The Connection Between $X^2 + 1$ and Balancing Numbers

Kelly Harris

Abstract. Balancing numbers as introduced by Behera and Panda [1] can be shown to be connected to the formula $x^2 + 1 = N$ in a very simple way. The goal of this paper is to show that if a balancing number exists for the balancing equation $1+2+...+(y-1) = (y+1)+(y+2)+...+(y+m)$, then there is a corresponding $(2y)^2 + 1 = N$, where N is composite. We will also show how this can be used to factor N.

1. Introduction

Balancing numbers are defined as solutions to Equation 1 below.

$$1 + 2 + ... + (y - 1) = (y + 1) + (y + 2) + ... + (y + m) \tag{1}$$

where y is the balancing number and m is called the balancer. The first few numbers for y are 6, 35, and 204 and 2, 14, and 84 are the corresponding balancers. One property of balancing equations is that the left hand side of the equation starts at 1. We will later extend this to include equations that don't.

2. Show the connection between $x^2 + 1$ and balancing numbers

If $x^2 + 1 = N$ (where $x = 2y$) is composite, then $x^2 + 1 = a^2 + b^2$. Now if we multiply both sides by 2, we get $2N = 2(a^2+b^2) = (a+b)^2+(a-b)^2$. Since N is odd, a and b have to be of opposite parity, so $a+b$ and $a-$b are both odd and can be written as $2r+1$ and $2s+1$ respectively. Now we have:

$$(x + 1)^2 + (x - 1)^2 = 2(x^2 + 1) = 2N = (a + b)^2 + (a - b)^2 \tag{2}$$

$$(x - 1)^2 - (a - b)^2 = (a + b)^2 - (x + 1)^2 \tag{3}$$

$$(2y - 1)^2 - (2s + 1)^2 = (2r + 1) - (2y + 1) \tag{4}$$

$$4y(y - 1) + 1 - 4s(s + 1) - 1 = 4r(r + 1) + 1 - 4y(y + 1) - 1 \tag{5}$$

$$y(y - 1) - s(s + 1) = r(r + 1) - y(y + 1) \tag{6}$$

Since all four terms in Equation 6 are oblong (pronic) numbers, we can divide them by 2 and convert them to triangular numbers.

$$T_{y-1} - T_s = T_r - T_y \tag{7}$$

There are two cases to consider. In the special case where $T_s = 0$ then we have:

$$T_{y-1} = T_r - T_y \tag{8}$$

Since $T_y + T_{y-1} = y^2$, Equation 8 shows how balancing numbers and square triangular numbers are related, as shown in [2] by O. Karaatlı and R. Keskin. But we instead will use the fact that triangular numbers are the sum of consecutive integers to expand Equation 8 as shown below.

$$1 + 2 + ... + (y - 1) = (y + 1) + (y + 2) + ... + r \tag{9}$$

If we let $r = y + m$, we have

$$1 + 2 + ... + (y - 1) = (y + 1) + (y + 2) + ... + (y + m) \tag{10}$$

which is Equation 1. So we have shown that $x^2 + 1$ is related to balancing equation.

Example 1:
Let $y = 6$, then $x = 2(6) = 12$, $N = x^2 + 1 = 145$, and $2N = 290 = 17^2 + 1$
$17 = 2(8) + 1$ and thus $r = 8$ and $m = 2$
$1 = 2(0) + 1$ and thus $s = 0$
1+2+3+4+5=7+8
We see from this example that y is a solution to Equation 1 and is a number for which N is composite.

Now we will consider the general case when s is not zero. We can then write Equation 7 as:

$$(s + 1) + (s + 2) + ... + (y - 1) = (y + 1) + (y + 2) + ...r \tag{11}$$

Since we have defined r as $y + m$ by counting from y to r, lets do the same for s. Counting back from y, we get $s = y - n - 1$. Equation 12 can now be written as:

$$(y - n) + (y - n + 1) + ... + (y - 1) = (y + 1) + (y + 2) + ... + (y + m) \tag{12}$$

Notice that the first term $y - n$ does not have to got to 1, so Equation 12 is a more general form of Equation 1. We started with a composite $x^2 + 1 = N$ and ended with the balancing equation (12), so if we do the process in reverse we go from a balancing equation with a solution to a composite N. Or stated another way for every y that is a solution to Equation 12 , there is a corresponding $x^2 + 1 = N$ equation. (Table 1 shows the first few examples).

Example 2: Let $y = 9$, and put it in Equation 12
$(9 - n) + (9 - n + 1) + ... + (9 - 1) = (9 + 1) + (9 + 2) + ... + (9 + m)$
This equation is solved when $n = 3$ and $m = 2$ (or $n = 6$ and $m = 3$)
$(9 - 3) + (9 - 3 + 1) + (9 - 3 + 2) = (9 + 1) + (9 + 2)$
$6 + 7 + 8 = 10 + 11$

								y					corresponding $x^2+1=N$ equation
							0	**1**	2				
							1	**2**	3				
						1	2	**3**	4	5			
					1	2	3	**4**	5	6	7		$8^2+1=65$ with $n=2$ and $m=1$
				1	2	3	4	**5**	6	7	8		
			1	2	3	4	5	**6**	7	8	9		$12^2+1=145$ with $n=5$ and $m=2$
		1	2	3	4	5	6	**7**	8	9	10		
	1	2	3	4	5	6	7	**8**	9	10	11		
1	2	3	4	5	6	7	8	**9**	10	11	12	13	$18^2+1=325$ with $n=3$ and $m=2$
1	2	3	4	5	6	7	8	**9**	10	11	12	13	$18^2+1=325$ with $n=6$ and $m=3$

Table 1 - The sum of the underlined numbers on the left side of the y column is equal to the underlined numbers on the right. Example $y=4$ and $2+3=5$

3. Factoring $x^2+1=N$

We will now show how the above section can be used to factor $x^2+1=N$ into two factors, starting with Equation 6.

$$y(y-1)-s(s+1)=r(r+1)-y(y+1) \tag{13}$$

Now replace s with $y-n-1$ and r with $y+m$ and simplify both sides, we have:

$$2ny-n^2-n=2my+m^2+m \tag{14}$$

Combine the terms with y and simplify:

$$y=\frac{m(m+1)+n(n+1)}{2(n-m)} \tag{15}$$

$$y=\frac{T_m+T_n}{n-m} \tag{16}$$

Now that we have solved for y, we can plug this into x^2+1

$$x^2+1=4\left[\frac{m(m+1)+n(n+1)}{2(n-m)}\right]^2+1=\left[\frac{m(m+1)+n(n+1)}{n-m}\right]^2+1 \tag{17}$$

$$=\left[\frac{[m(m+1)+n(n+1)]^2+(n-m)^2}{(n-m)^2}\right] \tag{18}$$

And this simplifies to:

$$x^2+1=\frac{[m^2+n^2][(m+1)^2+(n+1)^2]}{(n-m)^2} \tag{19}$$

Example 3: Let $y = 9$, then $(2y)^2 + 1 = 18^2 + 1 = 325$
From Table 1 $n = 6$ and $m = 3$

$$\frac{[m^2+n^2][(m+1)^2+(n+1)^2]}{(n-m)^2} = \frac{[3^2+6^2][4^2+7^2]}{3^2} = \frac{(45)(65)}{9} = 5(65) \tag{20}$$

Example 4: Let $y = 9$, then $(2y)^2 + 1 = 18^2 + 1 = 325$ From Table 1 $n = 3$ and $m = 2$

$$\frac{[m^2+n^2][(m+1)^2+(n+1)^2]}{(n-m)^2} = \frac{[2^2+3^2][3^2+4^2]}{1} = 13(25) \tag{21}$$

The two examples above were chosen to show that for each y that is a solution to Equation 12, there will be an m and n for each combination of factors of N.

4. Conclusion

The fact that there is a connection between balancing numbers and $x^2 + 1$ leads to a couple of interesting questions. Can this connection be used to show that there is an infinite number of primes of the form $x^2 + 1$ (one of the famous Landau problems [3])? Or can this be used to aid the factoring of Fermat Numbers [4], because we now know that all the factors has to have the form shown in Equation 19?

References

[1] Behera, A. - Panda, G. K.: On the square roots of triangular numbers, The Fibonacci Quart. 37 (1999), no. 2, 98–105.

[2] O. Karaatlı and R. Keskin: On some Diophantine equations related to square triangular and balancing numbers, J. Algebra, Number Theory: Adv. Appl. 4 (2010), 71–89.

[3] Landau's problems: https://en.wikipedia.org/wiki/Landau%27s_problems

[4] Fermat Numbers: https://en.wikipedia.org/wiki/Fermat_number