ and } d \text{ divisor of } x^2\}.$$

Let $x \in \mathbb{N}$ be now with x even, that is,

$$x = 2^n k,$$

with $n \in \mathbb{N}$ and $k \geq 1$ odd fixed. We define

$$P(x) = \{d \in \mathbb{N} \text{ such that } d = 2^s l, \text{ with } l \text{ divisor of } x^2 \text{ and}$$
$$s \in \{1, 2, ..., 2n - 1\}\}.$$

Finally, let $x \in \mathbb{N}$ and we define

$$C(x) = \begin{cases} D(x), & \text{if } x \text{ is odd,} \\ D(x) \cap P(x), & \text{if } x \text{ is even.} \end{cases}$$

We have the following theorem [1]:

Theorem 1.1. (x, y, z) *is a Pythagorean triple if and only if there exists a unique* $d \in C(x)$ *such that*

$$x = x, \quad y = \frac{x^2}{2d} - \frac{d}{2}, \quad z = \frac{x^2}{2d} + \frac{d}{2}. \qquad (1.1)$$

Proof. Let us suppose that (x, y, z) is a Pythagorean triple, that is, $x^2 + y^2 = z^2$, so that

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Summary of results extracted from the work [1] R. Amato, *A characterization of pythagorean triples*, JP Journal of Algebra, Number Theory and Applications, 39, (2017), 221–230

$$x^2 = z^2 - y^2 = (z - y)(z + y), \tag{1.2}$$

that we can write also in the following way:

$$2(z - y)\frac{(z + y)}{2} = x^2,$$

from which, setting

$$d = z - y, \tag{1.3}$$

we obtain

$$\frac{z + y}{2} = \frac{x^2}{2d}. \tag{1.4}$$

We prove that $d < x$ and moreover if x is even, then d must be divisible by 2^s with $s \in \{1, 2, ..., 2n - 1\}$.

In fact, from (1.2) and (1.3), we obtain

$$x^2 = d(z + y), \tag{1.5}$$

from which we have $\frac{z + y}{x} = \frac{x}{d}$, and taking into account that $z + y > x$, that is, $\frac{z + y}{x} > 1$, we obtain $\frac{x}{d} > 1$, so that $d < x$.

Moreover, if x is even, then we can write

$$x = 2^n k, \text{ with } k \text{ odd and } k \geq 1.$$

If, on the contrary, $d = z - y = 2^n l$, with $l \geq 1$ and l odd divisor of k^2, then from

$$x^2 = 2^{2n} k^2 = d(z + y),$$

we obtain that $z + y = \frac{k^2}{l}$ is odd, which is a contradiction because if $z - y$ is even, then $z + y$ is even. We obtain that if x is even, then it must be divisible by 2^s with $s \in \{1, 2, ..., 2n-1\}$. In this way, if $d = z - y = 2^s l$, then $z + y$ is even.

Obviously, if we choose $d = x$, by (1.1), we obtain the trivial triples

$$x = d, \quad y = 0, \quad z = x.$$

Now we prove that $z - y = d$ is unique. In fact, from (1.1), we consider the following system:

$$\begin{cases} d^2 + 2yd - x^2 = 0, \\ d^2 - 2zd + x^2 = 0, \end{cases} \tag{1.6}$$

from which we obtain, respectively,

$$d = -y \pm \sqrt{y^2 + x^2} = -y \pm z,$$

$$d = z \pm \sqrt{z^2 - x^2} = z \pm y,$$

from which it results $d = z - y$ the unique solution that satisfies system (1.6).

Now we prove that, fixed any $x \in \mathbb{N}$ and any $d \in C(x)$, every Pythagorean triple is given by (1.1). In fact, from (1.4), once subtracting $\frac{z-y}{2} = \frac{d}{2}$ and summing once in both members, we obtain, respectively,

$$\frac{z+y}{2} - \frac{z-y}{2} = \frac{x^2}{2d} - \frac{d}{2}, \text{ and then } y = \frac{x^2}{2d} - \frac{d}{2},$$

$$\frac{z+y}{2} + \frac{z-y}{2} = \frac{x^2}{2d} + \frac{d}{2}, \text{ and then } z = \frac{x^2}{2d} + \frac{d}{2}.$$

Since (1.4) is true for every Pythagorean triple, the assertion is proved.

Finally, we prove that (1.1) gives us every Pythagorean triple. Fixed each $x \in \mathbb{N}$ and each $d \in C(x)$ and in this order, let us observe that

$$x^2 + \left(\frac{x^2}{2d} - \frac{d}{2}\right)^2 = \left(\frac{x^2}{2d} + \frac{d}{2}\right)^2$$

is true for all $x \in \mathbb{N}$ and for each $d \in C(x)$.

Moreover, based on the previous results we have proved before, the following theorem holds [1]:

Theorem 1.2. *Each $x \in \mathbb{N}$ can be found as cathetus in at least one Pythagorean triple. Every $x \in \mathbb{N}$ can be represented in the form $x = \sqrt{z^2 - y^2}$ with $y, z \in \mathbb{N}$.*

Theorem 1.2 has also one geometrical interpretation for $x \in \mathbb{N}$ and $x > 2$. We consider a circumference *F* with diameter $MN = z$, $AB = x$ a chord of *F* perpendicular to *MN* in *H*, $x \in \mathbb{N}$ and $x > 2$ (see Figure 1).

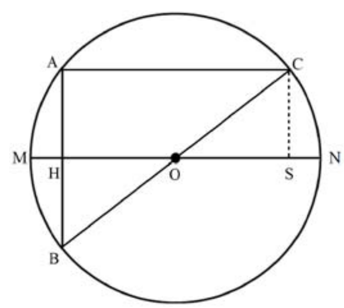

Figure 1

In point *A*, we consider the chord $AC = y$ parallel to *MN* and, because the triangle *ABC* has been inscribed in one semi-circumference, it results that *ABC* is right and $BC = MN$. Considering $MH = \frac{z - y}{2} = \frac{d}{2}$ and due to second theorem of Euclid $HN = \frac{x^2}{2d}$, we have $y = \frac{x^2}{2d} - \frac{d}{2}$ and $z = \frac{x^2}{2d} + \frac{d}{2}$.

Taking into account all the conditions, we have given in the previous section, (1.1) represents the measures of the sides related to the right triangle ABC expressed by integer numbers in which there is a predetermined cathetus $x = AB$. Finally, noticing that $MH = SN$, $BC = MN$, $MN - AC = MH + SN = d$ and that in one triangle $MN - AC < AB = x$, we obtain that $d < x$. Therefore, this completes the geometrical interpretation.

To prove the completeness of (1.1) we consider the following example.

Example 1.1. To demonstrate our method, we give the following table for $1 \le x \le 23$. Obviously, the table can be extended for each cathetus $x \in \mathbb{N}$, obtaining all right triangles whose sides have integer measures and one cathetus is given.

$x = 1$ $C(x) = \{1\}$ for $d = 1$ then $x = 1$, $y = 0$, $z = 1$	(1, 0, 1)
$x = 2$ $C(x) = \{2\}$ for $d = 2$ then $x = 2$, $y = 0$, $z = 0$	(2, 0, 2)
$x = 3$ $C(x) = \{1, 3\}$ for $d = 1$ then $x = 3$, $y = 4$, $z = 5$ for $d = 3$ then $x = 3$, $y = 0$, $z = 3$	(3, 4, 5) (3, 0, 3)
$x = 4$ $C(x) = \{2, 4\}$ for $d = 2$ then $x = 4$, $y = 3$, $z = 5$ for $d = 4$ then $x = 4$, $y = 0$, $z = 4$	(4, 3, 5) (4, 0, 4)
$x = 5$ $C(x) = \{1, 5\}$ for $d = 1$ then $x = 5$, $y = 12$, $z = 13$ for $d = 5$ then $x = 5$, $y = 0$, $z = 5$	(5, 12, 13) (5, 0, 5)
$x = 6$ $C(x) = \{2, 6\}$ for $d = 2$ then $x = 6$, $y = 8$, $z = 10$ for $d = 6$ then $x = 6$, $y = 0$, $z = 6$	(6, 8, 10) (6, 0, 6)
$x = 7$ $C(x) = \{1, 7\}$ for $d = 1$ then $x = 7$, $y = 24$, $z = 25$ for $d = 7$ then $x = 7$, $y = 0$, $z = 7$	(7, 24, 25) (7, 0, 7)

$x = 8 \quad C(x) = \{2, 4, 8\}$ for $d = 2$ then $x = 8,\ y = 15,\ z = 17$ for $d = 4$ then $x = 8,\ y = 6,\ z = 10$ for $d = 8$ then $x = 8,\ y = 0,\ z = 8$	 (8, 15, 17) (8, 6, 10) (8, 0, 8)
$x = 9 \quad C(x) = \{1, 3, 9\}$ for $d = 1$ then $x = 9,\ y = 40,\ z = 41$ for $d = 3$ then $x = 9,\ y = 12,\ z = 15$ for $d = 9$ then $x = 9,\ y = 0,\ z = 9$	 (9, 40, 41) (9, 12, 15) (9, 0, 9)
$x = 10 \quad C(x) = \{2, 10\}$ for $d = 2$ then $x = 10,\ y = 24,\ z = 26$ for $d = 10$ then $x = 10,\ y = 0,\ z = 10$	 (10, 24, 26) (10, 0, 10)
$x = 11 \quad C(x) = \{1, 11\}$ for $d = 1$ then $x = 11,\ y = 60,\ z = 61$ for $d = 11$ then $x = 11,\ y = 0,\ z = 11$	 (11, 60, 61) (11, 0, 11)
$x = 12 \quad C(x) = \{2, 4, 6, 8, 12\}$ for $d = 2$ then $x = 12,\ y = 35,\ z = 37$ for $d = 4$ then $x = 12,\ y = 16,\ z = 20$ for $d = 6$ then $x = 12,\ y = 9,\ z = 15$ for $d = 8$ then $x = 12,\ y = 5,\ z = 13$ for $d = 12$ then $x = 12,\ y = 0,\ z = 12$	 (12, 35, 37) (12, 16, 20) (12, 9, 15) (12, 5, 13) (12, 0, 12)
$x = 13 \quad C(x) = \{1, 13\}$ for $d = 1$ then $x = 13,\ y = 84,\ z = 85$ for $d = 13$ then $x = 13,\ y = 0,\ z = 13$	 (13, 84, 85) (13, 0, 13)
$x = 14 \quad C(x) = \{2, 14\}$ for $d = 2$ then $x = 14,\ y = 48,\ z = 50$ for $d = 14$ then $x = 14,\ y = 0,\ z = 14$	 (14, 48, 50) (14, 0, 14)
$x = 15 \quad C(x) = \{1, 3, 5, 9, 15\}$ for $d = 1$ then $x = 15,\ y = 112,\ z = 113$ for $d = 3$ then $x = 15,\ y = 36,\ z = 39$ for $d = 5$ then $x = 15,\ y = 20,\ z = 25$ for $d = 9$ then $x = 15,\ y = 8,\ z = 17$ for $d = 15$ then $x = 15,\ y = 0,\ z = 15$	 (15, 112, 113) (15, 36, 39) (15, 20, 25) (15, 8, 17) (15, 0, 15)

$x = 16 \quad C(x) = \{2, 4, 8, 16\}$	
for $d = 2$ then $x = 16,\ y = 63,\ z = 65$	(16, 63, 65)
for $d = 4$ then $x = 16,\ y = 30,\ z = 34$	(16, 30, 34)
for $d = 8$ then $x = 16,\ y = 12,\ z = 20$	(16, 12, 20)
for $d = 16$ then $x = 16,\ y = 0,\ z = 16$	(16, 0, 16)
$x = 17 \quad C(x) = \{1, 17\}$	
for $d = 1$ then $x = 17,\ y = 144,\ z = 145$	(17, 144, 145)
for $d = 17$ then $x = 17,\ y = 0,\ z = 17$	(17, 0, 17)
$x = 18 \quad C(x) = \{2, 6, 18\}$	
for $d = 2$ then $x = 18,\ y = 80,\ z = 82$	(18, 80, 82)
for $d = 6$ then $x = 18,\ y = 24,\ z = 30$	(18, 24, 30)
for $d = 18$ then $x = 18,\ y = 0,\ z = 18$	(18, 0, 18)
$x = 19 \quad C(x) = \{1, 19\}$	
for $d = 1$ then $x = 19,\ y = 180,\ z = 181$	(19, 180, 181)
for $d = 19$ then $x = 19,\ y = 0,\ z = 19$	(19, 0, 19)
$x = 20 \quad C(x) = \{2, 4, 8, 10, 20\}$	
for $d = 2$ then $x = 20,\ y = 99,\ z = 101$	(20, 99, 101)
for $d = 4$ then $x = 20,\ y = 48,\ z = 52$	(20, 48, 52)
for $d = 8$ then $x = 20,\ y = 21,\ z = 29$	(20, 21, 29)
for $d = 10$ then $x = 20,\ y = 15,\ z = 25$	(20, 15, 25)
for $d = 20$ then $x = 20,\ y = 0,\ z = 20$	(20, 0, 20)
$x = 21 \quad C(x) = \{1, 3, 7, 9, 21\}$	
for $d = 1$ then $x = 21,\ y = 220,\ z = 221$	(21, 220, 221)
for $d = 3$ then $x = 21,\ y = 72,\ z = 75$	(21, 72, 75)
for $d = 7$ then $x = 21,\ y = 28,\ z = 35$	(21, 28, 35)
for $d = 9$ then $x = 21,\ y = 20,\ z = 29$	(21, 20, 29)
for $d = 21$ then $x = 21,\ y = 0,\ z = 21$	(21, 0, 21)
$x = 22 \quad C(x) = \{2, 22\}$	
for $d = 2$ then $x = 22,\ y = 120,\ z = 122$	(22, 120, 122)
for $d = 22$ then $x = 22,\ y = 0,\ z = 22$	(22, 0, 22)
$x = 23 \quad C(x) = \{1, 23\}$	
for $d = 1$ then $x = 23,\ y = 264,\ z = 265$	(23, 264, 265)
for $d = 23$ then $x = 23,\ y = 0,\ z = 23$	(23, 0, 23)

References

[1] R. Amato, *A characterization of pythagorean triples*, JP Journal of Algebra, Number Theory and Applications, 39, (2017), 221–230