Reexamining the Thomas Precession

Marcelo Carvalho*

Departamento de Matemática

Universidade Federal de Santa Catarina

Florianópolis, 88.040-900, SC, Brasil

Abstract

We review the Thomas precession exhibiting the exact form of the Thomas rotation in the axis-angle parameterization. Assuming three inertial frames S, S', S'' moving with arbitrary velocities and with S, S'' having their axis parallel to the axis of S' we focus our attention on the two essential elements of the Thomas precession e.g., (i) there is a rotation between the axis of frames S, S'' and (ii) the combination of two Lorentz transformations from S to S' and from S' to S'' fails to produce a pure Lorentz transformation from S to S''. The physical consequence of (i) and (ii) refers to the impossibility of having arbitrary frames S, S', S'' moving with their axis mutually parallel. Then, we reexamine the validity of (i) and (ii) under the conjecture the time depends on the state of motion of the frames and we show that the Thomas precession assumes a different form as formulated in (i) and (ii).

Keywords

Special relativity, absolute time, Thomas precession.

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1 Introduction

Thomas precession originated with the solution provided by L H Thomas to a problem on the spectral structure of the Hydrogen atom investigated by Uhlenbeck and Gouldsmit who found for the energy of the spin-orbit interaction a value twice larger than the observed experimental value. Shortly after their work, Thomas obtained the correct experimental value by recalculating the spin-orbit energy using elaborate arguments involving the relativistic transformation of velocities and some approximations [1], [2]. In his analysis, Thomas was able to reduce the problem to an algebraic property of the Lorentz tranformations, e.g. that the composition of two Lorentz transformations (with non-parallel

*e-mail: m.carvalho@ufsc.br

velocities) is not a pure Lorentz transformation but it gives a Lorentz transformation followed by a rotation [1]. Since then, investigation concerning this behavior and its consequences allows us to restrict the Thomas precession within the domain of Special Relativity (SR) without reference to the setting of atomic physics where it has originally appeared. In this respect, a standard reference reviewing some aspects of Thomas' original problem is found in [3]. Among the many developments that followed the work of Thomas, we will focus on the proposal outlined by C Moller [4] that is distinguished for its clarity and simplicity and that we now summarize.

Given three inertial moving frames S, S', S'' with S, S'' having their axis parallel to S', we consider an event P registered as $(t, \vec{x})$, $(t', \vec{x}\,')$, $(t'', \vec{x}\,'')$ by S, S', S''. The first point to be noticed is that Thomas precession is a statement on the impossibility of having these frames S, S', S'' moving with axis mutually parallel [2] which is a consequence of the application of the expression for the relativistic addition of velocities to $\vec{\tilde{v}}_{SS''} = \vec{\tilde{v}}_{S'S''} \oplus \vec{\tilde{v}}_{SS'}$ (22) and to $\vec{\tilde{v}}_{S''S} = \vec{\tilde{v}}_{S'S} \oplus \vec{\tilde{v}}_{S''S'}$ (23) that gives

$$\vec{\tilde{v}}_{S''S} = -\mathcal{R}^{-1} \vec{\tilde{v}}_{SS''} , \tag{1}$$

with $\mathcal{R}^{-1}$ being a rotation. This equation reveals there is a rotation between the axis of the moving frames S and S'' that manifests itself in setting the relation between $(t, \vec{x})$ and $(t'', \vec{x}\,'')$ as given by $(t'', \vec{x}\,'') = \mathcal{R}^{-1} L(\vec{\tilde{v}}_{SS''})(t, \vec{x})$, where $L(\vec{\tilde{v}}_{SS''})$ denotes a Lorentz transformation. The second aspect to be noticed is that if S, S'' have axis parallel to S' then we have $(t', \vec{x}\,') = L(\vec{\tilde{v}}_{SS'})(t, \vec{x})$ and $(t'', \vec{x}\,'') = L(\vec{\tilde{v}}_{S'S''})(t', \vec{x}\,')$ and composing them we obtain $(t'', \vec{x}\,'') = L(\vec{\tilde{v}}_{S'S''}) L(\vec{\tilde{v}}_{SS'})(t, \vec{x})$. Then comparing these two transformations relating $(t, \vec{x})$ and $(t'', \vec{x}\,'')$ we obtain that

$$\mathcal{R}^{-1} L(\vec{\tilde{v}}_{SS''}) = L(\vec{\tilde{v}}_{S'S''}) L(\vec{\tilde{v}}_{SS'}) , \tag{2}$$

which is the algebraic expression of the Thomas precession.

It is the purpose of our work to fill in the details of Moller's development [4] emphasizing the role of equations (1), (2) as the fundamental equations of the Thomas precession in the sense that (1) is sufficient to determine the form of the rotation $\mathcal{R}$ that exists between the axis of the moving frames S, S'', while (2) is a consequence of this rotation and the fact there is no *obstruction* to compose two Lorentz transformations $L(\vec{\tilde{v}}_{SS'})$ and $L(\vec{\tilde{v}}_{S'S''})$. While in Moller's treatment equation (1) was used to obtain the *infinitesimal* form of the rotation and then employed to obtain the infinitesimal form of (2) in our work we will use (1) to obtain the *exact* form of the rotation $\mathcal{R}$ and (2) to confirm the validity of the expression we obtained. In fact, it seems few works exhibit the exact form of the Thomas rotation, see for instance [5], [6], and the expressions they obtained present differences that make it difficult to check their equivalence, therefore in our work we show the correcteness of our expression for the Thomas rotation $\mathcal{R}$ by providing an outline of the proof of (2).

Another purpose of our work is to examine if (1), (2) remain valid under the assumption that *time depends on the state of motion of the frames.* It seems this assumption was stated originally

[1] We are not claiming that L H Thomas was the first one to notice this.

[2] This, of course, under the common view of the SR, which assumes the time doesn't depend on the state of motion of the frames.

in [7] in a rather qualitative form without implementing it. In previous works [8], [9] we found a concrete way to implement this assumption using the concept of *absolute* time that we denote by τ. Then in our model we have two times: the standard time of the SR denoted by t and hencefort called *local* time and the absolute time τ. As a result we also have two types of velocities depending on we take derivatives relative to t or to τ, e.g.

$$\vec{\tilde{v}} := \frac{d\vec{x}}{dt} \text{ and } \vec{v} := \frac{d\vec{x}}{d\tau} .$$

In our work we introduce τ adopting an axiomatic approach that allow us to conceive $\tilde{v} = \tilde{v}(v)$ and

$$t = t(\tau, v) . \tag{3}$$

It is in the sense shown in (3) that we say the local time depends on the state of motion of the frames. Now, returning to the situation that has prompted the Thomas precession, if we consider three inertial frames moving in such a way that S, S'' have their axis parallel to the frame S' then when we calculate $\vec{\tilde{v}}_{SS''}$ we do so in such a way that it will depend on $v_{SS'}$ (80), while $\vec{\tilde{v}}_{S''S}$ will depend on $v_{S''S'}$ (89). Then, we cannot compare anymore $\vec{\tilde{v}}_{SS''}(v_{SS'}) = \vec{\tilde{v}}_{S'S''}(v_{SS'}) \oplus \vec{\tilde{v}}_{SS'}(v_{SS'})$ and $\vec{\tilde{v}}_{S''S}(v_{S''S'}) = \vec{\tilde{v}}_{S'S}(v_{S''S'}) \oplus \vec{\tilde{v}}_{S''S'}(v_{S''S'})$ as we did in (1), unless if, for instance, we first express $\vec{\tilde{v}}_{S'S}(v_{S''S'})$ and $\vec{\tilde{v}}_{S''S'}(v_{S''S'})$ in terms of $v_{SS'}$. When we do so we obtain

$$\vec{\tilde{v}}_{S''S}(v_{S''S'}) = -\Omega(v_{S''S'}, v_{SS'}) \vec{\tilde{v}}_{SS''}(v_{SS'}) \tag{4}$$

with Ω a scalar term, which shows there is no rotation between the axis of the frames S and S''. Then, at this point equation (1) is to be replaced by equation (4).

Now, let us shift to the task of composing two Lorentz transformations $L(\vec{\tilde{v}}_{SS'})$ and $L(\vec{\tilde{v}}_{S'S''})$. Fixing our attention on the local time used by S' we have that considering the pair $\{S, S'\}$ the time that S' employs depends on the relative velocity $\vec{v}_{SS'}$ and we denote it by $t'_{S'S}$. Considering the pair $\{S', S''\}$ the local time used by S' depends now on the velocity $\vec{v}_{S'S''}$ and we denote it by $t'_{S'S''}$. Then, we have $t'_{S'S} \neq t'_{S'S''}$ and this forbids us to compose $L(\vec{\tilde{v}}_{SS'})$ and $L(\vec{\tilde{v}}_{S'S''})$. However, the fact that $t'_{S'S}(\tau, \vec{v}_{SS'})$ and $t'_{S'S''}(\tau, \vec{v}_{S'S''})$ allows us to eliminate τ to write $t'_{S'S''}(t'_{S'S}, \vec{v}_{S'S}, \vec{v}_{S'S''})$, then we can define a map $K_{\{S'S''\},\{S',S\}}$ by $(t'_{S'S''}, \vec{x}\,') := K_{\{S'S''\},\{S',S\}}(t'_{S'S}, \vec{x}\,')$ with $t'_{S'S''} = t'_{S'S''}(t'_{S'S}, \vec{v}_{S'S}, \vec{v}_{S'S''})$ in terms of which we can write

$$L(\vec{\tilde{v}}_{S'S''}) K_{\{S',S''\},\{S'S\}} L(\vec{\tilde{v}}_{SS'}) .$$

Using this K-map defined as above we show that instead of (2) now the relation between $L(\vec{\tilde{v}}_{SS''})$, $L(\vec{\tilde{v}}_{S'S''})$ and $L(\vec{\tilde{v}}_{SS'})$ is given by

$$L(\vec{\tilde{v}}_{SS''}) = K_{\{S'',S\},\{S'',S'\}} L(\vec{\tilde{v}}_{S'S''}) K_{\{S',S''\},\{S',S\}} L(\vec{\tilde{v}}_{SS'}) K_{\{S,S'\},\{S,S''\}} . \tag{5}$$

Then, under the hypothesis that the local time depends on the state of motion of the frames, equations (4) and (5) become the new form for the Thomas precession (with the factor Ω replacing the rotation $\mathcal{R}$) and it allows us to envisage a situation where the three frames move with axis mutually parallel.

Our work is divided in two parts. In Part I we present the standard form of the Thomas precession and it consists of sections 2 and 3. In section 2 we review Moller's interpretation of the Lorentz transformation by considering spatial coordinates of events (as seen by two inertial frames) as represented by vectors in an abstract vector space $\mathcal{V}$ and the Lorentz transformation between the spatial coordinates is considered as an active transformation in $\mathcal{V}$. In section 3 we consider three inertial frames S, S', S'' with S, S'' having axis parallel to S' and show from the formula of the relativistic addition of velocities that there is a rotation between the axis of frames S and S'' which affects the form of the Lorentz transformation between S and S''. Then, we calculate the exact form of this rotation completing Moller's analysis of the Thomas Precession. In Part II we review the Thomas precession assuming the dependence of time with the state of motion of the frames. It consists of sections 4, 5 and 6. In section 4 we set the stage to reexamine Thomas precession by introducing axiomatically the concept of absolute time, which brings in a consistent way aspects of the Galilei and the SR that allows us to conceive the local time as depending on the state of motion of the frames. Then in section 5 we show how the Galilean law for the addition of velocities allow us to derive the corresponding relativistic law for the addition of velocities that we apply to the movement of the frames S, S', S'' to calculate $\vec{\tilde{v}}_{SS''}$ and $\vec{\tilde{v}}_{S''S}$. We show there is no Thomas rotation $\mathcal{R}$ between them, which is now replaced by a multiplicative scalar term Ω. In section 6 we introduce the K-map and show how it allows us to compose two Lorentz transformations in such a way that it produces a Lorentz transformation. We end our work with two appendixes disposed as sections 7, 8. In section 7 we give an outline of the proof of relation (2). In section 8 we prove the formula for the relativistic addition of velocities within the context of the usual formulation of SR. We do so in order to facilitate those willing to compare the same derivation we presented in section 5.

A word about notation. When we write $t'_{S'S}$ we intend to make it clear that it refers to the local time used by S' taking into consideration the state of motion between S' and S. In the same way $t'_{S'S''}$ refers to the local time used by S' considering the state of motion between S' and S''. In these expressions there is no role attributed to the position of the indexes S', S''. A different situation occurs when we write $\vec{\tilde{v}}_{S'S}$ to indicate the velocity of S relative to S' and $\vec{\tilde{v}}_{SS'}$ to indicate the velocity of S' relative to S, therefore for velocities the position of the indexes plays a crucial role.

Two vectors $\vec{u}$ and $\vec{v}$ are said parallel if they are proportional, i.e. $\vec{u} = \xi\vec{v},\ \xi \in \mathbb{R}$ and we denote it writting $\vec{u} \parallel \vec{v}$. Given a vector $\vec{X}$ written in terms of a basis vector $\{\hat{e}_i\}$ we write $X|_{\hat{e}_i}$ to denote the component of $\vec{X}$ relative to the basis vector $\hat{e}_i$.

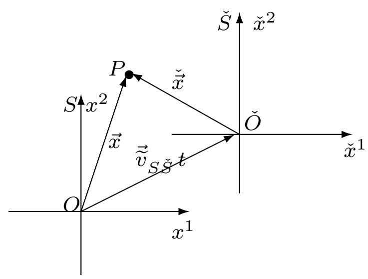
Finally, what we call Lorentz transformation is, in fact, what we understand to be a Lorentz boost. Given an event P we may write it as $(t_P, \vec{x}_P)$ or simply as $(t, \vec{x})$ if it is clear these coordinates refer to the event P.

Part I: The Standard presentation of the Thomas Preccesion

2 The Lorentz transformation of space coordinates represented in an abstract space

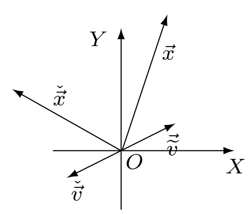
Let us consider two inertial observers S, $\check{S}$ that register an event P as $(t, \vec{x})$, $(\check{t}, \check{\vec{x}})$. The Lorentz transformation of the spatial coordinates is interpreted by Moller as follows. The space coordinates $\vec{x}$, $\check{\vec{x}}$ are represented as vectors $\vec{x}^*$, $\check{\vec{x}}^*$ in an abstract vector space $\mathcal{V}$. There are two forms for the association $x \to \vec{x}^* \in \mathcal{V}$, $\check{\vec{x}} \to \check{\vec{x}}^* \in \mathcal{V}$, which depends on the reference frames S, $\check{S}$ having their axis parallel or with a rotation.

2.1 The frames S, $\check{S}$ have parallel axis



(6)

Here, we assume the association $\vec{x}^* := \vec{x}$, $\check{\vec{x}}^* := \check{\vec{x}}$. We also represent the relative velocities $\vec{\tilde{v}}_{S\check{S}}$, $\vec{\tilde{v}}_{\check{S}S}$ as vectors in $\mathcal{V}$ by the association $\vec{\tilde{v}}_{S\check{S}} \to \vec{v}^* \in \mathcal{V}$, $\vec{\tilde{v}}_{\check{S}S} \to \check{\vec{v}}^* \in \mathcal{V}$ with $\vec{v}^* := \vec{\tilde{v}}_{S\check{S}}$, $\check{\vec{v}}^* := \vec{\tilde{v}}_{\check{S}S}$. For ease of notation we write $\vec{\tilde{v}} \equiv \vec{\tilde{v}}_{S\check{S}}$ and $\check{\vec{v}} \equiv \vec{\tilde{v}}_{\check{S}S}$. Then in the space $\mathcal{V}$ we have the following representation of the vectors $\vec{x}, \check{\vec{x}}, \vec{\tilde{v}}, \check{\vec{v}}$



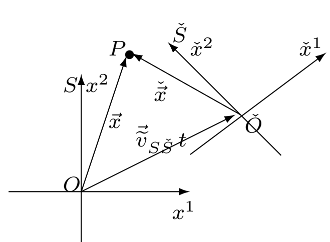
(7)

where the relation between $\vec{x}$ and $\check{\vec{x}}$ and $\vec{\tilde{v}}$ and $\check{\vec{v}}$ are given by

$$\check{\vec{x}} = \vec{x} - (1 - \gamma_{\tilde{v}}) \frac{\vec{x} \cdot \vec{\tilde{v}}}{\tilde{v}^2}\, \vec{\tilde{v}} - \gamma_{\tilde{v}}\, t\, \vec{\tilde{v}} \tag{8}$$

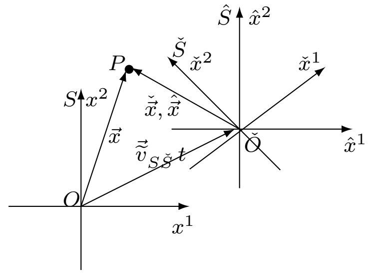
$$\check{\vec{v}} = -\vec{\tilde{v}} \tag{9}$$

2.2 The frames S, $\check{S}$ have a rotation



(10)

Here, in order to make the association $\vec{x} \to \vec{x}^* \in \mathcal{V}$, $\breve{\vec{x}} \to \breve{\vec{x}}^* \in \mathcal{V}$, we consider an auxiliary frame $\hat{S}$ having axis parallel to the axis of the frame S as shown in (11). Relative to $\hat{S}$, the event P is registered as $(\hat{t}, \hat{\vec{x}})$.



(11)

If $\mathcal{R}$ is the (passive) rotation transforming the frame $\hat{S}$ into $\check{S}$, then in the active view the relation between $\hat{\vec{x}}$ and $\breve{\vec{x}}$ is given by $\breve{\vec{x}} = \mathcal{R}^{-1}\hat{\vec{x}}$. Similarly, for the relative velocities we have $\vec{\tilde{v}}_{\check{S}S} = \mathcal{R}^{-1}\vec{\tilde{v}}_{\hat{S}S}$ and $\vec{\tilde{v}}_{\hat{S}S} = -\vec{\tilde{v}}_{S\hat{S}}$. But $\vec{\tilde{v}}_{S\hat{S}} = \vec{\tilde{v}}_{S\check{S}}$, then we obtain

$$\vec{\tilde{v}}_{\hat{S}S} = -\vec{\tilde{v}}_{S\check{S}} \tag{12}$$

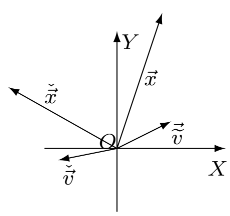
$$\vec{\tilde{v}}_{\check{S}S} = -\mathcal{R}^{-1}\vec{\tilde{v}}_{S\check{S}} \, . \tag{13}$$

But $\hat{\vec{x}}$ and $\vec{x}$ have a relation similar to that shown in (8), therefore denoting $\vec{\tilde{v}} \equiv \vec{\tilde{v}}_{S\check{S}}$, $\breve{\vec{v}} \equiv \vec{\tilde{v}}_{\check{S}S}$ we obtain

$$\breve{\vec{x}} = \mathcal{R}^{-1}\vec{x} - (1 - \gamma_{\tilde{v}})\frac{\vec{x} \cdot \vec{\tilde{v}}}{\tilde{v}^2}\mathcal{R}^{-1}\vec{\tilde{v}} - \gamma_{\tilde{v}}\, t\, \mathcal{R}^{-1}\vec{\tilde{v}} \tag{14}$$

$$\breve{\vec{v}} = -\mathcal{R}^{-1}\vec{\tilde{v}} \, . \tag{15}$$

Then, in much the same way as we did in §2.1 we ascribe the following association $\vec{x} \to \vec{x}^* \in \mathcal{V}$, $\breve{\vec{x}} \to \breve{\vec{x}}^* \in \mathcal{V}$ with $\vec{x}^* := \vec{x}$, $\breve{\vec{x}}^* := \mathcal{R}^{-1}\vec{x} - (1 - \gamma_{\tilde{v}})\frac{\vec{x} \cdot \vec{\tilde{v}}}{\tilde{v}^2}\mathcal{R}^{-1}\vec{\tilde{v}} - \gamma_{\tilde{v}}\, t\, \mathcal{R}^{-1}\vec{\tilde{v}}$, and $\vec{\tilde{v}}_{S\check{S}} \to \vec{v}^* \in \mathcal{V}$, $\vec{\tilde{v}}_{\check{S}S} \to \breve{\vec{v}}^* \in \mathcal{V}$ with $\vec{v}^* := \vec{\tilde{v}}$ and $\breve{\vec{v}}^* := \breve{\vec{v}} = -\mathcal{R}^{-1}\vec{\tilde{v}}$. They are represented in the space $\mathcal{V}$ as shown in figure (16)

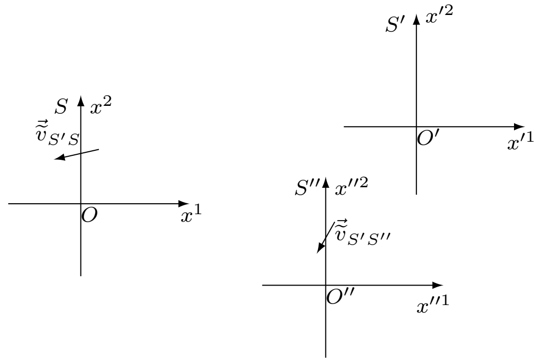


(16)

3 The analysis of Thomas precession according to Moller

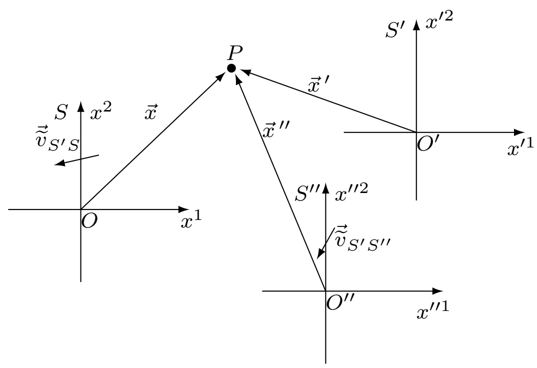
3.1 The basic configuration for the analysis

Let us consider three inertial reference frames S, S', S'' with S and S'' having axis parallel to S'. The frame S' will be used merely as an auxiliary frame to relate S and S'', and an observer in frame S' sees the following configuration between the frames



$$(17)$$

An event P is registered as $(t,\vec{x}),(t',\vec{x}\,'),(t'',\vec{x}\,'')$ respectively by S, S' and S''



$$(18)$$

where

$$S \overset{L(\vec{\tilde{v}}_{SS'})}{\longrightarrow} S' : \begin{cases} \vec{x}\,' = \vec{x} - (1-\gamma_{\tilde{v}_{SS'}})\frac{\vec{x}\cdot\vec{\tilde{v}}_{SS'}}{\tilde{v}^2_{SS'}}\vec{\tilde{v}}_{SS'} - \gamma_{\tilde{v}_{SS'}}\, t\, \vec{\tilde{v}}_{SS'} \\ t' = \gamma_{\tilde{v}_{SS'}}\left(t - \frac{\vec{x}\cdot\vec{\tilde{v}}_{SS'}}{c^2}\right) \end{cases} \tag{19}$$

$$S' \overset{L(\vec{\tilde{v}}_{S'S''})}{\longrightarrow} S'' : \begin{cases} \vec{x}\,'' = \vec{x}\,' - (1-\gamma_{\tilde{v}_{S'S''}})\frac{\vec{x}\,'\cdot\vec{\tilde{v}}_{S'S''}}{\tilde{v}^2_{S'S''}}\vec{\tilde{v}}_{S'S''} - \gamma_{\tilde{v}_{S'S''}}\, t'\, \vec{\tilde{v}}_{S'S''} \\ t'' = \gamma_{\tilde{v}_{S'S''}}\left(t' - \frac{\vec{x}\,'\cdot\vec{\tilde{v}}_{S'S''}}{c^2}\right) . \end{cases} \tag{20}$$

It is straightforward to show that from (19) and (20) we have

$$\begin{cases} \vec{\tilde{v}}_{S'S} = -\vec{\tilde{v}}_{SS'} \\ \vec{\tilde{v}}_{S''S'} = -\vec{\tilde{v}}_{S'S''} \ , \end{cases} \tag{21}$$

which confirms that S and S'' have their axis parallel to S'. Now, what can we say about S and S''? Do these frames have parallel axis? In order to analyze this, let us calculate the velocities $\vec{\tilde{v}}_{SS''}$ and $\vec{\tilde{v}}_{S''S}$ and compare them.

We can write $\vec{\tilde{v}}_{SS''}$ in terms of $\vec{\tilde{v}}_{SS'}$, $\vec{\tilde{v}}_{S'S''}$ using the expression for the relativistic addition of velocities (see Apendix B for a derivation of this formula).

$$\vec{\tilde{v}}_{SS''} := \vec{\tilde{v}}_{S'S''} \oplus \vec{\tilde{v}}_{SS'} = \frac{\vec{\tilde{v}}_{S'S''} + \gamma_{\tilde{v}_{SS'}}\vec{\tilde{v}}_{SS'} - (1-\gamma_{\tilde{v}_{SS'}})\frac{\vec{\tilde{v}}_{SS'}\cdot\vec{\tilde{v}}_{S'S''}}{\tilde{v}^2_{SS'}}\vec{\tilde{v}}_{SS'}}{\gamma_{\tilde{v}_{SS'}}\left(1 + \frac{\vec{\tilde{v}}_{SS'}\cdot\vec{\tilde{v}}_{S'S''}}{c^2}\right)} \ , \tag{22}$$

and there is a similar expression relating $\vec{\tilde{v}}_{S''S}$ with $\vec{\tilde{v}}_{S'S}$ and $\vec{\tilde{v}}_{S''S'}$,

$$\vec{\tilde{v}}_{S''S} = \vec{\tilde{v}}_{S'S} \oplus \vec{\tilde{v}}_{S''S'} := \frac{\vec{\tilde{v}}_{S'S} + \gamma_{\tilde{v}_{S''S'}} \vec{\tilde{v}}_{S''S'} - (1-\gamma_{\tilde{v}_{S''S'}}) \frac{\vec{\tilde{v}}_{S'S}\cdot\vec{\tilde{v}}_{S''S'}}{\tilde{v}^2_{S''S'}} \vec{\tilde{v}}_{S''S'}}{\gamma_{\tilde{v}_{S''S'}} \left(1 + \frac{\vec{\tilde{v}}_{S'S}\cdot\vec{\tilde{v}}_{S''S'}}{c^2}\right)} \ . \tag{23}$$

It is evident from (22), (23) that $\vec{\tilde{v}}_{S''S} \neq -\vec{\tilde{v}}_{SS''}$, but

$$\tilde{v}^2_{SS''} = \tilde{v}^2_{S''S} = \frac{|\vec{\tilde{v}}_{SS'} + \vec{\tilde{v}}_{S'S''}|^2 + \frac{1}{c^2}\Big((\vec{\tilde{v}}_{SS'}\cdot\vec{\tilde{v}}_{S'S''})^2 - \tilde{v}^2_{SS'}\tilde{v}^2_{S'S''}\Big)}{\left(1+\frac{\vec{\tilde{v}}_{SS'}\cdot\vec{\tilde{v}}_{S'S''}}{c^2}\right)^2} \ , \tag{24}$$

then $\vec{\tilde{v}}_{S''S}$ and $-\vec{\tilde{v}}_{SS''}$ are related by a rotation,

$$\vec{\tilde{v}}_{S''S} = -\mathcal{R}^{-1}\vec{\tilde{v}}_{SS''} \ , \tag{25}$$

which shows the axis of the moving frames S and S'' have a rotation, therefore the interpretation given in section 2.2 suggests that $\vec{x}$ and $\vec{x}\,''$ are related by a transformation similar to (14) and then that $(t, \vec{x})$ and $(t'', \vec{x}\,'')$ are related by

$$S \to S'' : \begin{cases} \vec{x}\,'' = \mathcal{R}^{-1}\vec{x} - (1-\gamma_{\tilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\tilde{v}}_{SS''}}{\tilde{v}^2_{SS''}}\mathcal{R}^{-1}\vec{\tilde{v}}_{SS''} - \gamma_{\tilde{v}_{SS''}} t \mathcal{R}^{-1}\vec{\tilde{v}}_{SS''} \\ t'' = \gamma_v\Big(t - \frac{\vec{x}\cdot\vec{\tilde{v}}_{SS''}}{c^2}\Big) \end{cases} \tag{26}$$

or, more compactly

$$(t'', \vec{x}\,'') = \mathcal{R}^{-1} \circ L(\vec{\tilde{v}}_{SS''})(t, \vec{x}) \ . \tag{27}$$

3.2 The composition of two Lorentz transformations

The same relation between the readings of frames S and S'' may be obtained by replacing the expressions for $\vec{x}\,'$ and t' given in (19) into (20). Here, we obtain

$$\vec{x}\,''\big(t'(t,\vec{x},\vec{\tilde{v}}_{SS'}), \vec{x}\,'(t,\vec{x},\vec{\tilde{v}}_{SS'}), \vec{\tilde{v}}_{S'S''}\big), \ \ t''\big(t'(t,\vec{x},\vec{\tilde{v}}_{SS'}), \vec{x}\,'(t,\vec{x},\vec{\tilde{v}}_{SS'}), \vec{\tilde{v}}_{S'S''}\big)$$

where

$$\begin{aligned} &\vec{x}\,''\big(t'(t,\vec{x},\vec{\tilde{v}}_{SS'}), \vec{x}\,'(t,\vec{x},\vec{\tilde{v}}_{SS'}), \vec{\tilde{v}}_{S'S''}\big) = \\ &\quad = \vec{x} - (1-\gamma_{\tilde{v}_{SS'}})\frac{\vec{x}\cdot\vec{\tilde{v}}_{SS'}}{\tilde{v}^2_{SS'}}\,\vec{\tilde{v}}_{SS'} - (1-\gamma_{\tilde{v}_{S'S''}})\frac{\vec{x}\cdot\vec{\tilde{v}}_{S'S''}}{\tilde{v}^2_{S'S''}}\vec{\tilde{v}}_{S'S''} + \\ &\qquad +(1-\gamma_{\tilde{v}_{SS'}})(1-\gamma_{\tilde{v}_{S'S''}})\frac{\vec{\tilde{v}}_{SS'}\cdot\vec{\tilde{v}}_{S'S''}}{\tilde{v}^2_{S'S''}}\frac{\vec{x}\cdot\vec{\tilde{v}}_{SS'}}{\tilde{v}^2_{SS'}}\,\vec{\tilde{v}}_{S'S''} + \gamma_{\tilde{v}_{SS'}}\gamma_{\tilde{v}_{S'S''}}\frac{\vec{x}\cdot\vec{\tilde{v}}_{SS'}}{c^2}\vec{\tilde{v}}_{S'S''} + \\ &\qquad +\Big(-\gamma_{\tilde{v}_{SS'}}\vec{\tilde{v}}_{SS'} + (1-\gamma_{\tilde{v}_{S'S''}})\gamma_{\tilde{v}_{SS'}}\frac{\vec{\tilde{v}}_{SS'}\cdot\vec{\tilde{v}}_{S'S''}}{\tilde{v}^2_{S'S''}}\vec{\tilde{v}}_{S'S''} - \gamma_{\tilde{v}_{S'S''}}\gamma_{\tilde{v}_{SS'}}\vec{\tilde{v}}_{S'S''}\Big)t \end{aligned} \tag{28}$$

$$\begin{aligned} &t''\big(t'(t,\vec{x},\vec{\tilde{v}}_{SS'}), \vec{x}\,'(t,\vec{x},\vec{\tilde{v}}_{SS'}), \vec{\tilde{v}}_{S'S''}\big) = \\ &\quad = \gamma_{\tilde{v}_{SS'}}\gamma_{\tilde{v}_{S'S''}}\Big(1+\frac{\vec{\tilde{v}}_{SS'}\cdot\vec{\tilde{v}}_{S'S''}}{c^2}\Big)t - \gamma_{\tilde{v}_{S'S''}}\frac{\vec{x}\cdot\vec{\tilde{v}}_{S'S''}}{c^2} - \gamma_{\tilde{v}_{SS'}}\gamma_{\tilde{v}_{S'S''}}\frac{\vec{x}\cdot\vec{\tilde{v}}_{SS'}}{c^2} + \\ &\qquad +\gamma_{\tilde{v}_{S'S''}}(1-\gamma_{\tilde{v}_{SS'}})\frac{\vec{\tilde{v}}_{SS'}\cdot\vec{\tilde{v}}_{S'S''}}{\tilde{v}^2_{SS'}}\frac{\vec{x}\cdot\vec{\tilde{v}}_{SS'}}{c^2} \ . \end{aligned} \tag{29}$$

This corresponds to the composition of two Lorentz transformations

$$(t'', \vec{x}\,'') = L(\vec{\tilde{v}}_{S'S''}) \circ L(\vec{\tilde{v}}_{SS'})(t, \vec{x}) \ . \tag{30}$$

3.3 The Thomas precession

Before we proceed, let us simplify our notation. We denote

$$\vec{v} \equiv \vec{\tilde{v}}_{SS'},\ \vec{u} \equiv \vec{\tilde{v}}_{S'S''},\ \vec{w} \equiv \vec{\tilde{v}}_{SS''} \ . \tag{31}$$

With this notation we write the velocities $\vec{\tilde{v}}_{SS''} = \vec{\tilde{v}}_{S'S''} \oplus \vec{\tilde{v}}_{SS'}$ and $\vec{\tilde{v}}_{S''S} = \vec{\tilde{v}}_{S'S} \oplus \vec{\tilde{v}}_{S''S'}$ given in (22), (23) as

$$\vec{u}\oplus\vec{v} := \frac{\vec{u} + \gamma_v \vec{v} - (1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\vec{v}}{\gamma_v\left(1+\frac{\vec{u}\cdot\vec{v}}{c^2}\right)} \ , \qquad (-\vec{v})\oplus(-\vec{u}) = \frac{-\vec{v} - \gamma_u \vec{u} + (1-\gamma_u)\frac{\vec{v}\cdot\vec{u}}{u^2}\vec{u}}{\gamma_u\left(1+\frac{\vec{v}\cdot\vec{u}}{c^2}\right)} \ . \tag{32}$$

and equation (25) becomes

$$\frac{-\vec{v} - \gamma_u \vec{u} + (1-\gamma_u)\frac{\vec{v}\cdot\vec{u}}{u^2}\vec{u}}{\gamma_u\left(1+\frac{\vec{v}\cdot\vec{u}}{c^2}\right)} = -\mathcal{R}^{-1}\ \frac{\vec{u} + \gamma_v \vec{v} - (1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\vec{v}}{\gamma_v\left(1+\frac{\vec{u}\cdot\vec{v}}{c^2}\right)} \ . \tag{33}$$

We also denote

$$\begin{aligned} (\check{t}, \check{\vec{x}}) &\equiv L(\vec{\tilde{v}}_{S'S''}) \circ L(\vec{\tilde{v}}_{SS'})(t, \vec{x}) \\ (\hat{t}, \hat{\vec{x}}) &\equiv L(\vec{\tilde{v}}_{SS''})(t, \vec{x}) \ , \end{aligned}$$

which reads explicitly as

$$\left\{ \begin{aligned} \check{\vec{x}} \ &= \ \vec{x} - (1-\gamma_v)\tfrac{\vec{x}\cdot\vec{v}}{v^2}\ \vec{v} - (1-\gamma_u)\tfrac{\vec{x}\cdot\vec{u}}{u^2}\vec{u} + (1-\gamma_u)(1-\gamma_v)\tfrac{\vec{x}\cdot\vec{v}}{v^2}\ \tfrac{\vec{u}\cdot\vec{v}}{u^2}\ \vec{u} + \gamma_u\gamma_v\tfrac{\vec{x}\cdot\vec{v}}{c^2}\vec{u} + \\ &\quad + \big(-\gamma_v\vec{v} + (1-\gamma_u)\gamma_v\tfrac{\vec{u}\cdot\vec{v}}{u^2}\vec{u} - \gamma_u\gamma_v\vec{u}\big)t \\ \check{t} \ &= \ \gamma_u\gamma_v\Big(1+\tfrac{\vec{u}\cdot\vec{v}}{c^2}\Big)t - \gamma_u\tfrac{\vec{x}\cdot\vec{u}}{c^2} - \gamma_u\gamma_v\tfrac{\vec{x}\cdot\vec{v}}{c^2} + \gamma_u(1-\gamma_v)\tfrac{\vec{u}\cdot\vec{v}}{v^2}\tfrac{\vec{x}\cdot\vec{v}}{c^2} \ , \end{aligned} \right. \tag{34}$$

and

$$\left\{ \begin{aligned} \hat{\vec{x}} \ &= \ \vec{x} - \Big[1-\gamma_u\gamma_v\Big(1+\tfrac{\vec{u}\cdot\vec{v}}{c^2}\Big)\Big]\frac{\left(\vec{x}\cdot\vec{u} + \gamma_v\,\vec{x}\cdot\vec{v} - (1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\,\vec{x}\cdot\vec{v}\right)}{\gamma_v^2\left(u^2+v^2+2\vec{u}\cdot\vec{v} - \frac{u^2v^2}{c^2} + \frac{(\vec{u}\cdot\vec{v})^2}{c^2}\right)}\big(\vec{u} + \gamma_v\vec{v} - (1-\gamma_v)\tfrac{\vec{u}\cdot\vec{v}}{v^2}\ \vec{v}\big) + \\ &\quad -\gamma_u\big(\vec{u} + \gamma_v\vec{v} - (1-\gamma_v)\tfrac{\vec{u}\cdot\vec{v}}{v^2}\ \vec{v}\big)t \\ \hat{t} \ &= \ \gamma_u\gamma_v\Big(1+\tfrac{\vec{u}\cdot\vec{v}}{c^2}\Big)t - \gamma_u\tfrac{\vec{x}\cdot\vec{u}}{c^2} - \gamma_u\gamma_v\tfrac{\vec{x}\cdot\vec{v}}{c^2} + \gamma_u(1-\gamma_v)\tfrac{\vec{u}\cdot\vec{v}}{v^2}\tfrac{\vec{x}\cdot\vec{v}}{c^2} \ . \end{aligned} \right. \tag{35}$$

where in obtaining (35) we have used the well know expressions

$$\begin{aligned} w^2 \ &= \ \frac{u^2 + v^2 + 2\vec{u}\cdot\vec{v} + \frac{1}{c^2}\Big((\vec{u}\cdot\vec{v})^2 - u^2v^2\Big)}{\Big(1+\frac{\vec{u}\cdot\vec{v}}{c^2}\Big)^2} \\ \gamma_w \ &= \ \gamma_u\gamma_v\Big(1+\frac{\vec{v}\cdot\vec{u}}{c^2}\Big) \ . \end{aligned}$$

From (27) and (30) we get

$$\boxed{\mathcal{R}^{-1} \circ L(\vec{\check{v}}_{SS''})(t,\vec{x}) = L(\vec{\check{v}}_{S'S''}) \circ L(\vec{\check{v}}_{SS'})(t,\vec{x})}\,, \tag{36}$$

and since $\check{t} = \hat{t}$ we see that relative to the space coordinate we have

$$\boxed{\hat{\vec{x}} = \mathcal{R}\,\check{\vec{x}}}\,, \tag{37}$$

which is the analytical content of the Thomas precession that we must check.

Remark: We rewrite (32) as

$$(-\vec{v})\oplus(-\vec{u}) = -\frac{\vec{v} + \gamma_u\vec{u} - (1-\gamma_u)\frac{\vec{v}\cdot\vec{u}}{u^2}\vec{u}}{\gamma_u\big(1+\frac{\vec{v}\cdot\vec{u}}{c^2}\big)} = -(\vec{v}\oplus\vec{u}) \tag{38}$$

then (33) becomes

$$\vec{v}\oplus\vec{u} = \mathcal{R}^{-1}(\vec{u}\oplus\vec{v}) \tag{39}$$

which indicates the anticommutativity of the relativistic adddition formula.

3.4 The form of the rotation $\mathcal{R}$

We describe rotations using the axis-angle parameterization. Recall from [10] that in this view a rotation $\mathcal{R}_{\widehat{n}}(\varphi)$ through an angle φ and along an axis whose direction is fixed by a unitary vector $\widehat{n}$ acts on a vector $\vec{X}$ by

$$\mathcal{R}_{\widehat{n}}(\varphi)\vec{X} = \cos\varphi\;\vec{X} + (1-\cos\varphi)\;(\widehat{n}\cdot\vec{X})\widehat{n} - \chi\sin\varphi\;(\widehat{n}\times\vec{X})\;, \tag{40}$$

where $|\chi| = 1$. Here we do not fix the form of χ in order to allow for the use of any of the two conventions discussed by Goldstein [10] (see formulas (4-92) and (4-92′) of [10]). We recall that using $\chi = 1$ we are measuring φ in a clockwise sense, while taking $\chi = -1$ we are measuring φ in a counterclockwise sense. Using that $\mathcal{R}_{\widehat{n}}^{-1}(\varphi) = \mathcal{R}_{\widehat{n}}(-\varphi)$ we rewrite (33) as

$$\frac{-\vec{v} - \gamma_u\vec{u} + (1-\gamma_u)\frac{\vec{v}\cdot\vec{u}}{u^2}\vec{u}}{\gamma_u\big(1+\frac{\vec{v}\cdot\vec{u}}{c^2}\big)} \;=\; -\mathcal{R}_{\widehat{n}}(-\varphi)\frac{\vec{u} + \gamma_v\vec{v} - (1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\vec{v}}{\gamma_v\big(1+\frac{\vec{u}\cdot\vec{v}}{c^2}\big)} \tag{41}$$

and from (40) we have that

$$\begin{aligned}
\frac{\vec{v} + \gamma_u\vec{u} - (1-\gamma_u)\frac{\vec{u}\cdot\vec{v}}{u^2}\,\vec{u}}{\gamma_u\big(1+\frac{\vec{u}\cdot\vec{v}}{c^2}\big)} \;=\;& \cos\varphi\;\frac{\vec{u} + \gamma_v\vec{v} - (1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\,\vec{v}}{\gamma_v\big(1+\frac{\vec{u}\cdot\vec{v}}{c^2}\big)} + \\
&+(1-\cos\varphi)\;\left(\widehat{n}\cdot\frac{\big(\vec{u} + \gamma_v\vec{v} - (1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\,\vec{v}\big)}{\gamma_v\big(1+\frac{\vec{u}\cdot\vec{v}}{c^2}\big)}\right)\widehat{n} \\
&+\chi\sin\varphi\;\widehat{n}\times\frac{\big(\vec{u} + \gamma_v\vec{v} - (1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\,\vec{v}\big)}{\gamma_v(1+\frac{\vec{u}\cdot\vec{v}}{c^2})}\;.
\end{aligned} \tag{42}$$

Here there are two cases to consider.

3.4.1 The vectors $\vec{u}$ and $\vec{v}$ are non-parallel

Taking the scalar product of both sides of (42) with $\widehat{n}$ we obtain

$$\widehat{n}\cdot\vec{v}\left[\gamma_v-\gamma_u\gamma_v+\gamma_u(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\right]+\widehat{n}\cdot\vec{u}\left[-\gamma_u+\gamma_u\gamma_v-\gamma_v(1-\gamma_u)\frac{\vec{u}\cdot\vec{v}}{u^2}\right] \;=\; 0$$

and since this condition should be verified for any choices of $\vec{u}$ e $\vec{v}$ we should have $\widehat{n}\cdot\vec{u}=\widehat{n}\cdot\vec{v}=0$, which fixes $\widehat{n}$ as

$$\widehat{n}=\lambda\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}\ , \tag{43}$$

with $|\lambda|=1$. We do not fix the direction of the unitary $\widehat{n}$ that could be parallel or antiparallel to $\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}$ in order to show that whatever choice we take it doesn't change the form of the rotation we will find at the end. Expression (42) then becomes

$$\begin{aligned}
&\frac{\vec{v}+\gamma_u\vec{u}-(1-\gamma_u)\frac{\vec{u}\cdot\vec{v}}{u^2}\,\vec{u}}{\gamma_u\big(1+\frac{\vec{u}\cdot\vec{v}}{c^2}\big)}=\\
&\quad=\cos\varphi\;\frac{\vec{u}+\gamma_v\vec{v}-(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\,\vec{v}}{\gamma_v\big(1+\frac{\vec{u}\cdot\vec{v}}{c^2}\big)}+\\
&\qquad+(1-\cos\varphi)\left(\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}\cdot\frac{\big(\vec{u}+\gamma_v\vec{v}-(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\,\vec{v}\big)}{\gamma_v\big(1+\frac{\vec{u}\cdot\vec{v}}{c^2}\big)}\right)\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}+\\
&\qquad+\sin\varphi\;\chi\lambda\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}\times\left(\frac{\big(\vec{u}+\gamma_v\vec{v}-(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\,\vec{v}\big)}{\gamma_v(1+\frac{\vec{u}\cdot\vec{v}}{c^2})}\right)
\end{aligned}$$

$\therefore$

$$\begin{aligned}
\vec{v}+\gamma_u\vec{u}-(1-\gamma_u)\frac{\vec{u}\cdot\vec{v}}{u^2}\,\vec{u} \;=\;& \cos\varphi\;\frac{\gamma_u}{\gamma_v}\vec{u}+\cos\varphi\;\gamma_u\vec{v}-\cos\varphi\;\frac{\gamma_u}{\gamma_v}(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\,\vec{v}+\\
&+\chi\lambda\sin\varphi\;\frac{\gamma_u}{\gamma_v}\left(\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}\right)\times\vec{u}+\chi\lambda\sin\varphi\;\gamma_u\left(\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}\right)\times\vec{v}+\\
&-\chi\lambda\sin\varphi\;\frac{\gamma_u}{\gamma_v}(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\left(\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}\right)\times\vec{v}\ .
\end{aligned}$$

Using

$$\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}\times\vec{u}=-\frac{1}{|\vec{v}\times\vec{u}|}[u^2\vec{v}-(\vec{v}\cdot\vec{u})\vec{u}]\ ,\qquad \frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}\times\vec{v}=\frac{1}{|\vec{v}\times\vec{u}|}[v^2\vec{u}-(\vec{u}\cdot\vec{v})\vec{v}]$$

we obtain

$$\begin{aligned}
0 \;=\;& \vec{u}\Big(-\gamma_u+(1-\gamma_u)\frac{\vec{u}\cdot\vec{v}}{u^2}+\frac{\gamma_u}{\gamma_v}\cos\varphi+\gamma_u\frac{\chi\lambda}{|\vec{v}\times\vec{u}|}\vec{v}\cdot\vec{u}\;\sin\varphi\;+\gamma_u\frac{\chi\lambda}{|\vec{v}\times\vec{u}|}v^2\;\sin\varphi\Big)+\\
&+\vec{v}\Big(-1+\gamma_u\cos\varphi-\frac{\gamma_u}{\gamma_v}(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\;\cos\varphi-\frac{\gamma_u}{\gamma_v}\frac{\chi\lambda}{|\vec{v}\times\vec{u}|}u^2\sin\varphi+\\
&\quad-\gamma_u\frac{\chi\lambda}{|\vec{v}\times\vec{u}|}(\vec{u}\cdot\vec{v})\sin\varphi+\frac{\gamma_u}{\gamma_v}(1-\gamma_v)\frac{(\vec{u}\cdot\vec{v})^2}{v^2}\frac{\chi\lambda}{|\vec{v}\times\vec{u}|}\sin\varphi\Big)\ .
\end{aligned}$$

Since this relation must be verified for any choice of $\vec{u}$ and $\vec{v}$ we should have

$$\gamma_u - (1-\gamma_u)\frac{\vec{u}\cdot\vec{v}}{u^2} = \frac{\gamma_u}{\gamma_v}\cos\varphi + \gamma_u \frac{\chi\lambda}{|\vec{v}\times\vec{u}|}\Big(\vec{u}\cdot\vec{v} + v^2\Big)\sin\varphi$$

and

$$1 = \Big(\gamma_u - \frac{\gamma_u}{\gamma_v}(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\Big)\cos\varphi - \frac{\gamma_u}{\gamma_v}\frac{\chi\lambda}{|\vec{v}\times\vec{u}|}\Big(u^2 + \gamma_v(\vec{u}\cdot\vec{v}) - (1-\gamma_v)\frac{(\vec{u}\cdot\vec{v})^2}{v^2}\Big)\sin\varphi\ ,$$

which solving gives

$$\left\{\begin{array}{l} \cos\varphi = \Big\{\gamma_v^{-1}u^2 + \gamma_u^{-1}v^2 + \big(1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1}\big)\vec{u}\cdot\vec{v} - \big(\gamma_u^{-1}-1\big)\frac{(\vec{u}\cdot\vec{v})^2}{u^2} + \\ \qquad -\big(\gamma_v^{-1}-1\big)\frac{(\vec{u}\cdot\vec{v})^2}{v^2} + (\gamma_u^{-1}-1)(\gamma_v^{-1}-1)\frac{(\vec{u}\cdot\vec{v})^3}{u^2v^2}\Big\}\Big/\Big\{u^2 + 2\,\vec{u}\cdot\vec{v} + v^2 - \frac{u^2v^2}{c^2} + \frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big\} \end{array}\right. \tag{44}$$

$$\left\{\begin{array}{l} \sin\varphi = -\frac{|\vec{v}\times\vec{u}|}{\chi\lambda}\Big\{-1 + \gamma_u^{-1}\gamma_v^{-1} + (\gamma_u^{-1}-1)\frac{\vec{u}\cdot\vec{v}}{u^2} + (\gamma_v^{-1}-1)\frac{\vec{u}\cdot\vec{v}}{v^2} + \\ \qquad -(\gamma_u^{-1}-1)(\gamma_v^{-1}-1)\frac{(\vec{u}\cdot\vec{v})^2}{u^2v^2}\Big\}\Big/\Big\{u^2 + 2\,\vec{u}\cdot\vec{v} + v^2 - \frac{u^2v^2}{c^2} + \frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big\}\ . \end{array}\right. \tag{45}$$

Then, transformation (37) becomes

$$\hat{\vec{x}} = \mathcal{R}_{\hat{n}}(\varphi)\check{\vec{x}} = \cos\varphi\ \check{\vec{x}} + (1-\cos\varphi)\left(\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}\cdot\check{\vec{x}}\right)\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|} - \chi\lambda\sin\varphi\left(\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}\right)\times\check{\vec{x}}\ , \tag{46}$$

with $\cos\varphi$ and $\sin\varphi$ given by (44), (45). In section 7 we will check the validity of (46).

3.4.2 The vectors $\vec{u}$ and $\vec{v}$ are parallel

Let us consider $\vec{u} = \xi\vec{v}$. We denote the rotation as $\mathcal{R}^*_{\hat{n}}(\varphi)$ in order to distinguish it from the form we obtained when considering $\vec{u} \neq \xi\vec{v}$. Then (41) becomes

$$\frac{(1+\xi)}{1+\xi\frac{v^2}{c^2}}\vec{v} = \mathcal{R}^*_{\hat{n}}(\varphi)\frac{(1+\xi)}{1+\xi\frac{v^2}{c^2}}\vec{v}\ , \tag{47}$$

where [3]

$$\mathcal{R}^*_{\hat{n}}(\varphi) = 1 \quad \text{if} \quad \xi \neq -1, \quad \xi \neq -\frac{c^2}{v^2}\ . \tag{48}$$

We discard the case $\xi = -\frac{c^2}{v^2}$ because from $\vec{u} = \xi\vec{v}$ we would have

$$\vec{u} = -\frac{c^2}{v^2}\vec{v}\ \therefore\ uv = c^2$$

which is not attainable since $u < c,\ v < c$. Before considering the case $\xi = -1$ we observe that if we allow $\vec{u} = \xi\vec{v}$ in the equations (44), (45) we get

$$\begin{aligned} \cos\varphi\big|_{\vec{u}=\xi\vec{v}} &= 1 \\ \sin\varphi\big|_{\vec{u}=\xi\vec{v}} &= -\frac{|\xi\vec{v}\times\vec{v}|}{\chi\lambda}\frac{(-1-\xi^{-1}+\gamma_{\xi v}^{-1}\xi^{-1}+\gamma_v^{-1})}{v^2(1+\xi)} = 0 \end{aligned} \tag{49}$$

[3] The rotation axis $\hat{n}$ is not specified in this case.

then $\mathcal{R}_{\widehat{n}}(\varphi)\big|_{\vec{u}=\xi\vec{v}} = 1$ which agrees with the form of $\mathcal{R}^*_{\widehat{n}}(\varphi)$ given in (48).

If $\xi = -1$ equation (47) is trivially verified without specifying the form of $\mathcal{R}_{\widehat{n}}(\varphi)$. From (49) we have

$$\lim_{\xi\to -1} \cos\varphi\big|_{\vec{u}=\xi\vec{v}} = 1 \ , \qquad \lim_{\xi\to -1} \sin\varphi\big|_{\vec{u}=\xi\vec{v}} = 0 \ ,$$

i.e.

$$\lim_{\xi\to -1} \mathcal{R}_{\widehat{n}}(\varphi)\big|_{\vec{u}=\xi\vec{v}} = 1 \ ,$$

then in the case $\xi = -1$ we may fix the form of the rotation $\mathcal{R}^*_{\widehat{n}}(\varphi)$ as indicated by this limit, i.e.

$$\mathcal{R}^*_{\widehat{n}}(\varphi) = 1 \quad \text{if} \quad \xi = -1 \ . \tag{50}$$

Then we may take (44), (45), (46) as the form of the Thomas rotation for arbitrary $\vec{u}$ and $\vec{v}$ provided that $u < c, v < c$.

3.5 The infinitesimal form of the rotation

Let us assume $\vec{u}$ is small compared to $\vec{v}$, more precisely, let us neglect terms of u^2 and consider $u \ll v$ in the expressions for $\cos\varphi$ (44) and $\sin\varphi$ (45) and then analyze the form of $\hat{\vec{x}} = \mathcal{R}\check{\vec{x}}$ given in (46). We have

$$\cos\varphi \simeq 1 \ , \qquad \sin\varphi \simeq -\frac{|\vec{v}\times\vec{u}|}{\lambda\,\chi}\frac{\left(\gamma_v^{-1}-1\right)}{v^2} \ ,$$

which renders (46) as

$$\hat{\vec{x}} = \check{\vec{x}} - \frac{\left(1-\gamma_v^{-1}\right)}{v^2}\left(\vec{v}\times\vec{u}\right)\times\check{\vec{x}} \ ,$$

which is the same expression found by C Moller [4].

Part II: Reviewing the Thomas Precession under the assumption the local time depends on the state of motion of the frame.

4 The absolute time and some of its consequences

We introduce the absolute time τ by setting the following assumptions (for more details see [8], [9]).

A1. Inertial frames S and S' are endowed with another time variable τ such that any event P is registered as $(\tau, t, \vec{x})$ and $(\tau, t', \vec{x}\,')$. Therefore one can calculate the following velocities

$$\vec{\vec{v}}_{SS'} := \frac{d\vec{x}_{O'}}{dt_{O'}}, \quad \vec{v}_{SS'} := \frac{d\vec{x}_{O'}}{d\tau_{O'}} \tag{51}$$

$$\vec{\vec{v}}_{S'S} := \frac{d\vec{x}\,'_O}{dt'_O}, \quad \vec{v}_{S'S} := \frac{d\vec{x}\,'_O}{d\tau_O} \ . \tag{52}$$

where O' (O) denotes the origin of frame S' (S).
A2. τ, t, t' satisfy the relation

$$\tau = \frac{\widetilde{v}_{SS'}}{v_{SS'}}\Big\{(1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS'}}{\widetilde{v}^2_{SS'}}+\gamma_{\widetilde{v}_{SS'}}\,t\Big\} = \frac{\widetilde{v}_{S'S}}{v_{S'S}}\Big\{(1-\gamma_{\widetilde{v}_{S'S}})\frac{\vec{x}\,'\cdot\vec{\widetilde{v}}_{S'S}}{\widetilde{v}^2_{S'S}}+\gamma_{\widetilde{v}_{S'S}}\,t'\Big\} \tag{53}$$

A3. $(t,\vec{x})$ and $(t',\vec{x}\,')$ are related by the Lorentz transformation (19).

Remark: From (53) we write

$$t = \gamma^{-1}_{\widetilde{v}_{SS'}}\left\{\frac{v_{SS'}}{\widetilde{v}_{SS'}}\tau-(1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS'}}{\widetilde{v}^2_{SS'}}\right\}, \tag{54}$$

which shows how the time t depends on the state of motion of the frames. We call t the *local time* and denote it as $t\equiv t_{SS'}$. Then every derivative of the position relative to the local time produces a velocity depending on the relative velocity between the frames, i.e.

$$\vec{\widetilde{u}} = \frac{d\vec{x}}{dt_{SS'}} \Rightarrow \vec{\widetilde{u}} \equiv \vec{\widetilde{u}}(v_{SS'}) \ . \tag{55}$$

In particular,

$$\vec{\widetilde{v}}_{SS'} = \vec{\widetilde{v}}_{SS'}(v_{SS'}) \ . \tag{56}$$

Result 1

i. $\vec{\widetilde{v}}_{S'S} = -\vec{\widetilde{v}}_{SS'}$

ii. $\vec{v}_{S'S} = -\vec{v}_{SS'}$

iii. $\vec{x}\,' = \vec{x} - \vec{v}_{SS'}\tau$

Proof.
i. This is an immediate consequence of (19). In fact, considering the movement of the origin of frame S as a sucession of events we have $\vec{x}_O = 0$ and

$$\begin{aligned}\vec{x}\,'_O &= -\gamma_{\widetilde{v}_{SS'}}t_O\,\vec{\widetilde{v}}_{SS'}\\ t'_O &= \gamma_{\widetilde{v}_{SS'}}t_O \ ,\end{aligned}$$

therefore

$$\vec{\widetilde{v}}_{S'S} = \frac{d\vec{x}\,'_O}{dt'_O} = \frac{d\vec{x}\,'_O}{dt_O}\frac{dt_O}{dt'_O} \ ,$$

then

$$\vec{\widetilde{v}}_{S'S} = -\vec{\widetilde{v}}_{SS'} \tag{57}$$

and

$$\widetilde{v}_{S'S} = \widetilde{v}_{SS'} \ . \qquad \blacksquare \tag{58}$$

ii. Taking $P \equiv O'$ in (53) we have

$$\frac{\widetilde{v}_{SS'}}{v_{SS'}}\left\{(1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{x}_{O'}\cdot\vec{\widetilde{v}}_{SS'}}{\widetilde{v}^2_{SS'}}+\gamma_{\widetilde{v}_{SS'}}t_{O'}\right\}=\frac{\widetilde{v}_{S'S}}{v_{S'S}}\gamma_{\widetilde{v}_{S'S}}t'_{O'}\ . \tag{59}$$

Considering the inverse transformation of (19) we have

$$\left.\begin{aligned}\vec{x}_{O'}&=-\gamma_{\widetilde{v}_{S'S}}t'_{O'}\vec{\widetilde{v}}_{S'S}\\ t_{O'}&=\gamma_{\widetilde{v}_{S'S}}t'_{O'}\end{aligned}\right\}\Rightarrow \vec{x}_{O'}=\vec{\widetilde{v}}_{SS'}t_{O'},\quad t'_{O'}=t_{O'}\gamma^{-1}_{\widetilde{v}_{S'S}} \tag{60}$$

which replacing in (59) gives

$$\frac{\widetilde{v}_{SS'}}{v_{SS'}}=\frac{\widetilde{v}_{S'S}}{v_{S'S}}$$

and from (58) we get

$$v_{S'S}=v_{SS'}\ . \tag{61}$$

We have

$$\vec{v}_{SS'}=\frac{d\vec{x}_{O'}}{dt_{O'}}\frac{dt_{O'}}{d\tau} \tag{62}$$

and from (53) we write

$$\frac{dt_{O'}}{d\tau}=\gamma^{-1}_{\widetilde{v}_{SS'}}\left\{\frac{v_{SS'}}{\widetilde{v}_{SS'}}-(1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{v}_{SS'}\cdot\vec{\widetilde{v}}_{SS'}}{\widetilde{v}^2_{SS'}}\right\} \tag{63}$$

then

$$\begin{aligned}\vec{v}_{SS'}&=\vec{\widetilde{v}}_{SS'}\gamma^{-1}_{\widetilde{v}_{SS'}}\left\{\frac{v_{SS'}}{\widetilde{v}_{SS'}}-(1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{v}_{SS'}\cdot\vec{\widetilde{v}}_{SS'}}{\widetilde{v}^2_{SS'}}\right\}\\ \therefore\qquad \frac{\vec{v}_{SS'}\cdot\vec{\widetilde{v}}_{SS'}}{\widetilde{v}^2_{SS'}}&=\frac{v_{SS'}}{\widetilde{v}_{SS'}}\end{aligned}$$

which replacing back into (63) gives

$$\frac{dt_{O'}}{d\tau}=\frac{v_{SS'}}{\widetilde{v}_{SS'}}$$

and then we obtain from (62) that

$$\vec{v}_{SS'}=\frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\vec{\widetilde{v}}_{SS'}(v_{SS'})\ . \tag{64}$$

We obtain a similar result to $\vec{v}_{S'S}$

$$\vec{v}_{S'S}=\frac{v_{S'S}}{\widetilde{v}_{S'S}(v_{S'S})}\vec{\widetilde{v}}_{S'S}(v_{S'S})\ . \tag{65}$$

Then from (57), (58), (64), (65) we obtain that

$$\vec{v}_{S'S} = -\vec{v}_{SS'}\ . \qquad \blacksquare \tag{66}$$

iii. From the Lorentz transformation (19) we write

$$\vec{x}\,' = \vec{x} - \Big\{(1-\gamma_{\tilde{v}_{SS'}})\frac{\vec{x}\cdot\vec{\tilde{v}}_{SS'}}{\tilde{v}^2_{SS'}} + \gamma_{\tilde{v}_{SS'}}t\Big\}\vec{\tilde{v}}_{SS'}$$

and from (53) we obtain

$$\vec{x}\,' = \vec{x} - \frac{v_{SS'}}{\tilde{v}_{SS'}}\tau\,\vec{\tilde{v}}_{SS'}\ .$$

Then, from (64) we get

$$\vec{x}\,' = \vec{x} - \vec{v}_{SS'}\tau\ . \qquad \blacksquare \tag{67}$$

Remark: It is straigthforward to check the consistency of equation (53) established in A2 using the Lorentz transformation (19).

4.1 Locating the problem

Having introduced the concept of the absolute time and clarified the dependence of the local time with the state of motion of the observer as expressed in equation (54), it is instructive now to show where and how the analysis of section 3 must be changed. There are two points to be reviewed.

4.1.1 Reviewing the formula for the addition of velocities

The formula for the addition of velocities (22), (23) is a straightforward exercise involving the chain rule and the Lorentz transformation as we will show in section 8. In the standard treatment of SR the calculation assumes that the local time doesn't depend on the state of motion of the frames, therefore on taking derivatives relative to the local time there is no behavior such as the one shown in (55). Then, we may compare directly $\vec{\tilde{v}}_{S''S} = \vec{\tilde{v}}_{S'S} \oplus \vec{\tilde{v}}_{S''S'}$ and $\vec{\tilde{v}}_{SS''} = \vec{\tilde{v}}_{S'S''} \oplus \vec{\tilde{v}}_{SS'}$ as given by their respective expressions (23) and (22).

A new situation arises when we assume the local time depends on the state of motion of the frames for, as we have pointed out in (55), if we fix the pair S, S' we have

$$\begin{aligned}
\vec{\tilde{v}}_{SS''} &= \frac{d\vec{x}_{O''}}{dt_{O''}(v_{SS'})} \Rightarrow \vec{\tilde{v}}_{SS''} \equiv \vec{\tilde{v}}_{SS''}(v_{SS'}) \\
\vec{\tilde{v}}_{S'S''} &= \frac{d\vec{x}\,'_{O''}}{dt'_{O''}(v_{SS'})} \Rightarrow \vec{\tilde{v}}_{S'S''} \equiv \vec{\tilde{v}}_{S'S''}(v_{SS'}) \\
\vec{\tilde{v}}_{SS'} &= \frac{d\vec{x}_{O'}}{dt_{O'}(v_{SS'})} \Rightarrow \vec{\tilde{v}}_{SS'} \equiv \vec{\tilde{v}}_{SS'}(v_{SS'})
\end{aligned}$$

and we expect that $\vec{\tilde{v}}_{SS''}(v_{SS'}) = \vec{\tilde{v}}_{S'S''}(v_{SS'}) \oplus \vec{\tilde{v}}_{SS'}(v_{SS'})$.

A similar analysis applies to $\vec{\tilde{v}}_{S''S} = \vec{\tilde{v}}_{S'S} \oplus \vec{\tilde{v}}_{S''S'}$ if we consider the pair S', S''. We have now

$$\begin{aligned}
\vec{\tilde{v}}_{S''S} &= \frac{d\vec{x}\,''_O}{dt''_O(v_{S''S'})} \Rightarrow \vec{\tilde{v}}_{S''S} \equiv \vec{\tilde{v}}_{S''S}(v_{S''S'}) \\
\vec{\tilde{v}}_{S'S} &= \frac{d\vec{x}\,'_O}{dt'_O(v_{S''S'})} \Rightarrow \vec{\tilde{v}}_{S'S} \equiv \vec{\tilde{v}}_{S'S}(v_{S''S'}) \\
\vec{\tilde{v}}_{S''S'} &= \frac{d\vec{x}\,''_{O'}}{dt''_{O'}(v_{S''S'})} \Rightarrow \vec{\tilde{v}}_{S''S'} \equiv \vec{\tilde{v}}_{S''S'}(v_{S''S'})
\end{aligned}$$

and we expect to have $\vec{\tilde{v}}_{S''S}(v_{S''S'}) = \vec{\tilde{v}}_{S'S}(v_{S''S'}) \oplus \vec{\tilde{v}}_{S''S'}(v_{S''S'})$.

Then, in order to compare $\vec{\tilde{v}}_{S''S}(v_{S''S'})$ and $\vec{\tilde{v}}_{SS''}(v_{SS'})$ we must find out equivalent expressions for $\vec{\tilde{v}}_{S'S}(v_{S''S'})$ and $\vec{\tilde{v}}_{S''S'}(v_{S''S'})$ in terms of $v_{SS'}$. This wil lead us ultimately to an expression of the form

$$\vec{\tilde{v}}_{S''S}(v_{S''S'}) = -\Omega(v_{S''S'}, v_{SS'})\, \vec{\tilde{v}}_{SS''}(v_{SS'}) \tag{68}$$

with Ω being a scalar factor that depends on the absolute velocities $\vec{v}_{SS'} = \frac{d\vec{x}_{O'}}{d\tau}$ and $\vec{v}_{S'S''} = \frac{d\vec{x}\,'_{O''}}{d\tau}$. Therefore, adopting the view the local time depends on the state of motion, we have that equation (68) replaces the form given in equation (25), i.e. **there is no rotation** between the frames S and S''.

4.1.2 Reviewing the formula for the composition of two Lorentz transformations

We also review the composition of Lorentz transformations as given in equation (36)

$$\mathcal{R}^{-1} \circ L(\vec{\tilde{v}}_{SS''})(t, \vec{x}) = L(\vec{\tilde{v}}_{S'S''}) \circ L(\vec{\tilde{v}}_{SS'})(t, \vec{x}) \; .$$

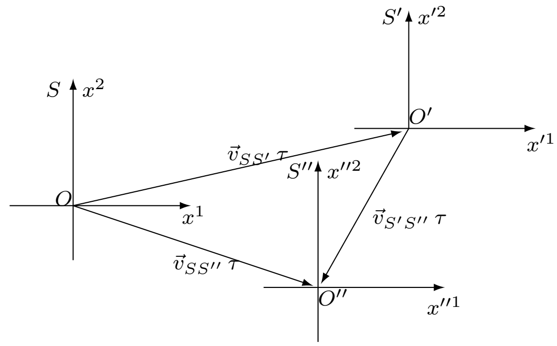
As shown in (68), if there is no rotation between the frames then the form given in equation (36) must be modified. In fact, considering the local time depends on the state of motion of the frames we have $t'_{S'S} \neq t'_{S'S''}$, therefore we cannot replace $t'_{S'S}$, obtained in the Lorentz transformation $(t'_{S'S}, \vec{x}\,') = L(\vec{\tilde{v}}_{SS'})(t_{SS'}, \vec{x})$, into the expression for the Lorentz transformation between S' and S'', e.g. $(t''_{S''S'}, \vec{x}\,'') = L(\vec{\tilde{v}}_{S'S''})(t'_{S'S''}, \vec{x}\,')$ as we did in (30). Therefore, we must rethink the way we compose Lorentz transformations in order to accommodate the fact that $t'_{S'S} \neq t'_{S'S''}$. In section 6 we will show how to compose two Lorentz transformations considering that the local time depends on the state of motion of the frames.

5 The formula for the addition of velocities under the assumption the local time depends on the state of motion of the frames

Here we show how the relativistic formula for the addition of velocities may be obtained from the corresponding Galilean formula. For details we refer the reader to [8, 9].

5.1 $\vec{v}_{SS''} = \vec{v}_{S'S''} + \vec{v}_{SS'} \Rightarrow \vec{\widetilde{v}}_{SS''}(v_{SS'}) = \vec{\widetilde{v}}_{S'S''}(v_{SS'}) \oplus \vec{\widetilde{v}}_{SS'}(v_{SS'})$

Let us suppose frames S, S', S'' moving as shown in the figure



(69)

In this configuration, the addition of velocities of the Galilei relativity gives

$$\vec{v}_{SS''} = \vec{v}_{S'S''} + \vec{v}_{SS'} \ . \tag{70}$$

We have

$$\vec{v}_{SS''} = \frac{d\vec{x}_{O''}}{d\tau} = \frac{d\vec{x}_{O''}}{dt_{O''}}\frac{dt_{O''}}{d\tau} = \vec{\widetilde{v}}_{SS''}\frac{dt_{O''}}{d\tau} \tag{71}$$

and considering the frames S, S' we write

$$\tau = \frac{\widetilde{v}_{SS'}(v_{SS'})}{v_{SS'}}\Big\{(1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{x}_{O''}\cdot\vec{\widetilde{v}}_{SS'}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})} + \gamma_{\widetilde{v}_{SS'}}t_{O''}\Big\} \ ,$$

then

$$\frac{dt_{O''}}{d\tau} = \frac{1}{\gamma_{\widetilde{v}_{SS'}}}\left\{\frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})} - (1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{v}_{SS''}\cdot\vec{\widetilde{v}}_{SS'}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})}\right\} \ ,$$

which replacing back into (71) gives

$$\vec{v}_{SS''} = \vec{\widetilde{v}}_{SS''}(v_{SS'})\frac{1}{\gamma_{\widetilde{v}_{SS'}}}\left\{\frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})} - (1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{v}_{SS''}\cdot\vec{\widetilde{v}}_{SS'}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})}\right\} \ . \tag{72}$$

Taking the scalar product of $\vec{v}_{SS''}$ with $\vec{\widetilde{v}}_{SS'}(v_{SS'})$ we obtain

$$\frac{\vec{v}_{SS''}\cdot\vec{\widetilde{v}}_{SS'}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})} = \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\frac{\vec{\widetilde{v}}_{SS''}(v_{SS'})\cdot\vec{\widetilde{v}}_{SS'}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})}}{\gamma_{\widetilde{v}_{SS'}} + (1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{SS''}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})}} \ ,$$

which replacing back into (72) gives

$$\vec{v}_{SS''} = \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\vec{\widetilde{v}}_{SS''}(v_{SS'})}{\gamma_{\widetilde{v}_{SS'}} + (1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{SS''}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})}} \ . \tag{73}$$

Now, let us take

$$\vec{v}_{S'S''} = \frac{d\vec{x}'_{O''}}{d\tau} = \frac{d\vec{x}'_{O''}}{dt'_{O''}}\frac{dt'_{O''}}{d\tau} = \vec{\widetilde{v}}_{S'S''}\frac{dt'_{O''}}{d\tau} \ , \tag{74}$$

and still considering the frames S, S' we write

$$\tau \ = \ \frac{\widetilde{v}_{S'S}(v_{S'S})}{v_{S'S}}\left\{(1-\gamma_{\widetilde{v}_{S'S}})\frac{\vec{x}'_{O''}\cdot\vec{\widetilde{v}}_{S'S}(v_{S'S})}{\widetilde{v}^2_{S'S}} + \gamma_{\widetilde{v}_{S'S}}t'_{O''}\right\} \ ,$$

then

$$\frac{dt'_{O''}}{d\tau} = \frac{1}{\gamma_{\widetilde{v}_{S'S}}}\left\{\frac{v_{S'S}}{\widetilde{v}_{S'S}(v_{S'S})} - (1-\gamma_{\widetilde{v}_{S'S}})\frac{\vec{v}_{S'S''}\cdot\vec{\widetilde{v}}_{S'S}(v_{S'S})}{\widetilde{v}^2_{S'S}(v_{S'S})}\right\}$$

which replacing back into (71) gives

$$\vec{v}_{S'S''} = \vec{\widetilde{v}}_{S'S''}(v_{S'S})\frac{1}{\gamma_{\widetilde{v}_{S'S}}}\left\{\frac{v_{S'S}}{\widetilde{v}_{S'S}(v_{S'S})} - (1-\gamma_{\widetilde{v}_{S'S}})\frac{\vec{v}_{S'S''}\cdot\vec{\widetilde{v}}_{S'S}(v_{S'S})}{\widetilde{v}^2_{S'S}(v_{S'S})}\right\} . \tag{75}$$

Taking the scalar product between $\vec{v}_{S'S''}$ and $\vec{\widetilde{v}}_{S'S}(v_{S'S})$ we obtain

$$\frac{\vec{v}_{S'S''}\cdot\vec{v}_{S'S}(v_{S'S})}{\widetilde{v}^2_{S'S}(v_{S'S})} = \frac{v_{S'S}}{\widetilde{v}_{S'S}(v_{S'S})}\frac{\frac{\vec{\widetilde{v}}_{S'S''}(v_{S'S})\cdot\vec{\widetilde{v}}_{S'S}(v_{S'S})}{\widetilde{v}^2_{S'S}(v_{S'S})}}{\gamma_{\widetilde{v}_{S'S}} + (1-\gamma_{\widetilde{v}_{S'S}})\frac{\vec{\widetilde{v}}_{S'S''}\cdot\vec{\widetilde{v}}_{S'S}(v_{S'S})}{\widetilde{v}^2_{S'S}(v_{S'S})}} \ ,$$

which replacing back into (75) gives

$$\vec{v}_{S'S''} = \frac{v_{S'S}}{\widetilde{v}_{S'S}(v_{S'S})}\frac{\vec{\widetilde{v}}_{S'S''}(v_{S'S})}{\gamma_{\widetilde{v}_{S'S}} + (1-\gamma_{\widetilde{v}_{S'S}})\frac{\vec{\widetilde{v}}_{S'S''}(v_{S'S})\cdot\vec{\widetilde{v}}_{S'S}(v_{S'S})}{\widetilde{v}^2_{S'S}(v_{S'S})}} \ . \tag{76}$$

We have seen that $v_{SS'} = v_{S'S}$ and $\vec{\widetilde{v}}_{S'S}(v_{S'S}) = -\vec{\widetilde{v}}_{SS'}(v_{SS'})$ then we write

$$\vec{v}_{S'S''} = \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\vec{\widetilde{v}}_{S'S''}(v_{SS'})}{\gamma_{\widetilde{v}_{SS'}} - (1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{\widetilde{v}}_{S'S''}(v_{SS'})\cdot\vec{\widetilde{v}}_{SS'}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})}} \ . \tag{77}$$

Replacing (64), (73) and (77) into (70) we obtain

$$\begin{aligned}
&\frac{\vec{\widetilde{v}}_{SS''}(v_{SS'})}{\gamma_{\widetilde{v}_{SS'}} + (1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{SS''}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})}} = \\
&= \frac{\vec{\widetilde{v}}_{S'S''}(v_{SS'}) + \gamma_{\widetilde{v}_{SS'}}\vec{\widetilde{v}}_{SS'}(v_{SS'}) - (1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})}\vec{\widetilde{v}}_{SS'}(v_{SS'})}{\gamma_{\widetilde{v}_{SS'}} - (1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})}} \ ,
\end{aligned} \tag{78}$$

then taking the scalar product of (78) with $\vec{\widetilde{v}}_{SS'}(v_{SS'})$ we obtain

$$\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{SS''}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})} = \frac{1 + \frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{SS'})}{v^2_{SS'}(v_{SS'})}}{1 + \frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{SS'})}{c^2}} \ ,$$

and the denominator of the left hand side of (78) becomes

$$\gamma_{\widetilde{v}_{SS'}}+(1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{SS''}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})}=\frac{\gamma_{\widetilde{v}_{SS'}}-(1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})}}{\gamma_{\widetilde{v}_{SS'}}\left(1+\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{SS'})}{c^2}\right)}\ , \tag{79}$$

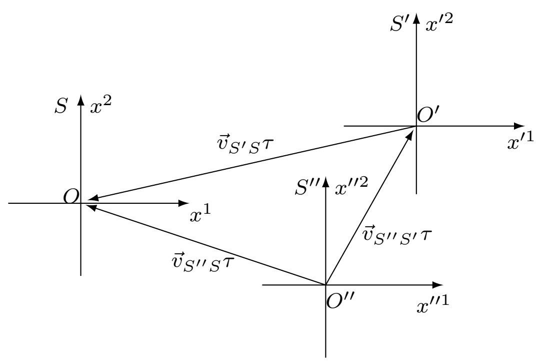
which replacing back into (78) gives

$$\vec{\widetilde{v}}_{SS''}(v_{SS'})=\frac{\vec{\widetilde{v}}_{S'S''}(v_{SS'})+\gamma_{\widetilde{v}_{SS'}}\vec{\widetilde{v}}_{SS'}(v_{SS'})-(1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})}\vec{\widetilde{v}}_{SS'}(v_{SS'})}{\gamma_{\widetilde{v}_{SS'}}\left(1+\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{SS'})}{c^2}\right)}\ , \tag{80}$$

which has a form similiar to (22).

5.2 $\vec{v}_{S''S}=\vec{v}_{S'S}+\vec{v}_{S''S'}\Rightarrow\vec{\widetilde{v}}_{S''S}(v_{S''S'})=\vec{\widetilde{v}}_{S'S}(v_{S''S'})\oplus\vec{\widetilde{v}}_{S''S'}(v_{S''S'})$

The configuration in figure (69) is equivalent to the one shown below



(81)

Here, the Galilei velocity law has the form

$$\vec{v}_{S''S}=\vec{v}_{S'S}+\vec{v}_{S''S'}\ . \tag{82}$$

We have

$$\vec{v}_{S''S}=\frac{d\vec{x}''_O}{d\tau}=\frac{d\vec{x}''_O}{dt''_O}\frac{dt''_O}{d\tau}=\vec{\widetilde{v}}_{S''S}\frac{dt''_O}{d\tau} \tag{83}$$

and considering the frames S', S'' we write

$$\tau=\frac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}\left\{(1-\gamma_{\widetilde{v}_{S''S'}})\frac{\vec{x}''_O\cdot\vec{\widetilde{v}}_{S''S'}(v_{S''S'})}{\widetilde{v}^2_{S''S'}(v_{S''S'})}+\gamma_{\widetilde{v}_{S''S'}}t''_O\right\}\ .$$

By a procedure similar to the one performed in the previous section, we obtain the analogue of equation (73) as

$$\vec{v}_{S''S}=\frac{v_{S''S'}}{\widetilde{v}_{S''S'}(v_{S''S'})}\frac{\vec{\widetilde{v}}_{S''S}(v_{S''S'})}{\gamma_{\widetilde{v}_{S''S'}}+(1-\gamma_{\widetilde{v}_{S''S'}})\frac{\vec{\widetilde{v}}_{S''S}(v_{S''S'})\cdot\vec{\widetilde{v}}_{S''S'}(v_{S''S'})}{\widetilde{v}^2_{S''S'}(v_{S''S'})}}\ . \tag{84}$$

Considering

$$\tau = \frac{\tilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}\Big\{(1-\gamma_{\tilde{v}_{S''S'}})\frac{\vec{x}''_{O'}\cdot\vec{\tilde{v}}_{S''S'}(v_{S''S'})}{\tilde{v}^2_{S''S'}(v_{S''S'})} + \gamma_{\tilde{v}_{S''S'}} t''_{O'}\Big\}$$

we obtain the analogue of (64) as

$$\vec{v}_{S''S'} = \frac{v_{S''S'}}{\tilde{v}_{S''S'}(v_{S''S'})}\vec{\tilde{v}}_{S''S'}(v_{S''S'}) \ . \tag{85}$$

We have

$$\vec{v}_{S'S} = \frac{d\vec{x}'_O}{d\tau} = \frac{d\vec{x}'_O}{dt'_O}\frac{dt'_O}{d\tau} = \vec{\tilde{v}}_{S'S}\frac{dt'_O}{d\tau} \ , \tag{86}$$

and considering the frames S', S'' we have

$$\tau \ \ = \ \ \frac{\tilde{v}_{S'S''}(v_{S'S''})}{v_{S'S''}}\Big\{(1-\gamma_{\tilde{v}_{S'S''}})\frac{\vec{x}'_O\cdot\vec{\tilde{v}}_{S'S''}(v_{S'S''})}{\tilde{v}^2_{S'S''}(v_{S'S''})} + \gamma_{\tilde{v}_{S'S''}} t'_O\Big\}$$

then we obtain the analogue of (77) as

$$\vec{v}_{S'S} = \frac{v_{S'S''}}{\tilde{v}_{S'S''}(v_{S'S''})}\frac{\vec{\tilde{v}}_{S'S}(v_{S'S''})}{\gamma_{\tilde{v}_{S'S''}} + (1-\gamma_{\tilde{v}_{S'S''}})\frac{\vec{\tilde{v}}_{S'S}(v_{S'S''})\cdot\vec{\tilde{v}}_{S'S''}(v_{S'S''})}{\tilde{v}^2_{S'S''}(v_{S'S''})}} \ . \tag{87}$$

Replacing (84), (85), (87) in (82) we obtain

$$\begin{aligned}
&\frac{\vec{\tilde{v}}_{S''S}(v_{S''S'})}{\gamma_{\tilde{v}_{S''S'}} + (1-\gamma_{\tilde{v}_{S''S'}})\frac{\vec{\tilde{v}}_{S''S}(v_{S''S'})\cdot\vec{\tilde{v}}_{S''S'}(v_{S''S'})}{\tilde{v}^2_{S''S'}(v_{S''S'})}} = \\
&\qquad = \frac{\vec{\tilde{v}}_{S'S}(v_{S'S''}) + \gamma_{\tilde{v}_{S'S''}}\vec{\tilde{v}}_{S''S'}(v_{S''S'}) + (1-\gamma_{\tilde{v}_{S'S''}})\frac{\vec{\tilde{v}}_{S'S}(v_{S'S''})\cdot\vec{\tilde{v}}_{S'S''}(v_{S'S''})}{\tilde{v}^2_{S'S''}(v_{S'S''})}\vec{\tilde{v}}_{S''S'}(v_{S''S'})}{\gamma_{\tilde{v}_{S'S''}} + (1-\gamma_{\tilde{v}_{S'S''}})\frac{\vec{\tilde{v}}_{S'S}(v_{S'S''})\cdot\vec{\tilde{v}}_{S'S''}(v_{S'S''})}{\tilde{v}^2_{S'S''}(v_{S'S''})}} \ .
\end{aligned} \tag{88}$$

Taking the scalar product with $\vec{\tilde{v}}_{S''S'}(v_{S''S'})$ and using that $\vec{\tilde{v}}_{S''S'}(v_{S''S'}) = -\vec{\tilde{v}}_{S'S''}(v_{S'S''})$ and $v_{S'S''} = v_{S''S'}$ (these are similar to the expressions we obtained in (57) and (61)) we obtain

$$\frac{\vec{\tilde{v}}_{S''S}(v_{S''S'})\cdot\vec{\tilde{v}}_{S''S'}(v_{S''S'})}{\tilde{v}^2_{S''S'}(v_{S''S'})} = \frac{1+\frac{\vec{\tilde{v}}_{S'S}\cdot\vec{\tilde{v}}_{S''S'}}{\tilde{v}^2_{S''S'}}[v_{S''S'}]}{1+\frac{\vec{\tilde{v}}_{S'S}\cdot\vec{\tilde{v}}_{S''S'}}{c^2}[v_{S''S'}]}$$

and replacing it back in (88) we obtain

$$\vec{\tilde{v}}_{S''S}(v_{S''S'}) = \frac{\vec{\tilde{v}}_{S'S}(v_{S''S'}) + \gamma_{\tilde{v}_{S''S'}}\vec{\tilde{v}}_{S''S'}(v_{S''S'}) - (1-\gamma_{\tilde{v}_{S''S'}})\frac{\vec{\tilde{v}}_{S'S}(v_{S''S'})\cdot\vec{\tilde{v}}_{S''S'}(v_{S''S'})}{\tilde{v}^2_{S''S'}(v_{S''S'})}\vec{\tilde{v}}_{S''S'}(v_{S''S'})}{\gamma_{\tilde{v}_{S''S'}}\Big(1+\frac{\vec{\tilde{v}}_{S'S}(v_{S''S'})\cdot\vec{\tilde{v}}_{S''S'}(v_{S''S'})}{c^2}\Big)} \ , \tag{89}$$

which has a form similar to (23).

5.3 $\vec{\widetilde{v}}_{S''S}(v_{S''S'}) = -\Omega(v_{S''S'}, v_{SS'})\vec{\widetilde{v}}_{SS''}(v_{SS'})$

We start with a result whose derivation is similar to the procedure adopted in sections 5.1 and 5.2.

Result 2

i. $$\vec{\widetilde{v}}_{SS''}(v_{SS''}) = \frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}} \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})} \frac{\gamma_{\widetilde{v}_{SS'}}\left(1 + \frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{SS'})}{c^2}\right)}{\gamma_{\widetilde{v}_{SS'}} - (1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})}} \vec{\widetilde{v}}_{SS''}(v_{SS'}) \tag{90}$$

ii. $$\vec{\widetilde{v}}_{S''S}(v_{S''S}) = \frac{\widetilde{v}_{S''S}(v_{S''S})}{v_{S''S}} \frac{v_{S''S'}}{\widetilde{v}_{S''S'}(v_{S''S'})} \frac{\gamma_{\widetilde{v}_{S''S'}}\left(1 + \frac{\vec{\widetilde{v}}_{S'S}(v_{S''S'})\cdot\vec{\widetilde{v}}_{S''S'}(v_{S''S'})}{c^2}\right)}{\gamma_{\widetilde{v}_{S''S'}} - (1-\gamma_{\widetilde{v}_{S''S'}})\frac{\vec{\widetilde{v}}_{S'S}(v_{S''S'})\cdot\vec{\widetilde{v}}_{S''S'}(v_{S''S'})}{\widetilde{v}^2_{S''S'}(v_{S''S'})}} \vec{\widetilde{v}}_{S''S}(v_{S''S'}) \tag{91}$$

Proof:

We have

$$\vec{v}_{SS''} = \frac{d\vec{x}_{O''}}{d\tau} = \frac{d\vec{x}_{O''}}{dt_{O''}}\frac{dt_{O''}}{d\tau} = \vec{\widetilde{v}}_{SS''}\frac{dt_{O''}}{d\tau} \tag{92}$$

and we consider now

$$\tau = \frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\left\{(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}_{O''}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})} + \gamma_{\widetilde{v}_{SS''}}t_{O''}\right\} ,$$

then following the same procedure that gave the expression (64) we now have

$$\vec{v}_{SS''} = \frac{v_{SS''}}{\widetilde{v}_{SS''}(v_{SS''})}\vec{\widetilde{v}}_{SS''}(v_{SS''}) \ . \tag{93}$$

Comparing the two expressions for $\vec{v}_{SS''}$ given in (73) and (93) we obtain that

$$\vec{\widetilde{v}}_{SS''}(v_{SS''}) = \frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}} \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})} \frac{\vec{\widetilde{v}}_{SS''}(v_{SS'})}{\gamma_{\widetilde{v}_{SS'}} + (1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{\widetilde{v}}_{SS''}(v_{SS'})\cdot\vec{\widetilde{v}}_{SS'}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})}} , \tag{94}$$

and using (79) we have

$$\vec{\widetilde{v}}_{SS''}(v_{SS''}) = \frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}} \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})} \frac{\gamma_{\widetilde{v}_{SS'}}\left(1 + \frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{SS'})}{c^2}\right)}{\gamma_{\widetilde{v}_{SS'}} - (1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})}} \vec{\widetilde{v}}_{SS''}(v_{SS'}) ,$$

which is the form we wanted to show in (90). In a similar way we obtain that

$$\vec{\widetilde{v}}_{S''S}(v_{S''S}) = \frac{\widetilde{v}_{S''S}(v_{S''S})}{v_{S''S}} \frac{v_{S''S'}}{\widetilde{v}_{S''S'}(v_{S''S'})} \frac{\gamma_{\widetilde{v}_{S''S'}}\left(1 + \frac{\vec{\widetilde{v}}_{S'S}(v_{S''S'})\cdot\vec{\widetilde{v}}_{S''S'}(v_{S''S'})}{c^2}\right)}{\gamma_{\widetilde{v}_{S''S'}} - (1-\gamma_{\widetilde{v}_{S''S'}})\frac{\vec{\widetilde{v}}_{S'S}(v_{S''S'})\cdot\vec{\widetilde{v}}_{S''S'}(v_{S''S'})}{\widetilde{v}^2_{S''S'}(v_{S''S'})}} \vec{\widetilde{v}}_{S''S}(v_{S''S'}) \ .$$

which is the form we wanted to show in (91), and this complete the proof of Result 2. ■

Now, what we have obtained in Result 1.i is also true for $\vec{\widetilde{v}}_{SS''}(v_{SS''})$ and $\vec{\widetilde{v}}_{S''S}(v_{S''S})$, i.e.

$$\vec{\widetilde{v}}_{S''S}(v_{S''S}) = -\vec{\widetilde{v}}_{SS''}(v_{SS''}) \tag{95}$$

then from (90), (91) and (95) we obtain

$$\vec{\widetilde{v}}_{S''S}(v_{S''S'}) = -\Omega(v_{S''S'}, v_{SS'})\vec{\widetilde{v}}_{SS''}(v_{SS'}) \tag{96}$$

with

$$\begin{aligned}\Omega(v_{S''S'}, v_{SS'}) &= \frac{\widetilde{v}_{S''S'}(v_{S''S'})/v_{S''S'}}{\widetilde{v}_{SS'}(v_{SS'})/v_{SS'}} \frac{\left\{1 - \frac{(1-\gamma_{\widetilde{v}_{S''S'}})}{\gamma_{\widetilde{v}_{S''S'}}} \frac{\vec{\widetilde{v}}_{S'S}(v_{S''S'})\cdot\vec{\widetilde{v}}_{S''S'}(v_{S''S'})}{\widetilde{v}^2_{S''S'}(v_{S''S'})}\right\}}{\left\{1 - \frac{(1-\gamma_{\widetilde{v}_{SS'}})}{\gamma_{\widetilde{v}_{SS'}}} \frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})}\right\}} \times \\ &\times \frac{\left\{1 + \frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{SS'})}{c^2}\right\}}{\left\{1 + \frac{\vec{\widetilde{v}}_{S'S}(v_{S''S'})\cdot\vec{\widetilde{v}}_{S''S'}(v_{S''S'})}{c^2}\right\}} \; .\end{aligned} \tag{97}$$

6 On the composition of Lorentz transformation

We now investigate the composition of Lorentz transformations. We search for a relation that replaces $(t'', \vec{x}\,'') = \mathcal{R}^{-1} \circ L(\vec{\widetilde{v}}_{SS''})(t, \vec{x}) = L(\vec{\widetilde{v}}_{S'S''}) \circ L(\vec{\widetilde{v}}_{SS'})(t, \vec{x})$.

6.1 A relation between the local times $t'_{S'S}$ and $t'_{S'S''}$

Considering the pair S', S we write

$$\tau = \frac{\widetilde{v}_{S'S}(v_{S'S})}{v_{S'S}} \left\{(1 - \gamma_{\widetilde{v}_{S'S}}) \frac{\vec{x}\,' \cdot \vec{\widetilde{v}}_{S'S}(v_{S'S})}{\widetilde{v}^2_{S'S}(v_{S'S})} + \gamma_{\widetilde{v}_{S'S}} t'_{S'S}\right\} ,$$

and considering the pair S', S'' we also write

$$\tau = \frac{\widetilde{v}_{S'S''}(v_{S'S''})}{v_{S'S''}} \left\{(1 - \gamma_{\widetilde{v}_{S'S''}}) \frac{\vec{x}\,' \cdot \vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} + \gamma_{\widetilde{v}_{S'S''}} t'_{S'S''}\right\} .$$

Then, eliminating the absolute time τ from these expressions we obtain

$$\begin{aligned} t'_{S'S''} &= \frac{v_{S'S''}}{\widetilde{v}_{S'S''}(v_{S'S''})} \frac{1}{\gamma_{\widetilde{v}_{S'S''}}} \left\{ - \frac{\widetilde{v}_{S'S''}(v_{S'S''})}{v_{S'S''}} (1 - \gamma_{\widetilde{v}_{S'S''}}) \frac{\vec{x}\,' \cdot \vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} + \right. \\ &\left. + \frac{\widetilde{v}_{S'S}(v_{S'S})}{v_{S'S}} (1 - \gamma_{\widetilde{v}_{S'S}}) \frac{\vec{x}\,' \cdot \vec{\widetilde{v}}_{S'S}(v_{S'S})}{\widetilde{v}^2_{S'S}(v_{S'S})} + \frac{\widetilde{v}_{S'S}(v_{S'S})}{v_{S'S}} \gamma_{\widetilde{v}_{S'S}} t'_{S'S} \right\} . \end{aligned} \tag{98}$$

Then, given two pair of frames $\{S', S\}$ and $\{S', S''\}$ we define a map

$$(t'_{S'S}, \vec{x}\,') \xrightarrow{K_{\{S',S''\},\{S',S\}}} (t'_{S'S''}, \vec{x}\,') := K_{\{S',S''\},\{S',S\}}(t'_{S'S}, \vec{x}\,') \tag{99}$$

where $t'_{S'S''} = t'_{S'S''}(t'_{S'S})$ is given by (98). It is immediate to check that

$$K^{-1}_{\{S',S''\},\{S',S\}} = K_{\{S',S\},\{S',S''\}} \; . \tag{100}$$

6.2 A useful expression for velocity

Let us consider again the formula for the addition of velocities in Galilei relativity (70)

$$\vec{v}_{SS''} = \vec{v}_{S'S''} + \vec{v}_{SS'} \ . \tag{101}$$

Using the same reasoning for obtaining (64) we have the following expressions

$$\begin{aligned}
\vec{v}_{SS'} &= \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\,\vec{\widetilde{v}}_{SS'}(v_{SS'}) \\
\vec{v}_{S'S''} &= \frac{v_{S'S''}}{\widetilde{v}_{S'S''}(v_{S'S''})}\,\vec{\widetilde{v}}_{S'S''}(v_{S'S''}) \\
\vec{v}_{SS''} &= \frac{v_{SS''}}{\widetilde{v}_{SS''}(v_{SS''})}\,\vec{\widetilde{v}}_{SS''}(v_{SS''}) \ .
\end{aligned}$$

Then we have

$$\frac{v_{SS''}}{\widetilde{v}_{SS''}(v_{SS''})}\,\vec{\widetilde{v}}_{SS''}(v_{SS''}) = \frac{v_{S'S''}}{\widetilde{v}_{S'S''}(v_{S'S''})}\,\vec{\widetilde{v}}_{S'S''}(v_{S'S''}) + \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\,\vec{\widetilde{v}}_{SS'}(v_{SS'}) \tag{102}$$

that we will use in section 6.4.

6.3 Composition of Lorentz transformations

Let us assume three moving frames S, S', S'' and the corresponding Lorentz transformation

$$S \longrightarrow S' : \begin{cases} (t'_{S'S}, \vec{x}\,') = L(\vec{\widetilde{v}}_{SS'})(t_{SS'}, \vec{x}) \\ \vec{x}\,' = \vec{x} - (1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS'}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})}\vec{\widetilde{v}}_{SS'}(v_{SS'}) - \gamma_{\widetilde{v}_{SS'}} t_{SS'}\,\vec{\widetilde{v}}_{SS'}(v_{SS'}) \\ t'_{S'S} = \gamma_{\widetilde{v}_{SS'}}\left(t_{SS'} - \frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS'}(v_{SS'})}{c^2}\right) \end{cases} \tag{103}$$

$$S' \longrightarrow S'' : \begin{cases} (t''_{S''S'}, \vec{x}\,'') = L(\vec{\widetilde{v}}_{S'S''})(t'_{S'S''}, \vec{x}\,') \\ \vec{x}\,'' = \vec{x}\,' - (1-\gamma_{\widetilde{v}_{S'S''}})\frac{\vec{x}\,'\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})}\vec{\widetilde{v}}_{S'S''}(v_{S'S''}) - \gamma_{\widetilde{v}_{S'S''}} t'_{S'S''}\,\vec{\widetilde{v}}_{S'S''}(v_{S'S''}) \\ t''_{S''S'} = \gamma_{\widetilde{v}_{S'S''}}\left(t'_{S'S''} - \frac{\vec{x}\,'\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{c^2}\right) \end{cases} \tag{104}$$

$$S \longrightarrow S'' : \begin{cases} (t''_{S''S}, \vec{x}\,'') = L(\vec{\widetilde{v}}_{SS''})(t_{SS''}, \vec{x}\,) \\ \vec{x}\,'' = \vec{x} - (1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}\vec{\widetilde{v}}_{SS''}(v_{SS''}) - \gamma_{\widetilde{v}_{SS''}} t_{SS''}\,\vec{\widetilde{v}}_{SS''}(v_{SS''}) \\ t''_{S''S} = \gamma_{\widetilde{v}_{SS''}}\left(t_{SS''} - \frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{c^2}\right) \ . \end{cases} \tag{105}$$

As the local time depends on the state of motion of the observers we have that $(t_{SS'}, \vec{x}), (t'_{S'S}, \vec{x}\,')$ designate how the pair S, S' describe the event P; in the same manner $(t'_{S'S''}, \vec{x}\,'), (t''_{S''S'}, \vec{x}\,'')$ refer how the pair S', S'' describe the event P, while $(t_{SS''}, \vec{x}\,), (t''_{S''S}, \vec{x}\,'')$ refer how the pair S, S'' describe the same event. We also observe a crucial difference between $(t''_{S''S}, \vec{x}\,'') = L(\vec{\widetilde{v}}_{SS''})(t_{SS''}, \vec{x}\,)$ given in (105) and the previous form $(t'', \vec{x}\,'') = \mathcal{R}^{-1}L(\vec{\widetilde{v}}_{SS''})(t, \vec{x}\,)$ given in (27).

Using the K-map defined in (99) and the form of the Lorentz transformations $L(\vec{\widetilde{v}}_{SS'})$ and $L(\vec{\widetilde{v}}_{S'S''})$ given in (103) and (104) we must have

$$(t''_{S''S}, \vec{x}\,'') = K_{\{S'',S\},\{S'',S'\}} L(\vec{\widetilde{v}}_{S'S''}) K_{\{S',S''\},\{S',S\}} L(\vec{\widetilde{v}}_{SS'})(t_{SS'}, \vec{x}) \ . \tag{106}$$

We also have

$$(t_{SS''}, \vec{x}) = K_{\{S,S''\},\{S,S'\}}(t_{SS'}, \vec{x})$$

and from (100)

$$K^{-1}_{\{S,S''\},\{S,S'\}} = K_{\{S,S'\},\{S,S''\}} \ ,$$

then

$$(t_{SS'}, \vec{x}) = K_{\{S,S'\},\{S,S''\}}(t_{SS''}, \vec{x}) \ ,$$

which replacing back in (106) gives

$$(t''_{S''S}, \vec{x}\,'') = K_{\{S'',S\},\{S'',S'\}} L(\vec{\widetilde{v}}_{S'S''}) K_{\{S',S''\},\{S',S\}} L(\vec{\widetilde{v}}_{SS'}) K_{\{S,S'\},\{S,S''\}}(t_{SS''}, \vec{x}) \ . \tag{107}$$

Finally, comparing this form (107) with the one shown in (105): $(t''_{S''S}, \vec{x}\,'') = L(\vec{\widetilde{v}}_{SS''})(t_{SS''}, \vec{x}\,)$, we obtain

$$\boxed{L(\vec{\widetilde{v}}_{SS''}) = K_{\{S'',S\},\{S'',S'\}} L(\vec{\widetilde{v}}_{S'S''}) K_{\{S',S''\},\{S',S\}} L(\vec{\widetilde{v}}_{SS'}) K_{\{S,S'\},\{S,S''\}}} \ , \tag{108}$$

and this expression is what replaces (36) when we assume the local time depends on the state of motion of the frames.

6.4 Composition of Lorentz transformations through the K-map

Since there is no guarantee that S and S'' are related by the Lorentz transformation (105) there is also no guarantee of the validity of (108), which we now check.

6.4.1 $(t_{SS''}, \vec{x}) \longrightarrow (t_{SS'}, \vec{x})$

We have

$$(t_{SS'}, \vec{x}) := K_{\{S,S'\},\{S,S''\}}(t_{SS''}, \vec{x}) \ . \tag{109}$$

Considering the pairs $\{S, S''\}, \{S, S'\}$ equation (98) gives

$$t_{SS'} = \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})} \frac{1}{\gamma_{\widetilde{v}_{SS'}}} \Bigg\{ -\frac{\widetilde{v}_{SS'}(v_{SS'})}{v_{SS'}}(1-\gamma_{\widetilde{v}_{SS'}}) \frac{\vec{x} \cdot \vec{\widetilde{v}}_{SS'}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})} + \\ + \frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{SS''}}) \frac{\vec{x} \cdot \vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})} + \frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}} \gamma_{\widetilde{v}_{SS''}} t_{SS''} \Bigg\} \ . \tag{110}$$

6.4.2 $(t_{SS'}, \vec{x}) \longrightarrow (t'_{S'S}, \vec{x}\,')$

We have

$$
\begin{aligned}
(t'_{S'S}, \vec{x}\,') &:= L(\vec{\widetilde{v}}_{SS'})(t_{SS'}, \vec{x}) \\
&= L(\vec{\widetilde{v}}_{SS'}) K_{\{S,S'\},\{S,S''\}}(t_{SS''}, \vec{x}) \ . \qquad (111)
\end{aligned}
$$

Replacing (110) in (103) we obtain for $\vec{x}\,'$

$$
\begin{aligned}
\blacktriangleright\ \vec{x}\,' &= \vec{x} - \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})} \frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}} (1-\gamma_{\widetilde{v}_{SS''}}) \frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})} \vec{\widetilde{v}}_{SS'}(v_{SS'}) + \\
&\quad - \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})} \frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}} \gamma_{\widetilde{v}_{SS''}}\, t_{SS''}\, \vec{\widetilde{v}}_{SS'}(v_{SS'}) \ , \qquad (112)
\end{aligned}
$$

and for $t'_{S'S}$ we have

$$
\begin{aligned}
t'_{S'S} &= -(1-\gamma_{\widetilde{v}_{SS'}}) \frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS'}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})} + \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})} \frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}} (1-\gamma_{\widetilde{v}_{SS''}}) \frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})} + \\
&\quad + \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})} \frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}} \gamma_{\widetilde{v}_{SS''}} t_{SS''} - \gamma_{\widetilde{v}_{SS'}} \frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS'}(v_{SS'})}{c^2} \ . \qquad (113)
\end{aligned}
$$

Using

$$
\frac{1}{c^2} = \frac{1-\gamma^{-2}_{\widetilde{v}_{SS'}}}{\widetilde{v}^2_{SS'}(v_{SS'})}
$$

we rewrite (113) as

$$
\begin{aligned}
\blacktriangleright\ t'_{S'S} &= \frac{(1-\gamma_{\widetilde{v}_{SS'}})}{\gamma_{\widetilde{v}_{SS'}}} \frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS'}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})} + \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})} \frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}} (1-\gamma_{\widetilde{v}_{SS''}}) \frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})} + \\
&\quad + \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})} \frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}} \gamma_{\widetilde{v}_{SS''}} t_{SS''} \ . \qquad (114)
\end{aligned}
$$

6.4.3 $(t'_{S'S}, \vec{x}\,') \longrightarrow (t'_{S'S''}, \vec{x}\,')$

We have

$$
\begin{aligned}
(t'_{S'S''}, \vec{x}\,') &:= K_{\{S',S''\},\{S',S\}}(t'_{S'S}, \vec{x}\,') \\
&= K_{\{S',S''\},\{S',S\}} L(\vec{\widetilde{v}}_{SS'}) K_{\{S,S'\},\{S,S''\}}(t_{SS''}, \vec{x}) \ . \qquad (115)
\end{aligned}
$$

Considering the pairs $\{S', S''\}, \{S', S\}$ the relation between $t'_{S'S''}$ and $t'_{S'S}$ is given by equation (98)

$$
\begin{aligned}
t'_{S'S''} = \frac{v_{S'S''}}{\widetilde{v}_{S'S''}(v_{S'S''})} \frac{1}{\gamma_{\widetilde{v}_{S'S''}}} \Bigg\{ &- \frac{\widetilde{v}_{S'S''}(v_{S'S''})}{v_{S'S''}} (1-\gamma_{\widetilde{v}_{S'S''}}) \frac{\vec{x}\,'\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} + \\
&+ \frac{\widetilde{v}_{S'S}(v_{S'S})}{v_{S'S}} (1-\gamma_{\widetilde{v}_{S'S}}) \frac{\vec{x}\,'\cdot\vec{\widetilde{v}}_{S'S}(v_{S'S})}{\widetilde{v}^2_{S'S}(v_{S'S})} + \frac{\widetilde{v}_{S'S}(v_{S'S})}{v_{S'S}} \gamma_{\widetilde{v}_{S'S}} t'_{S'S} \Bigg\} \ .
\end{aligned}
$$

Using (112) we obtain

$$\left\{\begin{array}{l} -\frac{\widetilde{v}_{S'S''}(v_{S'S''})}{v_{S'S''}}(1-\gamma_{\widetilde{v}_{S'S''}})\frac{\vec{x}\,'\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} = \\ \qquad = -\frac{\widetilde{v}_{S'S''}(v_{S'S''})}{v_{S'S''}}(1-\gamma_{\widetilde{v}_{S'S''}})\Big[\frac{\vec{x}\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} + \\ \qquad\quad -\frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} + \\ \qquad\quad -\frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{SS''}}t_{SS''}\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})}\Big] , \end{array}\right. \tag{116}$$

and

$$\left\{\begin{array}{l} \frac{\widetilde{v}_{S'S}(v_{S'S})}{v_{S'S}}(1-\gamma_{\widetilde{v}_{S'S}})\frac{\vec{x}\,'\cdot\vec{\widetilde{v}}_{S'S}(v_{S'S})}{\widetilde{v}^2_{S'S}(v_{S'S})} = \\ \qquad = \frac{\widetilde{v}_{S'S}(v_{S'S})}{v_{S'S}}(1-\gamma_{\widetilde{v}_{S'S}})\Big[-\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS'}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})} + \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})} + \\ \qquad\quad +\frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{SS''}}t_{SS''}\Big] . \end{array}\right. \tag{117}$$

Using (114) we obtain

$$\left\{\begin{array}{l} \frac{\widetilde{v}_{S'S}(v_{S'S})}{v_{S'S}}\gamma_{\widetilde{v}_{S'S}}t'_{S'S} = \frac{\widetilde{v}_{SS'}(v_{SS'})}{v_{SS'}}(1-\gamma_{\widetilde{v}_{SS'}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS'}(v_{SS'})}{\widetilde{v}^2_{SS'}(v_{SS'})} + \\ \qquad\qquad +\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{SS''}})\gamma_{\widetilde{v}_{S'S}}\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})} + \\ \qquad\qquad +\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{SS''}}\gamma_{\widetilde{v}_{S'S}}t_{SS''} . \end{array}\right. \tag{118}$$

Then, replacing (116), (117), (118) in the expression for $t'_{S'S''}$ given in (98) we obtain

$$\begin{aligned} \blacktriangleright\ t'_{S'S''} &= \frac{v_{S'S''}}{\widetilde{v}_{S'S''}(v_{S'S''})}\frac{1}{\gamma_{\widetilde{v}_{S'S''}}}\Big\{ -\frac{\widetilde{v}_{S'S''}(v_{S'S''})}{v_{S'S''}}(1-\gamma_{\widetilde{v}_{S'S''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} + \\ &\quad +\frac{\widetilde{v}_{S'S''}(v_{S'S''})}{v_{S'S''}}\frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{S'S''}})(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})} \times \\ &\quad \times\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} + \\ &\quad +\frac{\widetilde{v}_{S'S''}(v_{S'S''})}{v_{S'S''}}\frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{SS''}}(1-\gamma_{\widetilde{v}_{S'S''}})t_{SS''}\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} + \\ &\quad +\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})} + \frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{SS''}}t_{SS''}\Big\} . \end{aligned} \tag{119}$$

6.4.4 $(t'_{S'S''},\vec{x}\,') \longrightarrow (t''_{S''S'},\vec{x}\,'')$

We have

$$\begin{aligned} (t''_{S''S'},\vec{x}\,'') &:= L(\vec{\widetilde{v}}_{S'S''})(t'_{S'S''},\vec{x}\,') \\ &= L(\vec{\widetilde{v}}_{S'S''})K_{\{S',S''\},\{S',S\}}L(\vec{\widetilde{v}}_{SS'})K_{\{S,S'\},\{S,S''\}}(t_{SS''},\vec{x}) . \end{aligned}$$

Recall the Lorentz transformation (104),

$$\left\{\begin{array}{l} \vec{x}\,'' = \vec{x}\,' - (1-\gamma_{\widetilde{v}_{S'S''}})\frac{\vec{x}\,'\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})}\vec{\widetilde{v}}_{S'S''}(v_{S'S''}) - \gamma_{\widetilde{v}_{S'S''}}t'_{S'S''}\,\vec{\widetilde{v}}_{S'S''}(v_{S'S''}) \\ t''_{S''S'} = \gamma_{\widetilde{v}_{S'S''}}\left(t'_{S'S''} - \frac{\vec{x}\,'\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{c^2}\right)\,. \end{array}\right.$$

Using (112) we obtain

$$\left\{\begin{array}{l} -(1-\gamma_{\widetilde{v}_{S'S''}})\frac{\vec{x}\,'\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})}\vec{\widetilde{v}}_{S'S''}(v_{S'S''}) = \\ \qquad = -(1-\gamma_{\widetilde{v}_{S'S''}})\left[\frac{\vec{x}\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} + \right. \\ \qquad\quad - \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} + \\ \qquad\quad \left. - \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{SS''}}t_{SS''}\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})}\right]\vec{\widetilde{v}}_{S'S''}(v_{S'S''})\,, \end{array}\right. \tag{120}$$

and from (119) we have

$$\left\{\begin{array}{l} -\gamma_{\widetilde{v}_{S'S''}}t'_{S'S''}\,\vec{\widetilde{v}}_{S'S''}(v_{S'S''}) = \\ \qquad = -\gamma_{\widetilde{v}_{S'S''}}\frac{v_{S'S''}}{\widetilde{v}_{S'S''}(v_{S'S''})}\frac{1}{\gamma_{\widetilde{v}_{S'S''}}}\left[-\frac{\widetilde{v}_{S'S''}(v_{S'S''})}{v_{S'S''}}(1-\gamma_{\widetilde{v}_{S'S''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} + \right. \\ \qquad + \frac{\widetilde{v}_{S'S''}(v_{S'S''})}{v_{S'S''}}\frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{S'S''}})(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} + \\ \qquad + \frac{\widetilde{v}_{S'S''}(v_{S'S''})}{v_{S'S''}}\frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{SS''}}(1-\gamma_{\widetilde{v}_{S'S''}})t_{SS''}\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} + \\ \qquad \left. + \frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})} + \frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{SS''}}t_{SS''}\right]\vec{\widetilde{v}}_{S'S''}(v_{S'S''})\,. \end{array}\right. \tag{121}$$

Then, replacing (112), (120) and (121) in (104) we obtain

$$\begin{aligned} \vec{x}\,'' \;=\; & \vec{x} - \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\overbrace{(1-\gamma_{\widetilde{v}_{SS''}})\,\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}}^{\blacktriangle}\,\vec{\widetilde{v}}_{SS'}(v_{SS'}) + \\ & - \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\overbrace{\gamma_{\widetilde{v}_{SS''}}\,t_{SS''}}^{\blacktriangle\blacktriangle}\,\vec{\widetilde{v}}_{SS'}(v_{SS'}) + \\ & - \frac{v_{S'S''}}{\widetilde{v}_{S'S''}(v_{S'S''})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\overbrace{(1-\gamma_{\widetilde{v}_{SS''}})\,\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}}^{\blacktriangle}\,\vec{\widetilde{v}}_{S'S''}(v_{S'S''}) + \\ & - \frac{v_{S'S''}}{\widetilde{v}_{S'S''}(v_{S'S''})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\overbrace{\gamma_{\widetilde{v}_{SS''}}\,t_{SS''}}^{\blacktriangle\blacktriangle}\,\vec{\widetilde{v}}_{S'S''}(v_{S'S''})\,. \end{aligned} \tag{122}$$

But

$$\blacktriangle \;=\; -(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}\vec{\widetilde{v}}_{SS''}(v_{SS''})$$

and

$$\blacktriangle\blacktriangle \;=\; -\gamma_{\widetilde{v}_{SS''}}t_{SS''}\vec{\widetilde{v}}_{SS''}(v_{SS''})$$

then replacing ▲ and ▲▲ in (122) we obtain

$$\boxed{\vec{x}'' = \vec{x} - (1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}\vec{\widetilde{v}}_{SS''}(v_{SS''}) - \gamma_{\widetilde{v}_{SS''}} t_{SS''}\vec{\widetilde{v}}_{SS''}(v_{SS''})} \,. \tag{123}$$

Now, for obtaining $t''_{S''S'}$ we use (112) and

$$\frac{1}{c^2} = \frac{1-\gamma^{-2}_{\widetilde{v}_{S'S''}}}{\widetilde{v}^2_{S'S''}(v_{S'S''})} \,,$$

to obtain

$$\left\{ \begin{array}{l} \frac{\vec{x}'\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{c^2} = (1-\gamma^{-2}_{\widetilde{v}_{S'S''}})\Big[\frac{\vec{x}\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} + \\ \qquad -\frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} + \\ \qquad -\frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{SS''}} t_{SS''}\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})}\Big] \,. \end{array} \right. \tag{124}$$

Then, replacing (119), (124) in (104) we obtain

$$\begin{aligned} t''_{S''S'} \;=\; & \gamma_{\widetilde{v}_{S'S''}}\Bigg(- \frac{(1-\gamma_{\widetilde{v}_{S'S''}})}{\gamma_{\widetilde{v}_{S'S''}}}\;\overset{*}{\frac{\vec{x}\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})}} + \\ & + \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\frac{(1-\gamma_{\widetilde{v}_{S'S''}})}{\gamma_{\widetilde{v}_{S'S''}}}(1-\gamma_{\widetilde{v}_{SS''}})\;\overset{**}{\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}}\;\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} + \\ & + \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{SS''}}\frac{(1-\gamma_{\widetilde{v}_{S'S''}})}{\gamma_{\widetilde{v}_{S'S''}}} t_{SS''}\;\overset{***}{\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})}} + \\ & + \frac{v_{S'S''}}{\widetilde{v}_{S'S''}(v_{S'S''})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\frac{1}{\gamma_{\widetilde{v}_{S'S''}}}(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})} + \\ & + \frac{v_{S'S''}}{\widetilde{v}_{S'S''}(v_{S'S''})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\frac{1}{\gamma_{\widetilde{v}_{S'S''}}}\gamma_{\widetilde{v}_{SS''}} t_{SS''} - (1-\gamma^{-2}_{\widetilde{v}_{S'S''}})\;\overset{*}{\frac{\vec{x}\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})}} + \\ & + \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma^{-2}_{\widetilde{v}_{S'S''}})(1-\gamma_{\widetilde{v}_{SS''}})\;\overset{**}{\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}}\;\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} + \\ & + \frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma^{-2}_{\widetilde{v}_{S'S''}})\gamma_{\widetilde{v}_{SS''}} t_{SS''}\;\overset{***}{\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})}}\Bigg) \,. \end{aligned}$$

But

$$* \;=\; \frac{(1-\gamma_{\widetilde{v}_{S'S''}})}{\gamma^2_{\widetilde{v}_{S'S''}}}\frac{\vec{x}\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} \,,$$

$$** \;=\; -\frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\frac{(1-\gamma_{\widetilde{v}_{S'S''}})}{\gamma^2_{\widetilde{v}_{S'S''}}}(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} \,,$$

$$* * * \;=\; -\frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{SS''}}\frac{(1-\gamma_{\widetilde{v}_{S'S''}})}{\gamma^2_{\widetilde{v}_{S'S''}}}t_{SS''}\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})}\;,$$

then

$$\blacktriangleright\; t''_{S''S'} \;=\; \frac{(1-\gamma_{\widetilde{v}_{S'S''}})}{\gamma_{\widetilde{v}_{S'S''}}}\frac{\vec{x}\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})}+$$
$$-\frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\frac{(1-\gamma_{\widetilde{v}_{S'S''}})}{\gamma_{\widetilde{v}_{S'S''}}}(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})}+$$
$$-\frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{SS''}}\frac{(1-\gamma_{\widetilde{v}_{S'S''}})}{\gamma_{\widetilde{v}_{S'S''}}}t_{SS''}\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})}+$$
$$+\frac{v_{S'S''}}{\widetilde{v}_{S'S''}(v_{S'S''})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}+$$
$$+\frac{v_{S'S''}}{\widetilde{v}_{S'S''}(v_{S'S''})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{SS''}}t_{SS''}\;. \tag{125}$$

6.5 $(t''_{S''S'},\vec{x}\,'') \longrightarrow (t''_{S''S},\vec{x}\,'')$

We have

$$\begin{aligned}(t''_{S''S},\vec{x}\,'') &= K_{\{S'',S\},\{S'',S'\}}(t''_{S''S'},\vec{x}\,'')\\ &= K_{\{S'',S\},\{S'',S'\}}L(\vec{\widetilde{v}}_{S'S''})K_{\{S',S''\},\{S',S\}}L(\vec{\widetilde{v}}_{SS'})K_{\{S,S'\},\{S,S''\}}(t_{SS''},\vec{x})\;.\end{aligned}$$

Considering the pairs $\{S'',S\}$, $\{S'',S'\}$ and recalling that $\vec{\widetilde{v}}_{S''S}(v_{S''S})=-\vec{\widetilde{v}}_{SS''}(v_{SS''})$ (57), $\widetilde{v}_{S''S}=\widetilde{v}_{SS''}$ (58), and $v_{S''S}=v_{SS''}$ (61) then equation (98) becomes

$$t''_{S''S} \;=\; \frac{v_{SS''}}{\widetilde{v}_{SS''}(v_{SS''})}\frac{1}{\gamma_{\widetilde{v}_{SS''}}}\Bigg\{\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\,''\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}+$$
$$+\frac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}(1-\gamma_{\widetilde{v}_{S''S'}})\frac{\vec{x}\,''\cdot\vec{\widetilde{v}}_{S''S'}(v_{S''S'})}{\widetilde{v}^2_{S''S'}(v_{S''S'})}+\frac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}\gamma_{\widetilde{v}_{S''S'}}t''_{S''S'}\Bigg\}\;. \tag{126}$$

From (123) we have

$$\left\{\begin{array}{l}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\,''\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}=\\ \quad=\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}+\\ \qquad-\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{SS''}})^2\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}+\\ \qquad-\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{SS''}})\gamma_{\widetilde{v}_{SS''}}t_{SS''}\;,\end{array}\right. \tag{127}$$

and

$$\left\{\begin{array}{l}\frac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}(1-\gamma_{\widetilde{v}_{S''S'}})\frac{\vec{x}\,''\cdot\vec{\widetilde{v}}_{S''S'}(v_{S''S'})}{\widetilde{v}^2_{S''S'}(v_{S''S'})}=\\ \quad=\frac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}(1-\gamma_{\widetilde{v}_{S''S'}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{S''S'}(v_{S''S'})}{\widetilde{v}^2_{S''S'}(v_{S''S'})}+\\ \qquad-\frac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}(1-\gamma_{\widetilde{v}_{S''S'}})(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}\frac{\vec{\widetilde{v}}_{SS''}(v_{SS''})\cdot\vec{\widetilde{v}}_{S''S'}(v_{S''S'})}{\widetilde{v}^2_{S''S'}(v_{S''S'})}+\\ \qquad-\frac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}(1-\gamma_{\widetilde{v}_{S''S'}})\gamma_{\widetilde{v}_{SS''}}t_{SS''}\frac{\vec{\widetilde{v}}_{SS''}(v_{SS''})\cdot\vec{\widetilde{v}}_{S''S'}(v_{S''S'})}{\widetilde{v}^2_{S''S'}(v_{S''S'})}\;.\end{array}\right. \tag{128}$$

From (125) we have

$$
\left\{
\begin{aligned}
&\tfrac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}\gamma_{\widetilde{v}_{S''S'}}t''_{S''S'} = \\
&\qquad = \tfrac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}(1-\gamma_{\widetilde{v}_{S'S''}})\tfrac{\vec{x}\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})}+ \\
&\qquad\quad -\tfrac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}\tfrac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\tfrac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{S'S''}})(1-\gamma_{\widetilde{v}_{SS''}})\tfrac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}\times \\
&\qquad\qquad \times\tfrac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})}+ \\
&\qquad\quad -\tfrac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}\tfrac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\tfrac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{SS''}}(1-\gamma_{\widetilde{v}_{S'S''}})t_{SS''}\tfrac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})}+ \\
&\qquad\quad +\tfrac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{S''S'}}(1-\gamma_{\widetilde{v}_{SS''}})\tfrac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}+ \\
&\qquad\quad +\tfrac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{S''S'}}\gamma_{\widetilde{v}_{SS''}}t_{SS''}\ .
\end{aligned}
\right.
\tag{129}
$$

Then, replacing (127), (128), (129) in (126) we obtain

$$
\begin{aligned}
t''_{S''S} \;=\;& \frac{v_{SS''}}{\widetilde{v}_{SS''}(v_{SS''})}\frac{1}{\gamma_{\widetilde{v}_{SS''}}}\Bigg\{\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{SS''}})\ \overset{\clubsuit}{\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}} + \\
& -\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{SS''}})^2\ \overset{\clubsuit\clubsuit}{\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}} + \\
& -\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{SS''}})\gamma_{\widetilde{v}_{SS''}}\ \overset{\blacksquare}{t}_{SS''} + \\
& +\frac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}(1-\gamma_{\widetilde{v}_{S''S'}})\ \overset{\bigstar}{\frac{\vec{x}\cdot\vec{\widetilde{v}}_{S''S'}(v_{S''S'})}{\widetilde{v}^2_{S''S'}(v_{S''S'})}} + \\
& -\frac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}(1-\gamma_{\widetilde{v}_{S''S'}})(1-\gamma_{\widetilde{v}_{SS''}})\ \overset{\spadesuit}{\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}}\ \frac{\vec{\widetilde{v}}_{SS''}(v_{SS''})\cdot\vec{\widetilde{v}}_{S''S'}(v_{S''S'})}{\widetilde{v}^2_{S''S'}(v_{S''S'})} + \\
& -\frac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}(1-\gamma_{\widetilde{v}_{S''S'}})\gamma_{\widetilde{v}_{SS''}}\ \overset{\blacktriangle}{t}_{SS''}\ \frac{\vec{\widetilde{v}}_{SS''}(v_{SS''})\cdot\vec{\widetilde{v}}_{S''S'}(v_{S''S'})}{\widetilde{v}^2_{S''S'}(v_{S''S'})} + \\
& +\frac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}(1-\gamma_{\widetilde{v}_{S'S''}})\ \overset{\bigstar\bigstar}{\frac{\vec{x}\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})}} + \\
& -\frac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}\frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{S'S''}})(1-\gamma_{\widetilde{v}_{SS''}})\ \overset{\spadesuit\spadesuit}{\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}} \times \\
& \quad\times\frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} + \\
& -\frac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}\frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{SS''}}(1-\gamma_{\widetilde{v}_{S'S''}})\ \overset{\blacktriangle\blacktriangle}{t}_{SS''}\ \frac{\vec{\widetilde{v}}_{SS'}(v_{SS'})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} + \\
& +\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{S''S'}}(1-\gamma_{\widetilde{v}_{SS''}})\ \overset{\clubsuit\clubsuit\clubsuit}{\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}} + \\
& +\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{S''S'}}\gamma_{\widetilde{v}_{SS''}}\ \overset{\blacksquare\blacksquare}{t}_{SS''}\Bigg\}\,. \qquad (130)
\end{aligned}
$$

We have

$$\bigstar + \bigstar\bigstar \;=\; 0\,.$$

Let us denote

$$\Omega = \blacktriangle + \blacktriangle\blacktriangle + \blacksquare + \blacksquare\blacksquare\,.$$

Using (102), we write

$$\frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\vec{\widetilde{v}}_{SS'}(v_{SS'}) = \frac{v_{SS''}}{\widetilde{v}_{SS''}(v_{SS''})}\vec{\widetilde{v}}_{SS''}(v_{SS''}) - \frac{v_{S'S''}}{\widetilde{v}_{S'S''}(v_{S'S''})}\vec{\widetilde{v}}_{S'S''}(v_{S'S''})\,, \qquad (131)$$

then using (131) we have

$$\begin{aligned}
\blacktriangle\blacktriangle &= -\frac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{SS''}}(1-\gamma_{\widetilde{v}_{S'S''}})t_{SS''}\Big[\frac{v_{SS'}}{\widetilde{v}_{SS'}(v_{SS'})}\vec{\widetilde{v}}_{SS'}(v_{SS'})\Big]\cdot\frac{\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} \\
&= -\frac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{SS''}}(1-\gamma_{\widetilde{v}_{S'S''}})t_{SS''}\times \\
&\quad\times\Big[\frac{v_{SS''}}{\widetilde{v}_{SS''}(v_{SS''})}\vec{\widetilde{v}}_{SS''}(v_{SS''})-\frac{v_{S'S''}}{\widetilde{v}_{S'S''}(v_{S'S''})}\vec{\widetilde{v}}_{S'S''}(v_{S'S''})\Big]\cdot\frac{\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})} \\
&= -\frac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}\gamma_{\widetilde{v}_{SS''}}(1-\gamma_{\widetilde{v}_{S'S''}})t_{SS''}\frac{\vec{\widetilde{v}}_{SS''}(v_{SS''})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})}+\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{SS''}}(1-\gamma_{\widetilde{v}_{S'S''}})t_{SS''}\ ,
\end{aligned}$$

and using that $\vec{\widetilde{v}}_{S''S'}(v_{S''S'})=-\vec{\widetilde{v}}_{S'S''}(v_{S'S''})$ we have

$$\blacktriangle+\blacktriangle\blacktriangle = \frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{SS''}}(1-\gamma_{\widetilde{v}_{S'S''}})t_{SS''}\ . \tag{132}$$

Then

$$\begin{aligned}
\Omega &= -\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{SS''}})\gamma_{\widetilde{v}_{SS''}}\ \overset{\blacksquare}{t}_{SS''}+\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{S''S'}}\gamma_{\widetilde{v}_{SS''}}\ \overset{\blacksquare\blacksquare}{t}_{SS''}+ \\
&\quad+\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{SS''}}(1-\gamma_{\widetilde{v}_{S'S''}})\ \overset{\blacktriangle+\blacktriangle\blacktriangle}{t}_{SS''}
\end{aligned}$$

∴

$$\Omega = \frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma^2_{\widetilde{v}_{SS''}}t_{SS''}\ . \tag{133}$$

Next, we denote

$$\Upsilon = \clubsuit+\clubsuit\clubsuit+\clubsuit\clubsuit\clubsuit+\spadesuit+\spadesuit\spadesuit\ .$$

Using again (131) we obtain

$$\begin{aligned}
\spadesuit\spadesuit &= -\frac{\widetilde{v}_{S''S'}(v_{S''S'})}{v_{S''S'}}(1-\gamma_{\widetilde{v}_{S'S''}})(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}\frac{\vec{\widetilde{v}}_{SS''}(v_{SS''})\cdot\vec{\widetilde{v}}_{S'S''}(v_{S'S''})}{\widetilde{v}^2_{S'S''}(v_{S'S''})}+ \\
&\quad+\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{S'S''}})(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}\ ,
\end{aligned}$$

and

$$\spadesuit+\spadesuit\spadesuit = \frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{S'S''}})(1-\gamma_{\widetilde{v}_{SS''}})\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}\ .$$

Then,

$$\begin{aligned}
\Upsilon &= \frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{SS''}})\ \overset{\clubsuit}{\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}}-\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{SS''}})^2\ \overset{\clubsuit\clubsuit}{\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}}+ \\
&\quad+\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma_{\widetilde{v}_{S''S'}}(1-\gamma_{\widetilde{v}_{SS''}})\ \overset{\clubsuit\clubsuit\clubsuit}{\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}}+\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}(1-\gamma_{\widetilde{v}_{S'S''}})(1-\gamma_{\widetilde{v}_{SS''}})\ \overset{\spadesuit+\spadesuit\spadesuit}{\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{\widetilde{v}^2_{SS''}(v_{SS''})}}
\end{aligned}$$

$\therefore$

$$\Upsilon = -\frac{\widetilde{v}_{SS''}(v_{SS''})}{v_{SS''}}\gamma^2_{\widetilde{v}_{SS''}}\frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{c^2} \ . \tag{134}$$

Replacing (133) and (134) in (130) we get

$$\boxed{t''_{S''S} = \gamma_{\widetilde{v}_{SS''}}\left(t_{SS''} - \frac{\vec{x}\cdot\vec{\widetilde{v}}_{SS''}(v_{SS''})}{c^2}\right)} \ . \tag{135}$$

Then, we have obtained that

$$(t''_{S''S},\vec{x}\,'') = K_{\{S'',S\},\{S'',S'\}}L(\vec{\widetilde{v}}_{S'S''})K_{\{S',S''\},\{S',S\}}L(\vec{\widetilde{v}}_{SS'})K_{\{S,S'\},\{S,S''\}}(t_{SS''},\vec{x}) \tag{136}$$

with $t''_{S''S}$ and $\vec{x}\,''$ given by equations (135) and (123). However, we also identify these expressions with the Lorentz transformation (105), e.g.

$$(t''_{S''S},\vec{x}\,'') = L(\vec{\widetilde{v}}_{SS''})(t_{SS''},\vec{x}) \ ,$$

which then proves the form given in (108)

$$L(\vec{\widetilde{v}}_{SS''}) = K_{\{S'',S\},\{S'',S'\}}L(\vec{\widetilde{v}}_{S'S''})K_{\{S',S''\},\{S',S\}}L(\vec{\widetilde{v}}_{SS'})K_{\{S,S'\},\{S,S''\}} \ .$$

7 Appendix: A proof of the relation $\hat{\vec{x}} = \mathcal{R}_{\widehat{n}}(\varphi)\,\check{\vec{x}}$

In what follows we will denote $\mathcal{R} \equiv \mathcal{R}_{\widehat{n}}(\varphi)$. We want to show the validity of (37): $\hat{\vec{x}} = \mathcal{R}\,\check{\vec{x}}$. We understand this as a relation between vectors of $\mathcal{V}$. Then let us take

$$\frac{\vec{u}}{|\vec{u}|},\frac{\vec{v}}{|\vec{v}|},\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}$$

as basis vectors in $\mathcal{V}$. We have

$$\vec{x} = \frac{(v^2\,\vec{x}\cdot\vec{u} - \vec{u}\cdot\vec{v}\;\vec{x}\cdot\vec{v})}{u^2v^2-(\vec{u}\cdot\vec{v})^2}\,\vec{u} + \frac{(u^2\,\vec{x}\cdot\vec{v} - \vec{u}\cdot\vec{v}\;\vec{x}\cdot\vec{u})}{u^2v^2-(\vec{u}\cdot\vec{v})^2}\,\vec{v} + \vec{x}\cdot\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|} \ . \tag{137}$$

From (34), (46) and (137) we obtain

$$\begin{aligned}
\mathcal{R}\,\check{\vec{x}} \ = \ & \vec{u}\Bigg\{\Bigg[\frac{(v^2\,\vec{x}\cdot\vec{u} - \vec{u}\cdot\vec{v}\;\vec{x}\cdot\vec{v})}{u^2v^2-(\vec{u}\cdot\vec{v})^2} - (1-\gamma_u)\frac{\vec{x}\cdot\vec{u}}{u^2} + (1-\gamma_u)(1-\gamma_v)\frac{\vec{x}\cdot\vec{v}}{v^2}\,\frac{\vec{u}\cdot\vec{v}}{u^2} + \gamma_u\gamma_v\frac{\vec{x}\cdot\vec{v}}{c^2} + \\
& + \Big((1-\gamma_u)\gamma_v\frac{\vec{u}\cdot\vec{v}}{u^2} - \gamma_u\gamma_v\Big)t\Bigg]\cos\varphi + \\
& - \Bigg[\gamma_v\;\vec{x}\cdot\vec{v} - (1-\gamma_u)\;\frac{\vec{u}\cdot\vec{v}}{u^2}\;\vec{x}\cdot\vec{u} + \gamma_u\gamma_v\;\frac{\vec{u}\cdot\vec{v}}{c^2}\;\vec{x}\cdot\vec{v} + (1-\gamma_u)(1-\gamma_v)\;\frac{(\vec{u}\cdot\vec{v})^2}{u^2v^2}\;\vec{x}\cdot\vec{v} + \\
& + \Big(-\gamma_v v^2 - \gamma_u\gamma_v\vec{u}\cdot\vec{v} + (1-\gamma_u)\gamma_v\frac{(\vec{u}\cdot\vec{v})^2}{u^2}\Big)t\Bigg]\chi\lambda\frac{\sin\varphi}{|\vec{v}\times\vec{u}|}\Bigg\} \\
& + \vec{v}\Bigg\{\Bigg[\frac{(u^2\,\vec{x}\cdot\vec{v} - \vec{u}\cdot\vec{v}\;\vec{x}\cdot\vec{u})}{u^2v^2-(\vec{u}\cdot\vec{v})^2} - (1-\gamma_v)\frac{\vec{x}\cdot\vec{v}}{v^2} - \gamma_v t\Bigg]\cos\varphi + \\
& + \Bigg[\gamma_u\;\vec{x}\cdot\vec{u} - \gamma_u(1-\gamma_v)\;\frac{\vec{u}\cdot\vec{v}}{v^2}\;\vec{x}\cdot\vec{v} + \gamma_u\gamma_v\;\frac{u^2}{c^2}\;\vec{x}\cdot\vec{v} - \gamma_u\gamma_v\big(\vec{u}\cdot\vec{v} + u^2\big)t\Bigg]\chi\lambda\frac{\sin\varphi}{|\vec{v}\times\vec{u}|}\Bigg\} + \\
& + \frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}\left\{\vec{x}\cdot\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}\right\} ,
\end{aligned} \tag{138}$$

and from (35) we have

$$\hat{\vec{x}} = \vec{u}\left\{\frac{(v^2\,\vec{x}\cdot\vec{u}-\vec{u}\cdot\vec{v}\,\vec{x}\cdot\vec{v})}{u^2v^2-(\vec{u}\cdot\vec{v})^2}-\Big[1-\gamma_u\gamma_v\Big(1+\frac{\vec{u}\cdot\vec{v}}{c^2}\Big)\Big]\frac{\big(\vec{x}\cdot\vec{u}+\gamma_v\,\vec{x}\cdot\vec{v}-(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\,\vec{x}\cdot\vec{v}\big)}{\gamma_v^2\big(u^2+v^2+2\vec{u}\cdot\vec{v}-\frac{u^2v^2}{c^2}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}\big)}-\gamma_u t\right\}+$$
$$+\vec{v}\left\{\frac{(u^2\,\vec{x}\cdot\vec{v}-\vec{u}\cdot\vec{v}\,\vec{x}\cdot\vec{u})}{u^2v^2-(\vec{u}\cdot\vec{v})^2}-\Big[1-\gamma_u\gamma_v\Big(1+\frac{\vec{u}\cdot\vec{v}}{c^2}\Big)\Big]\frac{\big(\vec{x}\cdot\vec{u}+\gamma_v\,\vec{x}\cdot\vec{v}-(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\,\vec{x}\cdot\vec{v}\big)}{\gamma_v^2\big(u^2+v^2+2\vec{u}\cdot\vec{v}-\frac{u^2v^2}{c^2}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}\big)}\times\right.$$
$$\left.\times\Big[\gamma_v-(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\Big]-\gamma_u\Big[\gamma_v-(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\Big]t\right\}+$$
$$+\vec{v}\times\vec{u}\left\{\vec{x}\cdot\frac{(\vec{v}\times\vec{u})}{|\vec{v}\times\vec{u}|^2}\right\}. \qquad (139)$$

Having established these expressions for $\mathcal{R}\check{\vec{x}}$ and $\hat{\vec{x}}$ and since (37) is an equation between vectors in the vector space $\mathcal{V}$ we have that

$$R\check{\vec{x}}=\hat{\vec{x}}\Leftrightarrow\begin{cases}R\check{\vec{x}}\big|_{\vec{u}}=\hat{\vec{x}}\big|_{\vec{u}}\\ R\check{\vec{x}}\big|_{\vec{v}}=\hat{\vec{x}}\big|_{\vec{v}}\\ R\check{\vec{x}}\big|_{\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}}=\hat{\vec{x}}\big|_{\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}}\ .\end{cases} \qquad (140)$$

7.1 Verifying $R\check{\vec{x}}\big|_{\vec{u}}=\hat{\vec{x}}\big|_{\vec{u}}$

From (138) we have

$$R\check{\vec{x}}\big|_{\vec{u}} = \vec{x}\cdot\vec{u}\left\{\Big[\frac{v^2}{u^2v^2-(\vec{u}\cdot\vec{v})^2}-(1-\gamma_u)\frac{1}{u^2}\Big]\cos\varphi+(1-\gamma_u)\frac{\vec{u}\cdot\vec{v}}{u^2}\chi\lambda\frac{\sin\varphi}{|\vec{v}\times\vec{u}|}\right\}$$
$$+\vec{x}\cdot\vec{v}\left\{\Big[-\frac{\vec{u}\cdot\vec{v}}{u^2v^2-(\vec{u}\cdot\vec{v})^2}+(1-\gamma_u)(1-\gamma_v)\,\frac{\vec{u}\cdot\vec{v}}{u^2v^2}+\gamma_u\gamma_v\frac{1}{c^2}\Big]\cos\varphi+\right.$$
$$\left.-\Big[\gamma_v+\gamma_u\gamma_v\frac{\vec{u}\cdot\vec{v}}{c^2}+(1-\gamma_u)(1-\gamma_v)\frac{(\vec{u}\cdot\vec{v})^2}{u^2v^2}\Big]\chi\lambda\frac{\sin\varphi}{|\vec{v}\times\vec{u}|}\right\}+ \qquad (141)$$
$$+t\left\{\Big[(1-\gamma_u)\gamma_v\frac{\vec{u}\cdot\vec{v}}{u^2}-\gamma_u\gamma_v\Big]\cos\varphi-\Big[-\gamma_v v^2-\gamma_u\gamma_v\vec{u}\cdot\vec{v}+(1-\gamma_u)\gamma_v\frac{(\vec{u}\cdot\vec{v})^2}{u^2}\Big]\chi\lambda\frac{\sin\varphi}{|\vec{v}\times\vec{u}|}\right\},$$

and from (139) we have

$$\hat{\vec{x}}\big|_{\vec{u}} = \vec{x}\cdot\vec{u}\left\{\frac{v^2}{u^2v^2-(\vec{u}\cdot\vec{v})^2}-\Big[1-\gamma_u\gamma_v\Big(1+\frac{\vec{u}\cdot\vec{v}}{c^2}\Big)\Big]\frac{1}{\gamma_v^2\big(u^2+v^2+2\vec{u}\cdot\vec{v}-\frac{u^2v^2}{c^2}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}\big)}\right\}+$$
$$+\vec{x}\cdot\vec{v}\left\{-\frac{\vec{u}\cdot\vec{v}}{u^2v^2-(\vec{u}\cdot\vec{v})^2}-\Big[1-\gamma_u\gamma_v\Big(1+\frac{\vec{u}\cdot\vec{v}}{c^2}\Big)\Big]\frac{\big(\gamma_v-(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\big)}{\gamma_v^2\big(u^2+v^2+2\vec{u}\cdot\vec{v}-\frac{u^2v^2}{c^2}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}\big)}\right\}+$$
$$-t\gamma_u\ . \qquad (142)$$

Remark: Since $(t,\vec{x})$ are the coordinates of an arbitrary event P if we choose sucessively $\vec{x}\perp\vec{u}$, $\vec{x}\perp\vec{v}$ then to show that $R\check{\vec{x}}\big|_{\vec{u}}=\hat{\vec{x}}\big|_{\vec{u}}$ is equivalent to show the equality of the terms multiplying $\vec{x}\cdot\vec{u}$,

$\vec{x}\cdot\vec{v}$ and t, i.e.

$$R\check{\vec{x}}\big|_{\vec{u}} = \hat{\vec{x}}\big|_{\vec{u}} \Leftrightarrow \begin{cases} R\check{\vec{x}}\big|_{\vec{u}}\big]_{\vec{x}\cdot\vec{u}} = \hat{\vec{x}}\big|_{\vec{u}}\big]_{\vec{x}\cdot\vec{u}} \\ R\check{\vec{x}}\big|_{\vec{u}}\big]_{\vec{x}\cdot\vec{v}} = \hat{\vec{x}}\big|_{\vec{u}}\big]_{\vec{x}\cdot\vec{v}} \\ R\check{\vec{x}}\big|_{\vec{u}}\big]_{t} = \hat{\vec{x}}\big|_{\vec{u}}\big]_{t}\,. \end{cases} \tag{143}$$

7.1.1 $R\check{\vec{x}}\big|_{\vec{u}}\big]_{\vec{x}\cdot\vec{u}}$

Let us denote

$$\clubsuit \equiv R\check{\vec{x}}\big|_{\vec{u}}\big]_{\vec{x}\cdot\vec{u}}\,.$$

From (141) we have

$$\clubsuit \;=\; \Big[\frac{v^2}{u^2v^2-(\vec{u}\cdot\vec{v})^2} - (1-\gamma_u)\frac{1}{u^2}\Big]\cos\varphi + (1-\gamma_u)\frac{\vec{u}\cdot\vec{v}}{u^2}\chi\lambda\frac{\sin\varphi}{|\vec{v}\times\vec{u}|}\,,$$

and using (44), (45) we obtain

$$\begin{aligned}\clubsuit \;=\; & \Big\{\Big[v^2-(1-\gamma_u)\frac{1}{u^2}\big(u^2v^2-(\vec{u}\cdot\vec{v})^2\big)\Big]\Big[\gamma_v^{-1}u^2+\gamma_u^{-1}v^2+(1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})\vec{u}\cdot\vec{v}+ \\ & -(\gamma_u^{-1}-1)\frac{(\vec{u}\cdot\vec{v})^2}{u^2}-(\gamma_v^{-1}-1)\frac{(\vec{u}\cdot\vec{v})^2}{v^2}+(\gamma_u^{-1}-1)(\gamma_v^{-1}-1)\frac{(\vec{u}\cdot\vec{v})^3}{u^2v^2}\Big]+ \\ & -\Big[\big(u^2v^2-(\vec{u}\cdot\vec{v})^2\big)(1-\gamma_u)\frac{\vec{u}\cdot\vec{v}}{u^2}\Big]\Big[-1+\gamma_u^{-1}\gamma_v^{-1}+(\gamma_u^{-1}-1)\frac{\vec{u}\cdot\vec{v}}{u^2}+(\gamma_v^{-1}-1)\frac{\vec{u}\cdot\vec{v}}{v^2}+ \\ & -(\gamma_u^{-1}-1)(\gamma_v^{-1}-1)\frac{(\vec{u}\cdot\vec{v})^2}{u^2v^2}\Big]\Big\}\Big/\Big(u^2v^2-(\vec{u}\cdot\vec{v})^2\Big)\Big(u^2+v^2+2\vec{u}\cdot\vec{v}-\frac{u^2v^2}{c^2}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big)\,.\end{aligned}$$

But

$$v^2-(1-\gamma_u)\frac{1}{u^2}\big(u^2v^2-(\vec{u}\cdot\vec{v})^2\big) \;=\; \gamma_u v^2+(1-\gamma_u)\frac{1}{u^2}(\vec{u}\cdot\vec{v})^2$$

and

$$-\big(u^2v^2-(\vec{u}\cdot\vec{v})^2\big)(1-\gamma_u)\frac{\vec{u}\cdot\vec{v}}{u^2} = -(1-\gamma_u)v^2\vec{u}\cdot\vec{v}+(1-\gamma_u)\frac{(\vec{u}\cdot\vec{v})^3}{u^2}$$

then using these expressions we rewrite ♣ as follows

$$
\begin{aligned}
\clubsuit \;=\; & \Big\{ \gamma_u\gamma_v^{-1}\; \overset{\blacktriangle}{u^2v^2} + \overset{\blacktriangle}{v^4} + (1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})\gamma_u v^2\; \overset{\blacktriangle\blacktriangle}{\vec u\cdot\vec v} - \gamma_u(\gamma_u^{-1}-1)\frac{v^2}{u^2}\; \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec u\cdot\vec v)^2} + \\
& -\gamma_u(\gamma_v^{-1}-1)\; \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec u\cdot\vec v)^2} + \gamma_u(\gamma_u^{-1}-1)(\gamma_v^{-1}-1)\; \frac{\overset{\blacksquare}{(\vec u\cdot\vec v)^3}}{u^2} + \\
& +(1-\gamma_u)\gamma_v^{-1}\; \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec u\cdot\vec v)^2} + \gamma_u^{-1}(1-\gamma_u)\frac{v^2}{u^2}\; \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec u\cdot\vec v)^2} + \\
& +(1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})(1-\gamma_u)\frac{1}{u^2}\; \overset{\blacksquare}{(\vec u\cdot\vec v)^3} \\
& -(1-\gamma_u)(\gamma_u^{-1}-1)\frac{1}{u^4}\; \overset{\blacksquare\blacksquare}{(\vec u\cdot\vec v)^4} - (1-\gamma_u)(\gamma_v^{-1}-1)\frac{1}{u^2v^2}\; \overset{\blacksquare\blacksquare}{(\vec u\cdot\vec v)^4} + \\
& +(1-\gamma_u)(\gamma_u^{-1}-1)(\gamma_v^{-1}-1)\frac{1}{u^4v^2}\; \overset{\blacksquare\blacksquare\blacksquare}{(\vec u\cdot\vec v)^5} \\
& +(1-\gamma_u)v^2\; \overset{\blacktriangle\blacktriangle}{\vec u\cdot\vec v} - \gamma_u^{-1}(1-\gamma_u)\gamma_v^{-1}v^2\; \overset{\blacktriangle\blacktriangle}{\vec u\cdot\vec v} - (1-\gamma_u)(\gamma_u^{-1}-1)\frac{v^2}{u^2}\; \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec u\cdot\vec v)^2} + \\
& -(1-\gamma_u)(\gamma_v^{-1}-1)\; \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec u\cdot\vec v)^2} + (1-\gamma_u)(\gamma_u^{-1}-1)(\gamma_v^{-1}-1)\frac{1}{u^2}\; \overset{\blacksquare}{(\vec u\cdot\vec v)^3} \\
& -(1-\gamma_u)\; \frac{\overset{\blacksquare}{(\vec u\cdot\vec v)^3}}{u^2} + \gamma_u^{-1}(1-\gamma_u)\gamma_v^{-1}\; \frac{\overset{\blacksquare}{(\vec u\cdot\vec v)^3}}{u^2} + (1-\gamma_u)(\gamma_u^{-1}-1)\frac{1}{u^4}\; \overset{\blacksquare\blacksquare}{(\vec u\cdot\vec v)^4} \\
& +(1-\gamma_u)(\gamma_v^{-1}-1)\frac{1}{u^2v^2}\; \overset{\blacksquare\blacksquare}{(\vec u\cdot\vec v)^4} - (1-\gamma_u)(\gamma_u^{-1}-1)(\gamma_v^{-1}-1)\frac{1}{u^4v^2}\; \overset{\blacksquare\blacksquare\blacksquare}{(\vec u\cdot\vec v)^5} \Big\} \\
& \Big/ \Big\{ \Big(u^2v^2-(\vec u\cdot\vec v)^2\Big)\Big(u^2+v^2+2\vec u\cdot\vec v-\frac{u^2v^2}{c^2}+\frac{(\vec u\cdot\vec v)^2}{c^2}\Big) \Big\}\,.
\end{aligned}
$$

We have

$$
\begin{aligned}
\blacktriangle\big|_{\clubsuit} &= \gamma_u\gamma_v^{-1}u^2v^2+v^4 \\
&= c^4(1-\gamma_v^{-2})\big[-\gamma_u^{-1}\gamma_v^{-1}-\gamma_v^{-2}+1+\gamma_u\gamma_v^{-1}\big]
\end{aligned}
$$

$$
\begin{aligned}
\blacktriangle\blacktriangle\big|_{\clubsuit} &= \vec u\cdot\vec v\; v^2\big[(1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})\gamma_u+(1-\gamma_u)-\gamma_u^{-1}(1-\gamma_u)\gamma_v^{-1}\big] \\
&= \vec u\cdot\vec v\; c^2(1-\gamma_v^{-2})\big[-\gamma_u^{-1}\gamma_v^{-1}+2+\gamma_u\gamma_v^{-1}\big]
\end{aligned}
$$

$$
\begin{aligned}
\blacktriangle\blacktriangle\blacktriangle\big|_{\clubsuit} &= (\vec u\cdot\vec v)^2\Big[-\gamma_u(\gamma_u^{-1}-1)\frac{v^2}{u^2}-\gamma_u(\gamma_v^{-1}-1)+(1-\gamma_u)\gamma_v^{-1}+\gamma_u^{-1}(1-\gamma_u)\frac{v^2}{u^2}+ \\
&\qquad -(1-\gamma_u)(\gamma_u^{-1}-1)\frac{v^2}{u^2}-(1-\gamma_u)(\gamma_v^{-1}-1)\Big] \\
&= (\vec u\cdot\vec v)^2\big[1-\gamma_u\gamma_v^{-1}\big]
\end{aligned}
$$

$$
\begin{aligned}
\blacksquare\big|_{\clubsuit} &= (\vec u\cdot\vec v)^3\frac{1}{u^2}\big[\gamma_u(\gamma_u^{-1}-1)(\gamma_v^{-1}-1)+(1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})(1-\gamma_u) \\
&\qquad +(1-\gamma_u)(\gamma_u^{-1}-1)(\gamma_v^{-1}-1)-(1-\gamma_u)+\gamma_u^{-1}(1-\gamma_u)\gamma_v^{-1}\big] \\
&= -\frac{(\vec u\cdot\vec v)^3}{c^2}\gamma_u\gamma_v^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\blacksquare\blacksquare\big|_{\clubsuit} &= (\vec{u}\cdot\vec{v})^4\Big[-(1-\gamma_u)(\gamma_u^{-1}-1)\frac{1}{u^4}-(1-\gamma_u)(\gamma_v^{-1}-1)\frac{1}{u^2v^2}+(1-\gamma_u)(\gamma_u^{-1}-1)\frac{1}{u^4}+\\
&\qquad +(1-\gamma_u)(\gamma_v^{-1}-1)\frac{1}{u^2v^2}\Big]\\
&= 0
\end{aligned}
$$

$$
\blacksquare\blacksquare\blacksquare\big|_{\clubsuit} = 0
$$

$\therefore$

$$
\begin{aligned}
\clubsuit &= \Big\{\big(-\gamma_u^{-1}\gamma_v^{-1}-\gamma_v^{-2}+1+\gamma_u\gamma_v^{-1}\big)(1-\gamma_v^{-2})c^4+\\
&\quad +\big(-\gamma_u^{-1}\gamma_v^{-1}+2+\gamma_u\gamma_v^{-1}\big)(1-\gamma_v^{-2})c^2\vec{u}\cdot\vec{v}+\\
&\quad +\big(1-\gamma_u\gamma_v^{-1}\big)(\vec{u}\cdot\vec{v})^2-\gamma_u\gamma_v^{-1}\frac{(\vec{u}\cdot\vec{v})^3}{c^2}\Big\}\Big/\\
&\quad \Big/\Big\{\Big(u^2v^2-(\vec{u}\cdot\vec{v})^2\Big)\Big(u^2+v^2+2\vec{u}\cdot\vec{v}-\frac{u^2v^2}{c^2}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big)\Big\}\,. \qquad (144)
\end{aligned}
$$

7.1.2 $\hat{\vec{x}}\big|_{\vec{u}}\Big]_{\vec{x}\cdot\vec{u}}$

Let us denote

$$
\clubsuit\clubsuit \equiv \hat{\vec{x}}\big|_{\vec{u}}\Big]_{\vec{x}\cdot\vec{u}}\,.
$$

From (142) we obtain

$$
\begin{aligned}
\clubsuit\clubsuit &= \frac{v^2}{u^2v^2-(\vec{u}\cdot\vec{v})^2}-\Big[1-\gamma_u\gamma_v\Big(1+\frac{\vec{u}\cdot\vec{v}}{c^2}\Big)\Big]\frac{1}{\gamma_v^2\Big(u^2+v^2+2\vec{u}\cdot\vec{v}-\frac{u^2v^2}{c^2}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big)}\\
&= \Big\{\overset{\blacktriangle}{u^2}v^2+\overset{\blacktriangle}{v^4}+2v^2\,\overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}}-\frac{\overset{\blacktriangle}{u^2}v^4}{c^2}+\frac{v^2}{c^2}\,\overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2}-\gamma_v^{-2}\overset{\blacktriangle}{u^2}v^2+\gamma_u\gamma_v^{\overset{\blacktriangle}{-1}}u^2v^2+\gamma_u\gamma_v^{-1}u^2v^2\,\frac{\overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}}}{c^2}\\
&\quad +\gamma_v^{-2}\,\overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2}-\gamma_u\gamma_v^{-1}\,\overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2}-\gamma_u\gamma_v^{-1}\frac{(\vec{u}\cdot\vec{v})^3}{c^2}\Big]\Big\}\Big/\\
&\quad \Big/\Big\{\Big(u^2v^2-(\vec{u}\cdot\vec{v})^2\Big)\Big(u^2+v^2+2\vec{u}\cdot\vec{v}-\frac{u^2v^2}{c^2}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big)\Big\}\,.
\end{aligned}
$$

We have

$$
\begin{aligned}
\blacktriangle\big|_{\clubsuit\clubsuit} &= u^2v^2+v^4-\frac{u^2v^4}{c^2}-\gamma_v^{-2}u^2v^2+\gamma_u\gamma_v^{-1}u^2v^2\\
&= c^4(1-\gamma_v^{-2})\big[-\gamma_u^{-1}\gamma_v^{-1}-\gamma_v^{-2}+1+\gamma_u\gamma_v^{-1}\big]
\end{aligned}
$$

$$
\begin{aligned}
\blacktriangle\blacktriangle\big|_{\clubsuit\clubsuit} &= \vec{u}\cdot\vec{v}\,\Big[2v^2+\gamma_u\gamma_v^{-1}u^2v^2\frac{1}{c^2}\Big]\\
&= \vec{u}\cdot\vec{v}\,c^2(1-\gamma_v^{-2})\,\big[-\gamma_u^{-1}\gamma_v^{-1}+2+\gamma_u\gamma_v^{-1}\big]
\end{aligned}
$$

$$
\begin{aligned}
\blacktriangle\blacktriangle\blacktriangle\big|_{\clubsuit\clubsuit} &= (\vec{u}\cdot\vec{v})^2\Big[\frac{v^2}{c^2}+\gamma_v^{-2}-\gamma_u\gamma_v^{-1}\Big] \\
&= (\vec{u}\cdot\vec{v})^2\left[1-\gamma_u\gamma_v^{-1}\right]
\end{aligned}
$$

$\therefore$

$$
\begin{aligned}
\clubsuit\clubsuit \;=\; &\Big\{\big(-\gamma_u^{-1}\gamma_v^{-1}-\gamma_v^{-2}+1+\gamma_u\gamma_v^{-1}\big)(1-\gamma_v^{-2})c^4+ \\
&+\big(-\gamma_u^{-1}\gamma_v^{-1}+2+\gamma_u\gamma_v^{-1}\big)(1-\gamma_v^{-2})c^2\;\vec{u}\cdot\vec{v}+ \\
&+\big(1-\gamma_u\gamma_v^{-1}\big)(\vec{u}\cdot\vec{v})^2-\gamma_u\gamma_v^{-1}\frac{(\vec{u}\cdot\vec{v})^3}{c^2}\Big\}\Big/ \\
&\Big/\Big\{\Big(u^2v^2-(\vec{u}\cdot\vec{v})^2\Big)\Big(u^2+v^2+2\vec{u}\cdot\vec{v}-\frac{u^2v^2}{c^2}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big)\Big\}\,. \qquad (145)
\end{aligned}
$$

7.1.3

From (144) and (145) we have

$$\clubsuit=\clubsuit\clubsuit$$

$\therefore$

$$\boxed{R\check{\vec{x}}\big|_{\vec{u}}\Big]_{\vec{x}\cdot\vec{u}}=\hat{\vec{x}}\big|_{\vec{u}}\Big]_{\vec{x}\cdot\vec{u}}} \qquad (146)$$

7.1.4 $R\check{\vec{x}}|_{\vec{u}}]_{\vec{x}\cdot\vec{v}}$

Let us denote

$$\bigstar\;\equiv\;R\check{\vec{x}}\big|_{\vec{u}}\big]_{\vec{x}\cdot\vec{v}}\,.$$

From (141) we have

$$
\begin{aligned}
\bigstar \;=\; &\Big[-\frac{\vec{u}\cdot\vec{v}}{u^2v^2-(\vec{u}\cdot\vec{v})^2}+(1-\gamma_u)(1-\gamma_v)\,\frac{\vec{u}\cdot\vec{v}}{u^2v^2}+\gamma_u\gamma_v\frac{1}{c^2}\Big]\cos\varphi+ \\
&-\Big[\gamma_v+\gamma_u\gamma_v\frac{\vec{u}\cdot\vec{v}}{c^2}+(1-\gamma_u)(1-\gamma_v)\frac{(\vec{u}\cdot\vec{v})^2}{u^2v^2}\Big]\chi\lambda\frac{\sin\varphi}{|\vec{v}\times\vec{u}|}\;,
\end{aligned}
$$

and using (44) and (45) we obtain

$$
\begin{aligned}
\bigstar \;=\; &\Big\{\Big(-\vec{u}\cdot\vec{v}+(1-\gamma_u)(1-\gamma_v)(u^2v^2-(\vec{u}\cdot\vec{v})^2)\frac{\vec{u}\cdot\vec{v}}{u^2v^2}+\gamma_u\gamma_v(u^2v^2-(\vec{u}\cdot\vec{v})^2)\frac{1}{c^2}\Big)\times \\
&\times\Big[\gamma_v^{-1}u^2+\gamma_u^{-1}v^2+(1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})\vec{u}\cdot\vec{v}-(\gamma_u^{-1}-1)\frac{(\vec{u}\cdot\vec{v})^2}{u^2}+ \\
&-(\gamma_v^{-1}-1)\frac{(\vec{u}\cdot\vec{v})^2}{v^2}+(\gamma_u^{-1}-1)(\gamma_v^{-1}-1)\frac{(\vec{u}\cdot\vec{v})^3}{u^2v^2}\Big]+ \\
&+\big(u^2v^2-(\vec{u}\cdot\vec{v})^2\big)\Big(\gamma_v+\gamma_u\gamma_v\frac{\vec{u}\cdot\vec{v}}{c^2}+(1-\gamma_u)(1-\gamma_v)\frac{(\vec{u}\cdot\vec{v})^2}{u^2v^2}\Big)\Big[-1+\gamma_u^{-1}\gamma_v^{-1}+ \\
&+(\gamma_u^{-1}-1)\frac{\vec{u}\cdot\vec{v}}{u^2}+(\gamma_v^{-1}-1)\frac{\vec{u}\cdot\vec{v}}{v^2}-(\gamma_u^{-1}-1)(\gamma_v^{-1}-1)\frac{(\vec{u}\cdot\vec{v})^2}{u^2v^2}\Big]\Big\}\Big/ \\
&\Big/\Big\{\big(u^2v^2-(\vec{u}\cdot\vec{v})^2\big)\Big(u^2+v^2+2\vec{u}\cdot\vec{v}-\frac{u^2v^2}{c^2}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big)\Big\}\,.
\end{aligned}
$$

We have

$$\bigstar \;=\; \Big\{ -\gamma_v^{-1}u^2\ \overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} -\gamma_u^{-1}v^2\ \overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} -(1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})\ \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} +(\gamma_u^{-1}-1)\ \frac{\overset{\blacksquare}{(\vec{u}\cdot\vec{v})^3}}{u^2} +$$

$$+(\gamma_v^{-1}-1)\ \frac{\overset{\blacksquare}{(\vec{u}\cdot\vec{v})^3}}{v^2} -(\gamma_u^{-1}-1)(\gamma_v^{-1}-1)\ \frac{\overset{\blacksquare\blacksquare}{(\vec{u}\cdot\vec{v})^4}}{u^2v^2} +(1-\gamma_u)\gamma_v^{-1}(1-\gamma_v)u^2\ \overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} +$$

$$-(1-\gamma_u)\gamma_v^{-1}(1-\gamma_v)\ \frac{\overset{\blacksquare}{(\vec{u}\cdot\vec{v})^3}}{v^2} +\gamma_u^{-1}(1-\gamma_u)(1-\gamma_v)v^2\ \overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} -\gamma_u^{-1}(1-\gamma_u)(1-\gamma_v)\ \frac{\overset{\blacksquare}{(\vec{u}\cdot\vec{v})^3}}{u^2} +$$

$$+(1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})(1-\gamma_u)(1-\gamma_v)\ \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} +$$

$$-(1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})(1-\gamma_u)(1-\gamma_v)\ \frac{\overset{\blacksquare\blacksquare}{(\vec{u}\cdot\vec{v})^4}}{u^2v^2} -(\gamma_u^{-1}-1)(1-\gamma_u)(1-\gamma_v)\ \frac{\overset{\blacksquare}{(\vec{u}\cdot\vec{v})^3}}{u^2} +$$

$$+(\gamma_u^{-1}-1)(1-\gamma_u)(1-\gamma_v)\ \frac{\overset{\blacksquare\blacksquare\blacksquare}{(\vec{u}\cdot\vec{v})^5}}{u^4v^2} -(1-\gamma_u)(\gamma_v^{-1}-1)(1-\gamma_v)\ \frac{\overset{\blacksquare}{(\vec{u}\cdot\vec{v})^3}}{v^2} +$$

$$+(1-\gamma_u)(\gamma_v^{-1}-1)(1-\gamma_v)\ \frac{\overset{\blacksquare\blacksquare\blacksquare}{(\vec{u}\cdot\vec{v})^5}}{u^2v^4} +(\gamma_u^{-1}-1)(1-\gamma_u)(\gamma_v^{-1}-1)(1-\gamma_v)\ \frac{\overset{\blacksquare\blacksquare}{(\vec{u}\cdot\vec{v})^4}}{u^2v^2} +$$

$$-(\gamma_u^{-1}-1)(1-\gamma_u)(\gamma_v^{-1}-1)(1-\gamma_v)\ \frac{\overset{\blacklozenge}{(\vec{u}\cdot\vec{v})^6}}{u^4v^4} +\gamma_u\ \frac{\overset{\blacktriangle}{u^4v^2}}{c^2} -\gamma_u\frac{u^2}{c^2}\ \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} +\gamma_v\ \frac{\overset{\blacktriangle}{u^2v^4}}{c^2} +$$

$$-\gamma_v\frac{v^2}{c^2}\ \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} +(1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})\gamma_u\gamma_v\frac{u^2v^2}{c^2}\ \overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} +$$

$$-(1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})\gamma_u\gamma_v\frac{1}{c^2}\ \overset{\blacksquare}{(\vec{u}\cdot\vec{v})^3} -\gamma_u(\gamma_u^{-1}-1)\gamma_v\frac{v^2}{c^2}\ \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} +$$

$$+\gamma_u(\gamma_u^{-1}-1)\gamma_v\frac{1}{c^2}\ \frac{\overset{\blacksquare\blacksquare}{(\vec{u}\cdot\vec{v})^4}}{u^2} -\gamma_u\gamma_v(\gamma_v^{-1}-1)\frac{u^2}{c^2}\ \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} +\gamma_u\gamma_v(\gamma_v^{-1}-1)\frac{1}{c^2}\ \frac{\overset{\blacksquare\blacksquare}{(\vec{u}\cdot\vec{v})^4}}{v^2} +$$

$$+\gamma_u(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\frac{1}{c^2}\ \overset{\blacksquare}{(\vec{u}\cdot\vec{v})^3} -\gamma_u(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\frac{1}{c^2}\ \frac{\overset{\blacksquare\blacksquare\blacksquare}{(\vec{u}\cdot\vec{v})^5}}{u^2v^2} -\gamma_v\ \overset{\blacktriangle}{u^2v^2} +$$

$$+\gamma_v\ \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} + \gamma_u^{-1}\overset{\blacktriangle}{u^2v^2} -\gamma_u^{-1}\ \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} +(\gamma_u^{-1}-1)\gamma_v v^2\ \overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} -(\gamma_u^{-1}-1)\gamma_v\ \frac{\overset{\blacksquare}{(\vec{u}\cdot\vec{v})^3}}{u^2} +$$

$$+\gamma_v(\gamma_v^{-1}-1)u^2\ \overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} -\gamma_v(\gamma_v^{-1}-1)\ \frac{\overset{\blacksquare}{(\vec{u}\cdot\vec{v})^3}}{v^2} -(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\ \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} +$$

$$+(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\ \frac{\overset{\blacksquare\blacksquare}{(\vec{u}\cdot\vec{v})^4}}{u^2v^2} -\gamma_u\gamma_v u^2v^2\ \frac{\overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}}}{c^2} +\gamma_u\gamma_v\ \frac{\overset{\blacksquare}{(\vec{u}\cdot\vec{v})^3}}{c^2} +u^2v^2\ \frac{\overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}}}{c^2} +$$

$$-\ \frac{\overset{\blacksquare}{(\vec{u}\cdot\vec{v})^3}}{c^2} +\gamma_u(\gamma_u^{-1}-1)\gamma_v\frac{v^2}{c^2}\ \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} -\gamma_u(\gamma_u^{-1}-1)\gamma_v\ \frac{\overset{\blacksquare\blacksquare}{(\vec{u}\cdot\vec{v})^4}}{c^2u^2} +\gamma_u\gamma_v(\gamma_v^{-1}-1)\frac{u^2}{c^2}\ \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} +$$

$$
-\gamma_u\gamma_v(\gamma_v^{-1}-1)\ \overset{\blacksquare\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^4}{c^2v^2}} -\gamma_u(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\ \overset{\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^3}{c^2}} +\gamma_u(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\ \overset{\blacksquare\blacksquare\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^5}{c^2u^2v^2}} +
$$

$$
-(1-\gamma_u)(1-\gamma_v)\ \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} +(1-\gamma_u)(1-\gamma_v)\ \overset{\blacksquare\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^4}{u^2v^2}} +\gamma_u^{-1}(1-\gamma_u)\gamma_v^{-1}(1-\gamma_v)\ \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} +
$$

$$
-\gamma_u^{-1}(1-\gamma_u)\gamma_v^{-1}(1-\gamma_v)\ \overset{\blacksquare\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^4}{u^2v^2}} +(\gamma_u^{-1}-1)(1-\gamma_u)(1-\gamma_v)\ \overset{\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^3}{u^2}} +
$$

$$
-(\gamma_u^{-1}-1)(1-\gamma_u)(1-\gamma_v)\ \overset{\blacksquare\blacksquare\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^5}{u^4v^2}} +(1-\gamma_u)(\gamma_v^{-1}-1)(1-\gamma_v)\ \overset{\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^3}{v^2}} +
$$

$$
-(1-\gamma_u)(\gamma_v^{-1}-1)(1-\gamma_v)\ \overset{\blacksquare\blacksquare\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^5}{u^2v^4}} -(\gamma_u^{-1}-1)(1-\gamma_u)(\gamma_v^{-1}-1)(1-\gamma_v)\ \overset{\blacksquare\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^4}{u^2v^2}} +
$$

$$
+(\gamma_u^{-1}-1)(1-\gamma_u)(\gamma_v^{-1}-1)(1-\gamma_v)\ \overset{\blacklozenge}{\frac{(\vec{u}\cdot\vec{v})^6}{u^4v^4}}\ \Big\}\Big/
$$

$$
\Big/\left\{(u^2v^2-(\vec{u}\cdot\vec{v})^2)\left(u^2+v^2+2\vec{u}\cdot\vec{v}-\frac{u^2v^2}{c^2}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}\right)\right\}.
$$

We have

$$
\begin{aligned}
\blacktriangle\big|_\bigstar &= \gamma_u\frac{u^4v^2}{c^2}+\gamma_v\frac{u^2v^4}{c^2}-\gamma_v u^2v^2+\gamma_u^{-1}u^2v^2 \\
&= c^4(1-\gamma_u^{-2})(1-\gamma_v^{-2})\big[-\gamma_v^{-1}+\gamma_u\big]
\end{aligned}
$$

$$
\begin{aligned}
\blacktriangle\blacktriangle\big|_\bigstar &= \vec{u}\cdot\vec{v}\Big[-\gamma_v^{-1}u^2-\gamma_u^{-1}v^2+(1-\gamma_u)\gamma_v^{-1}(1-\gamma_v)u^2+\gamma_u^{-1}(1-\gamma_u)(1-\gamma_v)v^2\Big] \\
&= \vec{u}\cdot\vec{v}c^2\Big[\gamma_u^{-2}\gamma_v^{-1}+\gamma_u^{-1}\gamma_v^{-2}+\gamma_u^{-1}\gamma_v^{-1}-2\gamma_u^{-1}+\gamma_v^{-2}-\gamma_v^{-1}-1-\gamma_u\gamma_v^{-2}-\gamma_u\gamma_v^{-1}+2\gamma_u\Big]
\end{aligned}
$$

$$
\begin{aligned}
\blacktriangle\blacktriangle\blacktriangle\big|_\bigstar &= (\vec{u}\cdot\vec{v})^2\Big[-(1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})+(1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})(1-\gamma_u)(1-\gamma_v)+ \\
&\qquad -\gamma_u\frac{u^2}{c^2}-\gamma_v\frac{v^2}{c^2}-\gamma_u(\gamma_u^{-1}-1)\gamma_v\frac{v^2}{c^2}-\gamma_u\gamma_v(\gamma_v^{-1}-1)\frac{u^2}{c^2}+\gamma_v+ \\
&\qquad -\gamma_u^{-1}-(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)+\gamma_u(\gamma_u^{-1}-1)\gamma_v\frac{v^2}{c^2}+ \\
&\qquad +\gamma_u\gamma_v(\gamma_v^{-1}-1)\frac{u^2}{c^2}-(1-\gamma_u)(1-\gamma_v)+\gamma_u^{-1}(1-\gamma_u)\gamma_v^{-1}(1-\gamma_v)\Big] \\
&= (\vec{u}\cdot\vec{v})^2\Big[\gamma_u^{-1}\gamma_v^{-1}-\gamma_u^{-1}+\gamma_v^{-1}-2-\gamma_u\gamma_v^{-1}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\blacksquare\big|_\star &= (\vec{u}\cdot\vec{v})^3\Big[(\gamma_u^{-1}-1)\frac{1}{u^2}+(\gamma_v^{-1}-1)\frac{1}{v^2}-(1-\gamma_u)\gamma_v^{-1}(1-\gamma_v)\frac{1}{v^2}-\gamma_u^{-1}(1-\gamma_u)(1-\gamma_v)\frac{1}{u^2}\\
&\quad -(\gamma_u^{-1}-1)(1-\gamma_u)(1-\gamma_v)\frac{1}{u^2}-(1-\gamma_u)(\gamma_v^{-1}-1)(1-\gamma_v)\frac{1}{v^2}\\
&\quad -(1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})\gamma_u\gamma_v\frac{1}{c^2}+\gamma_u(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\frac{1}{c^2}-(\gamma_u^{-1}-1)\gamma_v\frac{1}{u^2}+\\
&\quad -\gamma_v(\gamma_v^{-1}-1)\frac{1}{v^2}+\gamma_u\gamma_v\frac{1}{c^2}-\frac{1}{c^2}-\gamma_u(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\frac{1}{c^2}\\
&\quad +(\gamma_u^{-1}-1)(1-\gamma_u)(1-\gamma_v)\frac{1}{u^2}+(1-\gamma_u)(\gamma_v^{-1}-1)(1-\gamma_v)\frac{1}{v^2}\Big]\\
&= \frac{(\vec{u}\cdot\vec{v})^3}{c^2}\frac{1}{(\gamma_v^2-1)}\big[\gamma_v-\gamma_v^2+\gamma_u+\gamma_u\gamma_v-2\gamma_u\gamma_v^2\big]
\end{aligned}
$$

$$
\begin{aligned}
\blacksquare\blacksquare\big|_\star &= (\vec{u}\cdot\vec{v})^4\Big[-(\gamma_u^{-1}-1)(\gamma_v^{-1}-1)\frac{1}{u^2v^2}-(1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})(1-\gamma_u)(1-\gamma_v)\frac{1}{u^2v^2}+\\
&\quad +(\gamma_u^{-1}-1)(1-\gamma_u)(\gamma_v^{-1}-1)(1-\gamma_v)\frac{1}{u^2v^2}+\gamma_u(\gamma_u^{-1}-1)\gamma_v\frac{1}{c^2u^2}+\\
&\quad +\gamma_u\gamma_v(\gamma_v^{-1}-1)\frac{1}{c^2v^2}+(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\frac{1}{u^2v^2}-\gamma_u(\gamma_u^{-1}-1)\gamma_v\frac{1}{c^2u^2}\\
&\quad -\gamma_u\gamma_v(\gamma_v^{-1}-1)\frac{1}{c^2v^2}+(1-\gamma_u)(1-\gamma_v)\frac{1}{u^2v^2}-\gamma_u^{-1}(1-\gamma_u)\gamma_v^{-1}(1-\gamma_v)\frac{1}{u^2v^2}+\\
&\quad -(\gamma_u^{-1}-1)(1-\gamma_u)(\gamma_v^{-1}-1)(1-\gamma_v)\frac{1}{u^2v^2}\Big]\\
&= -(\vec{u}\cdot\vec{v})^4\frac{\gamma_u\gamma_v}{c^4(\gamma_v+1)}
\end{aligned}
$$

$$
\begin{aligned}
\blacksquare\blacksquare\blacksquare\big|_\star &= (\vec{u}\cdot\vec{v})^5\Big[(\gamma_u^{-1}-1)(1-\gamma_u)(1-\gamma_v)\frac{1}{u^4v^2}+(1-\gamma_u)(\gamma_v^{-1}-1)(1-\gamma_v)\frac{1}{u^2v^4}+\\
&\quad -\gamma_u(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\frac{1}{c^2}\frac{1}{u^2v^2}+\gamma_u(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\frac{1}{c^2u^2v^2}+\\
&\quad -(\gamma_u^{-1}-1)(1-\gamma_u)(1-\gamma_v)\frac{1}{u^4v^2}-(1-\gamma_u)(\gamma_v^{-1}-1)(1-\gamma_v)\frac{1}{u^2v^4}\Big]\\
&= 0
\end{aligned}
$$

$$
\blacklozenge\big|_\star = 0
$$

$\therefore$

$$
\begin{aligned}
\star &= c^4(1-\gamma_u^{-2})(1-\gamma_v^{-2})\big[-\gamma_v^{-1}+\gamma_u\big]+\\
&\quad c^2\vec{u}\cdot\vec{v}\big[\gamma_u^{-2}\gamma_v^{-1}+\gamma_u^{-1}\gamma_v^{-2}+\gamma_u^{-1}\gamma_v^{-1}-2\gamma_u^{-1}+\gamma_v^{-2}-\gamma_v^{-1}-1-\gamma_u\gamma_v^{-2}-\gamma_u\gamma_v^{-1}+2\gamma_u\big]\\
&\quad +(\vec{u}\cdot\vec{v})^2\big[\gamma_u^{-1}\gamma_v^{-1}-\gamma_u^{-1}+\gamma_v^{-1}-2-\gamma_u\gamma_v^{-1}\big]+\\
&\quad +\frac{(\vec{u}\cdot\vec{v})^3}{c^2}\frac{1}{(\gamma_v^2-1)}\big[\gamma_v+\gamma_u-\gamma_v^2+\gamma_u\gamma_v-2\gamma_u\gamma_v^2\big]+\\
&\quad -(\vec{u}\cdot\vec{v})^4\frac{\gamma_u\gamma_v}{c^4(\gamma_v+1)}\ . \qquad (147)
\end{aligned}
$$

7.1.5 $\hat{\vec{x}}\big|_{\vec{u}}\big]_{\vec{x}\cdot\vec{v}}$

Let us denote

$$\bigstar\bigstar \equiv \hat{\vec{x}}\big|_{\vec{u}}\big]_{\vec{x}\cdot\vec{v}}\,.$$

From (142) we obtain

$$\begin{aligned}
\bigstar\bigstar &= -\frac{\vec{u}\cdot\vec{v}}{u^2v^2-(\vec{u}\cdot\vec{v})^2} - \Big[1-\gamma_u\gamma_v\Big(1+\frac{\vec{u}\cdot\vec{v}}{c^2}\Big)\Big]\frac{\Big(\gamma_v-(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\Big)}{\gamma_v^2\Big(u^2+v^2+2\vec{u}\cdot\vec{v}-\frac{u^2v^2}{c^2}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big)} \\
&= \Bigg\{-u^2\ \vec{u}\overset{\blacktriangle\blacktriangle}{\cdot}\vec{v} - v^2\ \vec{u}\overset{\blacktriangle\blacktriangle}{\cdot}\vec{v} - 2\ (\vec{u}\overset{\blacktriangle\blacktriangle\blacktriangle}{\cdot}\vec{v})^2 + \frac{u^2v^2}{c^2}\ \vec{u}\overset{\blacktriangle\blacktriangle}{\cdot}\vec{v} - \frac{(\vec{u}\overset{\blacksquare}{\cdot}\vec{v})^3}{c^2} + \\
&\quad -\gamma_v^{-1}\ u^{\overset{\blacktriangle}{2}}v^2 + \gamma_v^{-1}\ (\vec{u}\overset{\blacktriangle\blacktriangle\blacktriangle}{\cdot}\vec{v})^2 + \gamma_u\ u^{\overset{\blacktriangle}{2}}v^2 - \gamma_u\ (\vec{u}\overset{\blacktriangle\blacktriangle\blacktriangle}{\cdot}\vec{v})^2 + \gamma_u\frac{u^2v^2}{c^2}\ \vec{u}\overset{\blacktriangle\blacktriangle}{\cdot}\vec{v} - \gamma_u\ \frac{(\vec{u}\overset{\blacksquare}{\cdot}\vec{v})^3}{c^2} \\
&\quad +\gamma_v^{-2}(1-\gamma_v)u^2\ \vec{u}\overset{\blacktriangle\blacktriangle}{\cdot}\vec{v} - \gamma_v^{-2}(1-\gamma_v)\ \frac{(\vec{u}\overset{\blacksquare}{\cdot}\vec{v})^3}{v^2} - \gamma_u\gamma_v^{-1}(1-\gamma_v)u^2\ \vec{u}\overset{\blacktriangle\blacktriangle}{\cdot}\vec{v} + \gamma_u\gamma_v^{-1}(1-\gamma_v)\ \frac{(\vec{u}\overset{\blacksquare}{\cdot}\vec{v})^3}{v^2} + \\
&\quad -\gamma_u\gamma_v^{-1}(1-\gamma_v)\frac{u^2}{c^2}\ (\vec{u}\overset{\blacktriangle\blacktriangle\blacktriangle}{\cdot}\vec{v})^2 + \gamma_u\gamma_v^{-1}(1-\gamma_v)\frac{1}{c^2v^2}\ (\vec{u}\overset{\blacksquare\blacksquare}{\cdot}\vec{v})^4\Bigg\}\Bigg/ \\
&\qquad\qquad \Bigg/\Bigg\{(u^2v^2-(\vec{u}\cdot\vec{v})^2)\Big(u^2+v^2+2\vec{u}\cdot\vec{v}-\frac{u^2v^2}{c^2}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big)\Bigg\}\,.
\end{aligned}$$

We have

$$\begin{aligned}
\blacktriangle\big|_{\star\star} &= -\gamma_v^{-1}u^2v^2+\gamma_u u^2v^2 \\
&= c^4(1-\gamma_u^{-2})(1-\gamma_v^{-2})(-\gamma_v^{-1}+\gamma_u)
\end{aligned}$$

$$\begin{aligned}
\blacktriangle\blacktriangle\big|_{\star\star} &= \vec{u}\cdot\vec{v}\Big[-u^2-v^2+\frac{u^2v^2}{c^2}+\gamma_u\frac{u^2v^2}{c^2}+\gamma_v^{-2}(1-\gamma_v)u^2-\gamma_u\gamma_v^{-1}(1-\gamma_v)u^2\Big] \\
&= c^2\vec{u}\cdot\vec{v}\Big[\gamma_u^{-2}\gamma_v^{-1}+\gamma_u^{-1}\gamma_v^{-2}+\gamma_u^{-1}\gamma_v^{-1}-2\gamma_u^{-1}+\gamma_v^{-2}-\gamma_v^{-1}-1-\gamma_u\gamma_v^{-2}-\gamma_u\gamma_v^{-1}+2\gamma_u\Big]
\end{aligned}$$

$$\begin{aligned}
\blacktriangle\blacktriangle\blacktriangle\big|_{\star\star} &= (\vec{u}\cdot\vec{v})^2\Big[-2+\gamma_v^{-1}-\gamma_u-\gamma_u\gamma_v^{-1}(1-\gamma_v)\frac{u^2}{c^2}\Big] \\
&= (\vec{u}\cdot\vec{v})^2\big[\gamma_u^{-1}\gamma_v^{-1}-\gamma_u^{-1}+\gamma_v^{-1}-2-\gamma_u\gamma_v^{-1}\big]
\end{aligned}$$

$$\begin{aligned}
\blacksquare\big|_{\star\star} &= (\vec{u}\cdot\vec{v})^3\Big[-\frac{1}{c^2}-\gamma_u\frac{1}{c^2}-\gamma_v^{-2}(1-\gamma_v)\frac{1}{v^2}+\gamma_u\gamma_v^{-1}(1-\gamma_v)\frac{1}{v^2}\Big] \\
&= \frac{(\vec{u}\cdot\vec{v})^3}{c^2}\frac{1}{(\gamma_v^2-1)}\big[\gamma_v-\gamma_v^2+\gamma_u+\gamma_u\gamma_v-2\gamma_u\gamma_v^2\big]
\end{aligned}$$

$$\begin{aligned}
\blacksquare\blacksquare\big|_{\star\star} &= \gamma_u\gamma_v^{-1}(1-\gamma_v)\frac{1}{c^2v^2}(\vec{u}\cdot\vec{v})^4 \\
&= -\frac{\gamma_u\gamma_v}{c^4(\gamma_v+1)}(\vec{u}\cdot\vec{v})^4
\end{aligned}$$

$\therefore$

$$\begin{aligned}
\star\star &= c^4(1-\gamma_u^{-2})(1-\gamma_v^{-2})(-\gamma_v^{-1}+\gamma_u) + \\
&+c^2\vec{u}\cdot\vec{v}\left[\gamma_u^{-2}\gamma_v^{-1}+\gamma_u^{-1}\gamma_v^{-2}+\gamma_u^{-1}\gamma_v^{-1}+\gamma_v^{-2}-2\gamma_u^{-1}-\gamma_v^{-1}-\gamma_u\gamma_v^{-2}-1-\gamma_u\gamma_v^{-1}+2\gamma_u\right] \\
&+(\vec{u}\cdot\vec{v})^2\left[\gamma_u^{-1}\gamma_v^{-1}-\gamma_u^{-1}+\gamma_v^{-1}-2-\gamma_u\gamma_v^{-1}\right] + \\
&+\frac{(\vec{u}\cdot\vec{v})^3}{c^2}\frac{1}{(\gamma_v^2-1)}\left[\gamma_v+\gamma_u-\gamma_v^2+\gamma_u\gamma_v-2\gamma_u\gamma_v^2\right] + \\
&-(\vec{u}\cdot\vec{v})^4\frac{\gamma_u\gamma_v}{c^4(\gamma_v+1)}\,. \qquad (148)
\end{aligned}$$

7.1.6

From (147) and (148) we have

$$\mathcal{R}\check{\vec{x}}\left|_{\vec{u}}\right]_{\vec{x}\cdot\vec{v}} = \star = \star\star = \hat{\vec{x}}\left|_{\vec{u}}\right]_{\vec{x}\cdot\vec{v}}$$

$\therefore$

$$\boxed{\mathcal{R}\check{\vec{x}}\left|_{\vec{u}}\right]_{\vec{x}\cdot\vec{v}} = \hat{\vec{x}}\left|_{\vec{u}}\right]_{\vec{x}\cdot\vec{v}}} \qquad (149)$$

7.1.7 $R\check{\vec{x}}\big|_{\vec{u}}\big]_t$

Let us denote

$$\spadesuit \equiv R\check{\vec{x}}\big|_{\vec{u}}\big]_t\,.$$

From (141) we have

$$\spadesuit = \left[(1-\gamma_u)\gamma_v\frac{\vec{u}\cdot\vec{v}}{u^2}-\gamma_u\gamma_v\right]\cos\varphi - \left[-\gamma_v v^2-\gamma_u\gamma_v\vec{u}\cdot\vec{v}+(1-\gamma_u)\gamma_v\frac{(\vec{u}\cdot\vec{v})^2}{u^2}\right]\chi\lambda\frac{\sin\varphi}{|\vec{v}\times\vec{u}|}$$

and using (44) and (45) we obtain

$$
\begin{aligned}
\spadesuit \;=\; & \Big\{(1-\gamma_u)\;\vec{u}\overset{\blacktriangle\blacktriangle}{\cdot}\vec{v} + \gamma_u^{-1}(1-\gamma_u)\gamma_v\frac{v^2}{u^2}\;\vec{u}\overset{\blacktriangle\blacktriangle}{\cdot}\vec{v} + (1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})(1-\gamma_u)\gamma_v\;\frac{(\vec{u}\overset{\blacktriangle\blacktriangle\blacktriangle}{\cdot}\vec{v})^2}{u^2} + \\
& -(1-\gamma_u)(\gamma_u^{-1}-1)\gamma_v\;\frac{(\vec{u}\overset{\blacksquare}{\cdot}\vec{v})^3}{u^4} - (1-\gamma_u)\gamma_v(\gamma_v^{-1}-1)\;\frac{(\vec{u}\overset{\blacksquare}{\cdot}\vec{v})^3}{u^2v^2} + \\
& +(1-\gamma_u)(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\;\frac{(\vec{u}\overset{\blacksquare\blacksquare}{\cdot}\vec{v})^4}{u^4v^2} - \gamma_u\overset{\blacktriangle}{u}^2 - \gamma_v\overset{\blacktriangle}{v}^2 + \\
& -(1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})\gamma_u\gamma_v\;\vec{u}\overset{\blacktriangle\blacktriangle}{\cdot}\vec{v} + \gamma_u(\gamma_u^{-1}-1)\gamma_v\;\frac{(\vec{u}\overset{\blacktriangle\blacktriangle\blacktriangle}{\cdot}\vec{v})^2}{u^2} + \gamma_u\gamma_v(\gamma_v^{-1}-1)\;\frac{(\vec{u}\overset{\blacktriangle\blacktriangle\blacktriangle}{\cdot}\vec{v})^2}{v^2} \\
& -\gamma_u(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\;\frac{(\vec{u}\overset{\blacksquare}{\cdot}\vec{v})^3}{u^2v^2} + \gamma_v\overset{\blacktriangle}{v}^2 - \gamma_u^{-1}\overset{\blacktriangle}{v}^2 - (\gamma_u^{-1}-1)\gamma_v\frac{v^2}{u^2}\;\vec{u}\overset{\blacktriangle\blacktriangle}{\cdot}\vec{v} + \\
& -\gamma_v(\gamma_v^{-1}-1)\;\vec{u}\overset{\blacktriangle\blacktriangle}{\cdot}\vec{v} + (\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\;\frac{(\vec{u}\overset{\blacktriangle\blacktriangle\blacktriangle}{\cdot}\vec{v})^2}{u^2} + \gamma_u\gamma_v\;\vec{u}\overset{\blacktriangle\blacktriangle}{\cdot}\vec{v} - \vec{u}\overset{\blacktriangle\blacktriangle}{\cdot}\vec{v} + \\
& -\gamma_u(\gamma_u^{-1}-1)\gamma_v\;\frac{(\vec{u}\overset{\blacktriangle\blacktriangle\blacktriangle}{\cdot}\vec{v})^2}{u^2} - \gamma_u\gamma_v(\gamma_v^{-1}-1)\;\frac{(\vec{u}\overset{\blacktriangle\blacktriangle\blacktriangle}{\cdot}\vec{v})^2}{v^2} + \gamma_u(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\;\frac{(\vec{u}\overset{\blacksquare}{\cdot}\vec{v})^3}{u^2v^2} + \\
& -(1-\gamma_u)\gamma_v\;\frac{(\vec{u}\overset{\blacktriangle\blacktriangle\blacktriangle}{\cdot}\vec{v})^2}{u^2} + \gamma_u^{-1}(1-\gamma_u)\;\frac{(\vec{u}\overset{\blacktriangle\blacktriangle\blacktriangle}{\cdot}\vec{v})^2}{u^2} + (1-\gamma_u)(\gamma_u^{-1}-1)\gamma_v\;\frac{(\vec{u}\overset{\blacksquare}{\cdot}\vec{v})^3}{u^4} + \\
& +(1-\gamma_u)\gamma_v(\gamma_v^{-1}-1)\;\frac{(\vec{u}\overset{\blacksquare}{\cdot}\vec{v})^3}{u^2v^2} - (1-\gamma_u)(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\;\frac{(\vec{u}\overset{\blacksquare\blacksquare}{\cdot}\vec{v})^4}{u^4v^2}\;\Big\}\Big/ \\
& \Big/\Big\{(u^2+v^2+2\vec{u}\cdot\vec{v} - \frac{u^2v^2}{c^2} + \frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big\}\,.
\end{aligned}
$$

We have

$$
\begin{aligned}
\blacktriangle\big|_{\spadesuit} &= -\gamma_u u^2 - \gamma_v v^2 + \gamma_v v^2 - \gamma_u^{-1}v^2 \\
&= c^2\big[\gamma_u^{-1}\gamma_v^{-2} - \gamma_u\big]
\end{aligned}
$$

$$
\begin{aligned}
\blacktriangle\blacktriangle\big|_{\spadesuit} &= (1-\gamma_u)\vec{u}\cdot\vec{v} + \gamma_u^{-1}(1-\gamma_u)\gamma_v\frac{v^2}{u^2}\vec{u}\cdot\vec{v} - (1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})\gamma_u\gamma_v\vec{u}\cdot\vec{v} + \\
& -(\gamma_u^{-1}-1)\gamma_v\frac{v^2}{u^2}\vec{u}\cdot\vec{v} - (\gamma_v^{-1}-1)\gamma_v\vec{u}\cdot\vec{v} + \gamma_u\gamma_v\vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{v} \\
&= -2\gamma_u\vec{u}\cdot\vec{v}
\end{aligned}
$$

$$
\begin{aligned}
\blacktriangle\blacktriangle\blacktriangle\big|_{\spadesuit} &= (1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})(1-\gamma_u)\gamma_v\frac{(\vec{u}\cdot\vec{v})^2}{u^2}+\gamma_u(\gamma_u^{-1}-1)\gamma_v\frac{(\vec{u}\cdot\vec{v})^2}{u^2}+ \\
&\quad +\gamma_u\gamma_v(\gamma_v^{-1}-1)\frac{(\vec{u}\cdot\vec{v})^2}{v^2}+(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\frac{(\vec{u}\cdot\vec{v})^2}{u^2}+ \\
&\quad -\gamma_u(\gamma_u^{-1}-1)\gamma_v\frac{(\vec{u}\cdot\vec{v})^2}{u^2}-\gamma_u\gamma_v(\gamma_v^{-1}-1)\frac{(\vec{u}\cdot\vec{v})^2}{v^2} \\
&\quad -(1-\gamma_u)\gamma_v\frac{(\vec{u}\cdot\vec{v})^2}{u^2}+\gamma_u^{-1}(1-\gamma_u)\frac{(\vec{u}\cdot\vec{v})^2}{u^2} \\
&= -\gamma_u\frac{(\vec{u}\cdot\vec{v})^2}{c^2}
\end{aligned}
$$

$$
\blacksquare\big|_{\spadesuit} = 0
$$

$$
\blacksquare\blacksquare\big|_{\spadesuit} = 0
$$

∴

$$
\spadesuit = -\gamma_u\left\{c^2\left[1-\gamma_u^{-2}\gamma_v^{-2}\right]+2\vec{u}\cdot\vec{v}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}\right\}\Big/\left\{u^2+v^2+2\vec{u}\cdot\vec{v}-\frac{u^2v^2}{c^2}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}\right\}.
$$

But

$$
u^2+v^2-\frac{u^2v^2}{c^2} = c^2[1-\gamma_u^{-2}\gamma_v^{-2}]
$$

and the denominator of $\spadesuit$ rewrites as

$$
u^2+v^2+2\vec{u}\cdot\vec{v}-\frac{u^2v^2}{c^2}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}=c^2[1-\gamma_u^{-2}\gamma_v^{-2}]+2\vec{u}\cdot\vec{v}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}
$$

then

$$
\spadesuit = -\gamma_u\,. \tag{150}
$$

7.1.8 $\hat{\vec{x}}\,|_{\vec{u}}\big]_t$

Let us denote

$$
\spadesuit\spadesuit \equiv \hat{\vec{x}}\,|_{\vec{u}}\big]_t\,.
$$

From (142) we have

$$
\spadesuit\spadesuit = -\gamma_u \tag{151}
$$

7.1.9

From (150) and (151) we have

$$
R\check{\vec{x}}|_{\vec{u}}\big]_t = \spadesuit = \spadesuit\spadesuit = \hat{\vec{x}}\,|_{\vec{u}}\big]_t
$$

∴

$$
\boxed{R\check{\vec{x}}|_{\vec{u}}\big]_t = \hat{\vec{x}}\,|_{\vec{u}}\big]_t} \tag{152}
$$

7.1.10

From (146), (149), (152) and from (143) we have

$$\left.\begin{array}{l} \clubsuit = \clubsuit\clubsuit \rightleftharpoons R\check{\vec{x}}\,\big|_{\vec{u}}\Big]_{\vec{x}\cdot\vec{u}} = \hat{\vec{x}}\big|_{\vec{u}}\Big]_{\vec{x}\cdot\vec{u}} \\ \bigstar = \bigstar\bigstar \rightleftharpoons R\check{\vec{x}}\,\big|_{\vec{u}}\Big]_{\vec{x}\cdot\vec{v}} = \hat{\vec{x}}\big|_{\vec{u}}\Big]_{\vec{x}\cdot\vec{v}} \\ \spadesuit = \spadesuit\spadesuit \rightleftharpoons R\check{\vec{x}}\,\big|_{\vec{u}}\Big]_{t} = \hat{\vec{x}}\big|_{\vec{u}}\Big]_{t} \end{array}\right\} \Rightarrow \boxed{R\check{\vec{x}}\big|_{\vec{u}} = \hat{\vec{x}}\big|_{\vec{u}}} \tag{153}$$

7.2 Verifying $R\check{\vec{x}}\big|_{\vec{v}} = \hat{\vec{x}}\big|_{\vec{v}}$

From (138) we have

$$\begin{aligned} R\check{\vec{x}}\big|_{\vec{v}} \;&=\; \vec{x}\cdot\vec{u}\Big\{-\frac{\vec{u}\cdot\vec{v}}{u^2v^2-(\vec{u}\cdot\vec{v})^2}\cos\varphi+\gamma_u\chi\lambda\frac{\sin\varphi}{|\vec{v}\times\vec{u}|}\Big\}+ \\ &+\vec{x}\cdot\vec{v}\Big\{\Big[\frac{u^2}{u^2v^2-(\vec{u}\cdot\vec{v})^2}-(1-\gamma_v)\frac{1}{v^2}\Big]\cos\varphi+\Big[-\gamma_u(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}+\gamma_u\gamma_v\frac{u^2}{c^2}\Big]\chi\lambda\frac{\sin\varphi}{|\vec{v}\times\vec{u}|}\Big\} \\ &+t\Big\{-\gamma_v\cos\varphi-\gamma_u\gamma_v(\vec{u}\cdot\vec{v}+u^2)\chi\lambda\frac{\sin\varphi}{|\vec{v}\times\vec{u}|}\Big\} \end{aligned} \tag{154}$$

and from (139) we have

$$\begin{aligned} \hat{\vec{x}}\big|_{\vec{v}} \;&=\; \vec{x}\cdot\vec{u}\Big\{-\frac{\vec{u}\cdot\vec{v}}{u^2v^2-(\vec{u}\cdot\vec{v})^2}-\frac{\Big[1-\gamma_u\gamma_v\Big(1+\frac{\vec{u}\cdot\vec{v}}{c^2}\Big)\Big]\Big[\gamma_v-(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\Big]}{\gamma_v^2\Big(u^2+v^2+2\vec{u}\cdot\vec{v}-\frac{u^2v^2}{c^2}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big)}\Big\}+ \\ &+\vec{x}\cdot\vec{v}\Big\{\frac{u^2}{u^2v^2-(\vec{u}\cdot\vec{v})^2}-\frac{\Big[1-\gamma_u\gamma_v\Big(1+\frac{\vec{u}\cdot\vec{v}}{c^2}\Big)\Big]\Big[\gamma_v-(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\Big]^2}{\gamma_v^2\Big(u^2+v^2+2\vec{u}\cdot\vec{v}-\frac{u^2v^2}{c^2}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big)}\Big\} \\ &+t\Big\{-\gamma_u\gamma_v+\gamma_u(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\Big\}\,. \end{aligned} \tag{155}$$

Here, due to the arbitrariness of the event P we have a similar criteria as the one shown in (143), i.e.

$$R\check{\vec{x}}\big|_{\vec{v}} = \hat{\vec{x}}\big|_{\vec{v}} \Leftrightarrow \left\{\begin{array}{l} R\check{\vec{x}}\big|_{\vec{v}}\big]_{\vec{x}\cdot\vec{u}} = \hat{\vec{x}}\big|_{\vec{v}}\big]_{\vec{x}\cdot\vec{u}} \\ R\check{\vec{x}}\big|_{\vec{v}}\big]_{\vec{x}\cdot\vec{v}} = \hat{\vec{x}}\big|_{\vec{v}}\big]_{\vec{x}\cdot\vec{v}} \\ R\check{\vec{x}}\big|_{\vec{v}}\big]_{t} = \hat{\vec{x}}\big|_{\vec{v}}\big]_{t}\,. \end{array}\right. \tag{156}$$

7.2.1 $R\check{\vec{x}}\big|_{\vec{v}}\Big]_{\vec{x}\cdot\vec{u}}$

Let us denote

$$\clubsuit \equiv R\check{\vec{x}}\big|_{\vec{v}}\Big]_{\vec{x}\cdot\vec{u}}\,.$$

From (154) we have

$$\clubsuit = -\frac{\vec{u}\cdot\vec{v}}{u^2v^2-(\vec{u}\cdot\vec{v})^2}\cos\varphi+\gamma_u\chi\lambda\frac{\sin\varphi}{|\vec{v}\times\vec{u}|}$$

and using (44) and (45) we obtain

$$
\begin{aligned}
\clubsuit \;=\; & \Big\{ -\gamma_v^{-1}u^2 \overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} - \gamma_u^{-1}v^2 \overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} - (1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1}) \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} + (\gamma_u^{-1}-1) \overset{\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^3}{u^2}} + \\
& + (\gamma_v^{-1}-1) \overset{\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^3}{v^2}} - (\gamma_u^{-1}-1)(\gamma_v^{-1}-1) \overset{\blacksquare\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^4}{u^2v^2}} + \\
& + \gamma_u \overset{\blacktriangle}{u^2} v^2 - \gamma_v^{-1} \overset{\blacktriangle}{u^2} v^2 - (1-\gamma_u)v^2 \overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} - \gamma_u(\gamma_v^{-1}-1)u^2 \overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} + \\
& + (1-\gamma_u)(\gamma_v^{-1}-1) \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} - \gamma_u \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} + \gamma_v^{-1} \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} + (1-\gamma_u) \overset{\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^3}{u^2}} + \\
& + \gamma_u(\gamma_v^{-1}-1) \overset{\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^3}{v^2}} - (1-\gamma_u)(\gamma_v^{-1}-1) \overset{\blacksquare\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^4}{u^2v^2}} \Big\} \Big/ \\
& \qquad\qquad \Big/ \Big\{ \big(u^2v^2 - (\vec{u}\cdot\vec{v})^2\big)\Big(u^2+v^2+2\vec{u}\cdot\vec{v} - \frac{u^2v^2}{c^2} + \frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big) \Big\} .
\end{aligned}
$$

We have

$$
\begin{aligned}
\blacktriangle\big|_{\clubsuit} &= \gamma_u u^2 v^2 - \gamma_v^{-1} u^2 v^2 \\
&= u^2 v^2 (-\gamma_v^{-1} + \gamma_u)
\end{aligned}
$$

$$
\begin{aligned}
\blacktriangle\blacktriangle\big|_{\clubsuit} &= \vec{u}\cdot\vec{v}\Big[-\gamma_v^{-1}u^2 - \gamma_u^{-1}v^2 - (1-\gamma_u)v^2 - \gamma_u(\gamma_v^{-1}-1)u^2\Big] \\
&= c^2\vec{u}\cdot\vec{v}\Big[\gamma_u^{-2}\gamma_v^{-1} + \gamma_u^{-1}\gamma_v^{-2} + \gamma_u^{-1}\gamma_v^{-1} - 2\gamma_u^{-1} + \gamma_v^{-2} - \gamma_v^{-1} - 1 - \gamma_u\gamma_v^{-2} - \gamma_u\gamma_v^{-1} + 2\gamma_u\Big]
\end{aligned}
$$

$$
\begin{aligned}
\blacktriangle\blacktriangle\blacktriangle\big|_{\clubsuit} &= (\vec{u}\cdot\vec{v})^2\Big[-(1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1}) + (1-\gamma_u)(\gamma_v^{-1}-1) - \gamma_u + \gamma_v^{-1}\Big] \\
&= (\vec{u}\cdot\vec{v})^2\Big[\gamma_u^{-1}\gamma_v^{-1} - \gamma_u^{-1} + \gamma_v^{-1} - 2 - \gamma_u\gamma_v^{-1}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\blacksquare\big|_{\clubsuit} &= (\vec{u}\cdot\vec{v})^3\Big[(\gamma_u^{-1}-1)\frac{1}{u^2} + (\gamma_v^{-1}-1)\frac{1}{v^2} + (1-\gamma_u)\frac{1}{u^2} + \gamma_u(\gamma_v^{-1}-1)\frac{1}{v^2}\Big] \\
&= \frac{(\vec{u}\cdot\vec{v})^3}{c^2}\Big[\frac{-\gamma_v - \gamma_u - 2\gamma_u\gamma_v}{(\gamma_v+1)}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\blacksquare\blacksquare\big|_{\clubsuit} &= \frac{(\vec{u}\cdot\vec{v})^4}{u^2v^2}\Big[-(\gamma_u^{-1}-1)(\gamma_v^{-1}-1) - (1-\gamma_u)(\gamma_v^{-1}-1)\Big] \\
&= -\frac{(\vec{u}\cdot\vec{v})^4}{c^4}\frac{1}{(\gamma_v+1)}\gamma_u\gamma_v
\end{aligned}
$$

$\therefore$

$$
\begin{aligned}
\clubsuit \;=\;& u^2v^2(-\gamma_v^{-1}+\gamma_u)+\\
&+c^2\vec{u}\cdot\vec{v}\Big[\gamma_u^{-2}\gamma_v^{-1}+\gamma_u^{-1}\gamma_v^{-2}+\gamma_u^{-1}\gamma_v^{-1}-2\gamma_u^{-1}+\gamma_v^{-2}-\gamma_v^{-1}-1-\gamma_u\gamma_v^{-2}-\gamma_u\gamma_v^{-1}+2\gamma_u\Big]+\\
&+(\vec{u}\cdot\vec{v})^2\Big[\gamma_u^{-1}\gamma_v^{-1}-\gamma_u^{-1}+\gamma_v^{-1}-2-\gamma_u\gamma_v^{-1}\Big]+\\
&+\frac{(\vec{u}\cdot\vec{v})^3}{c^2}\Big[\frac{-\gamma_v-\gamma_u-2\gamma_u\gamma_v}{(\gamma_v+1)}\Big]+\\
&-\frac{(\vec{u}\cdot\vec{v})^4}{c^4}\frac{\gamma_u\gamma_v}{(\gamma_v+1)}\,. \qquad (157)
\end{aligned}
$$

7.2.2 $\hat{\vec{x}}\big|_{\vec{v}}\Big]_{\vec{x}\cdot\vec{u}}$

Let us denote

$$\clubsuit\clubsuit \equiv \vec{x}\,\big|_{\vec{v}}\Big]_{\vec{x}\cdot\vec{u}}\,.$$

From (155) we obtain

$$
\begin{aligned}
\clubsuit\clubsuit \;=\;& -\frac{\vec{u}\cdot\vec{v}}{u^2v^2-(\vec{u}\cdot\vec{v})^2}-\frac{\Big[1-\gamma_u\gamma_v\Big(1+\frac{\vec{u}\cdot\vec{v}}{c^2}\Big)\Big]\Big[\gamma_v-(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\Big]}{\gamma_v^2\Big(u^2+v^2+2\vec{u}\cdot\vec{v}-\frac{u^2v^2}{c^2}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big)}\\
=\;& \Bigg\{-u^2\,\overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}}-v^2\,\overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}}-2\,\overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2}+\frac{u^2v^2}{c^2}\,\overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}}-\overset{\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^3}{c^2}}+\\
&-\gamma_v^{-1}\,\overset{\blacktriangle}{u^2v^2}+\gamma_v^{-2}(1-\gamma_v)u^2\,\overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}}+\gamma_u\,\overset{\blacktriangle}{u^2v^2}+\gamma_u\frac{u^2v^2}{c^2}\,\overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}}+\\
&-\gamma_u\gamma_v^{-1}(1-\gamma_v)u^2\,\overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}}-\gamma_u\gamma_v^{-1}(1-\gamma_v)\frac{u^2}{c^2}\,\overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2}+\\
&+\gamma_v^{-1}\,\overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2}-\gamma_v^{-2}(1-\gamma_v)\,\overset{\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^3}{v^2}}-\gamma_u\,\overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2}-\gamma_u\,\overset{\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^3}{c^2}}+\\
&+\gamma_u\gamma_v^{-1}(1-\gamma_v)\,\overset{\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^3}{v^2}}+\gamma_u\gamma_v^{-1}(1-\gamma_v)\,\overset{\blacksquare\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^4}{c^2v^2}}\Bigg\}\Big/\\
&\Big/\Bigg\{\big(u^2v^2-(\vec{u}\cdot\vec{v})^2\big)\Big(u^2+v^2+2\vec{u}\cdot\vec{v}-\frac{u^2v^2}{c^2}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big)\Bigg\}\,.
\end{aligned}
$$

We have

$$\blacktriangle\big|_{\clubsuit\clubsuit} \;=\; u^2v^2\big[-\gamma_v^{-1}+\gamma_u\big]$$

$$
\begin{aligned}
\blacktriangle\blacktriangle\big|_{\clubsuit\clubsuit} \;=\;& \vec{u}\cdot\vec{v}\Big[-u^2-v^2+\frac{u^2v^2}{c^2}+\gamma_v^{-2}(1-\gamma_v)u^2+\gamma_u\frac{u^2v^2}{c^2}-\gamma_u\gamma_v^{-1}(1-\gamma_v)u^2\Big]\\
=\;& c^2\vec{u}\cdot\vec{v}\Big[\gamma_u^{-2}\gamma_v^{-1}+\gamma_u^{-1}\gamma_v^{-2}+\gamma_u^{-1}\gamma_v^{-1}-2\gamma_u^{-1}+\gamma_v^{-2}-\gamma_v^{-1}-1-\gamma_u\gamma_v^{-2}-\gamma_u\gamma_v^{-1}+2\gamma_u\Big]
\end{aligned}
$$

$$\blacktriangle\blacktriangle\blacktriangle\big|_{\clubsuit\clubsuit} = (\vec{u}\cdot\vec{v})^2\big[-2-\gamma_u\gamma_v^{-1}(1-\gamma_v)\frac{u^2}{c^2}+\gamma_v^{-1}-\gamma_u\big]$$
$$= (\vec{u}\cdot\vec{v})^2\big[\gamma_u^{-1}\gamma_v^{-1}-\gamma_u^{-1}+\gamma_v^{-1}-2-\gamma_u\gamma_v^{-1}\big]$$

$$\blacksquare\big|_{\clubsuit\clubsuit} = (\vec{u}\cdot\vec{v})^3\Big[-\frac{1}{c^2}-\gamma_v^{-2}(1-\gamma_v)\frac{1}{v^2}-\gamma_u\frac{1}{c^2}+\gamma_u\gamma_v^{-1}(1-\gamma_v)\frac{1}{v^2}\Big]$$
$$= \frac{(\vec{u}\cdot\vec{v})^3}{c^2}\frac{1}{(\gamma_v+1)}\big[-\gamma_v-\gamma_u-2\gamma_u\gamma_v\big]$$

$$\blacksquare\blacksquare\big|_{\clubsuit\clubsuit} = (\vec{u}\cdot\vec{v})^4\gamma_u\gamma_v^{-1}(1-\gamma_v)\frac{1}{c^2v^2}$$
$$= -\frac{(\vec{u}\cdot\vec{v})^4}{c^4}\frac{\gamma_u\gamma_v}{(\gamma_v+1)}$$

$\therefore$

$$\begin{aligned}\clubsuit\clubsuit = {} & u^2v^2\big[-\gamma_v^{-1}+\gamma_u\big]+\\ & +c^2\vec{u}\cdot\vec{v}\big[\gamma_u^{-2}\gamma_v^{-1}+\gamma_u^{-1}\gamma_v^{-2}+\gamma_u^{-1}\gamma_v^{-1}-2\gamma_u^{-1}+\gamma_v^{-2}-\gamma_v^{-1}-1-\gamma_u\gamma_v^{-2}-\gamma_u\gamma_v^{-1}+2\gamma_u\big]+\\ & +(\vec{u}\cdot\vec{v})^2\big[\gamma_u^{-1}\gamma_v^{-1}-\gamma_u^{-1}+\gamma_v^{-1}-2-\gamma_u\gamma_v^{-1}\big]+\\ & +\frac{(\vec{u}\cdot\vec{v})^3}{c^2}\frac{\big[-\gamma_v-\gamma_u-2\gamma_u\gamma_v\big]}{(\gamma_v+1)}+\\ & -\frac{(\vec{u}\cdot\vec{v})^4}{c^4}\frac{\gamma_u\gamma_v}{(\gamma_v+1)}\,.\end{aligned} \tag{158}$$

7.2.3

From (157) and (158) we have

$$R\check{\vec{x}}\big|_{\vec{v}}\Big]_{\vec{x}\cdot\vec{u}} = \clubsuit = \clubsuit\clubsuit = \hat{\vec{x}}\,\big|_{\vec{v}}\Big]_{\vec{x}\cdot\vec{u}}$$

$\therefore$

$$\boxed{R\check{\vec{x}}\big|_{\vec{v}}\Big]_{\vec{x}\cdot\vec{u}} = \hat{\vec{x}}\,\big|_{\vec{v}}\Big]_{\vec{x}\cdot\vec{u}}} \tag{159}$$

7.2.4 $R\check{\vec{x}}\big|_{\vec{v}}\Big]_{\vec{x}\cdot\vec{v}}$

Let us denote

$$\bigstar \equiv R\check{\vec{x}}\big|_{\vec{v}}\Big]_{\vec{x}\cdot\vec{v}}\,.$$

From (154) we have

$$\bigstar = \Big[\frac{u^2}{u^2v^2-(\vec{u}\cdot\vec{v})^2}-(1-\gamma_v)\frac{1}{v^2}\Big]\cos\varphi+\Big[-\gamma_u(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}+\gamma_u\gamma_v\frac{u^2}{c^2}\Big]\chi\lambda\frac{\sin\varphi}{|\vec{v}\times\vec{u}|}$$

and using (44) and (45) we obtain

$$
\begin{aligned}
\bigstar \;=\; & \Bigg\{ \overset{\blacktriangle}{u^4} + \gamma_u^{-1}\gamma_v \, \overset{\blacktriangle}{u^2 v^2} + (1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})\gamma_v u^2 \, \overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} - (\gamma_u^{-1}-1)\gamma_v \, \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} + \\
& -\gamma_v(\gamma_v^{-1}-1)\frac{u^2}{v^2} \, \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} + (\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1) \, \frac{\overset{\blacksquare}{(\vec{u}\cdot\vec{v})^3}}{v^2} + \gamma_v^{-1}(1-\gamma_v)\frac{u^2}{v^2} \, \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} + \\
& +\gamma_u^{-1}(1-\gamma_v) \, \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} + (1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})(1-\gamma_v) \, \frac{\overset{\blacksquare}{(\vec{u}\cdot\vec{v})^3}}{v^2} + \\
& -(\gamma_u^{-1}-1)(1-\gamma_v) \, \frac{\overset{\blacksquare\blacksquare}{(\vec{u}\cdot\vec{v})^4}}{u^2 v^2} - (1-\gamma_v)(\gamma_v^{-1}-1) \, \frac{\overset{\blacksquare\blacksquare}{(\vec{u}\cdot\vec{v})^4}}{v^4} + (\gamma_u^{-1}-1)(1-\gamma_v)(\gamma_v^{-1}-1) \, \frac{\overset{\blacksquare\blacksquare\blacksquare}{(\vec{u}\cdot\vec{v})^5}}{u^2 v^4} + \\
& -\gamma_u(1-\gamma_v)u^2 \, \overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} + \gamma_v^{-1}(1-\gamma_v)u^2 \, \overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} + \gamma_u(\gamma_u^{-1}-1)(1-\gamma_v) \, \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} + \\
& +\gamma_u(1-\gamma_v)(\gamma_v^{-1}-1)\frac{u^2}{v^2} \, \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} - \gamma_u(\gamma_u^{-1}-1)(1-\gamma_v)(\gamma_v^{-1}-1) \, \frac{\overset{\blacksquare}{(\vec{u}\cdot\vec{v})^3}}{v^2} + \gamma_u\gamma_v \, u^4 \overset{\blacktriangle}{\frac{v^2}{c^2}} + \\
& - u^4 \overset{\blacktriangle}{\frac{v^2}{c^2}} - \gamma_u(\gamma_u^{-1}-1)\gamma_v u^2 \frac{v^2}{c^2} \, \overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} - \gamma_u\gamma_v(\gamma_v^{-1}-1)\frac{u^4}{c^2} \, \overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} + \gamma_u(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\frac{u^2}{c^2} \, \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} + \\
& +\gamma_u(1-\gamma_v) \, \frac{\overset{\blacksquare}{(\vec{u}\cdot\vec{v})^3}}{v^2} - \gamma_v^{-1}(1-\gamma_v) \, \frac{\overset{\blacksquare}{(\vec{u}\cdot\vec{v})^3}}{v^2} - \gamma_u(\gamma_u^{-1}-1)(1-\gamma_v) \, \frac{\overset{\blacksquare\blacksquare}{(\vec{u}\cdot\vec{v})^4}}{u^2 v^2} + \\
& -\gamma_u(1-\gamma_v)(\gamma_v^{-1}-1) \, \frac{\overset{\blacksquare\blacksquare}{(\vec{u}\cdot\vec{v})^4}}{v^4} + \gamma_u(\gamma_u^{-1}-1)(1-\gamma_v)(\gamma_v^{-1}-1) \, \frac{\overset{\blacksquare\blacksquare\blacksquare}{(\vec{u}\cdot\vec{v})^5}}{u^2 v^4} - \gamma_u\gamma_v\frac{u^2}{c^2} \, \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} + \\
& +\frac{u^2}{c^2} \, \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec{u}\cdot\vec{v})^2} + \gamma_u(\gamma_u^{-1}-1)\gamma_v\frac{1}{c^2} \, \overset{\blacksquare}{(\vec{u}\cdot\vec{v})^3} + \gamma_u\gamma_v(\gamma_v^{-1}-1)\frac{u^2}{c^2 v^2} \, \overset{\blacksquare}{(\vec{u}\cdot\vec{v})^3} + \\
& -\gamma_u(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\frac{1}{c^2 v^2} \, \overset{\blacksquare\blacksquare}{(\vec{u}\cdot\vec{v})^4} \Bigg\} \Big/ \\
& \Big/ \left\{ \Big(u^2 v^2 - (\vec{u}\cdot\vec{v})^2\Big)\Big(u^2 + v^2 + 2\vec{u}\cdot\vec{v} - \frac{u^2 v^2}{c^2} + \frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big) \right\} .
\end{aligned}
$$

We have

$$
\begin{aligned}
\blacktriangle\Big|_{\bigstar} &= u^4 + \gamma_u^{-1}\gamma_v u^2 v^2 + \gamma_u\gamma_v u^4 \frac{v^2}{c^2} - u^4\frac{v^2}{c^2} \\
&= \gamma_u\gamma_v u^2 v^2 + u^4 - \frac{u^4 v^2}{c^2}
\end{aligned}
$$

$$
\begin{aligned}
\blacktriangle\blacktriangle\Big|_{\bigstar} &= \vec{u}\cdot\vec{v}\Big[(1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})\gamma_v u^2 - \gamma_u(1-\gamma_v)u^2 + \gamma_v^{-1}(1-\gamma_v)u^2 + \\
&\qquad -\gamma_u(\gamma_u^{-1}-1)\gamma_v u^2\frac{v^2}{c^2} - \gamma_u\gamma_v(\gamma_v^{-1}-1)\frac{u^4}{c^2}\Big] \\
&= u^2\,\vec{u}\cdot\vec{v}\Big[2\gamma_v^{-1} - \gamma_u\gamma_v^{-1} - 2\gamma_u + 3\gamma_u\gamma_v\Big]
\end{aligned}
$$

$$\begin{aligned}
\blacktriangle\blacktriangle\blacktriangle\big|_{\star} &= (\vec{u}\cdot\vec{v})^2\Big[-(\gamma_u^{-1}-1)\gamma_v-\gamma_v(\gamma_v^{-1}-1)\frac{u^2}{v^2}+\gamma_v^{-1}(1-\gamma_v)\frac{u^2}{v^2}+\gamma_u^{-1}(1-\gamma_v)+\\
&\quad +\gamma_u(\gamma_u^{-1}-1)(1-\gamma_v)+\gamma_u(1-\gamma_v)(\gamma_v^{-1}-1)\frac{u^2}{v^2}+\gamma_u(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\frac{u^2}{c^2}+\\
&\quad -\gamma_u\gamma_v\frac{u^2}{c^2}+\frac{u^2}{c^2}\Big]\\
&= (\vec{u}\cdot\vec{v})^2\Big[\frac{u^2}{v^2}\Big(-2+\gamma_v+\gamma_v^{-1}+\gamma_u\gamma_v^{-1}-2\gamma_u+\gamma_u\gamma_v\Big)-2\gamma_u^{-1}\gamma_v+\gamma_u^{-1}+1-\gamma_u+\gamma_u\gamma_v+\\
&\quad +\frac{u^2}{c^2}\Big(2-\gamma_v-\gamma_u\Big)\Big]\ .
\end{aligned}$$

Using that

$$\left.\begin{aligned} u^2 &= c^2(1-\gamma_u^{-2})\\ v^2 &= c^2(1-\gamma_v^{-2})\end{aligned}\right\}\Rightarrow\frac{u^2}{v^2}=\frac{(1-\gamma_u^{-2})}{(1-\gamma_v^{-2})}\ , \tag{160}$$

we end up with

$$\blacktriangle\blacktriangle\blacktriangle\big|_{\star} = (\vec{u}\cdot\vec{v})^2\frac{1}{(1+\gamma_v)}\big[-2\gamma_u^{-2}+2\gamma_u^{-1}+\gamma_u^{-1}\gamma_v-3\gamma_u^{-1}\gamma_v^2+3+\gamma_v-2\gamma_u-2\gamma_u\gamma_v+2\gamma_u\gamma_v^2\big]$$

$$\begin{aligned}
\blacksquare\big|_{\star} &= (\vec{u}\cdot\vec{v})^3\Big[(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\frac{1}{v^2}+(1+\gamma_u^{-1}+\gamma_v^{-1}-\gamma_u^{-1}\gamma_v^{-1})(1-\gamma_v)\frac{1}{v^2}+\\
&\quad -\gamma_u(\gamma_u^{-1}-1)(1-\gamma_v)(\gamma_v^{-1}-1)\frac{1}{v^2}+\gamma_u(1-\gamma_v)\frac{1}{v^2}-\gamma_v^{-1}(1-\gamma_v)\frac{1}{v^2}+\\
&\quad +\gamma_u(\gamma_u^{-1}-1)\gamma_v\frac{1}{c^2}+\gamma_u\gamma_v(\gamma_v^{-1}-1)\frac{u^2}{c^2v^2}\Big]\\
&= (\vec{u}\cdot\vec{v})^3\frac{1}{c^2(\gamma_v^2-1)}\Big[-\gamma_u^{-1}\gamma_v+2\gamma_u^{-1}\gamma_v^2-\gamma_u^{-1}\gamma_v^3-2\gamma_v+2\gamma_v^2+2\gamma_u\gamma_v-2\gamma_u\gamma_v^3\Big]
\end{aligned}$$

$$\begin{aligned}
\blacksquare\blacksquare\big|_{\star} &= (\vec{u}\cdot\vec{v})^4\Big\{-(\gamma_u^{-1}-1)(1-\gamma_v)\frac{1}{u^2v^2}-(1-\gamma_v)(\gamma_v^{-1}-1)\frac{1}{v^4}-\gamma_u(\gamma_u^{-1}-1)(1-\gamma_v)\frac{1}{u^2v^2}+\\
&\quad -\gamma_u(1-\gamma_v)(\gamma_v^{-1}-1)\frac{1}{v^4}-\gamma_u(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\frac{1}{c^2v^2}\Big\}\\
&= (\vec{u}\cdot\vec{v})^4\frac{1}{c^2v^2}\frac{1}{(1+\gamma_v)}\Big\{-1+\gamma_v+2\gamma_u+\gamma_u\gamma_v-3\gamma_u\gamma_v^2\Big\}
\end{aligned}$$

$$\begin{aligned}
\blacksquare\blacksquare\blacksquare\big|_{\star} &= (\gamma_u^{-1}-1)(1-\gamma_v)(\gamma_v^{-1}-1)\frac{(\vec{u}\cdot\vec{v})^5}{u^2v^4}+\gamma_u(\gamma_u^{-1}-1)(1-\gamma_v)(\gamma_v^{-1}-1)\frac{(\vec{u}\cdot\vec{v})^5}{u^2v^4}\\
&= -\gamma_u\gamma_v(\gamma_v^{-1}-1)^2\frac{(\vec{u}\cdot\vec{v})^5}{c^2v^4}
\end{aligned}$$

$$\begin{aligned}\bigstar &= \gamma_u\gamma_v u^2v^2 + u^4 - \frac{u^4v^2}{c^2} + \\ &+\vec u\cdot\vec v\; u^2\Big[2\gamma_v^{-1} - \gamma_u\gamma_v^{-1} - 2\gamma_u + 3\gamma_u\gamma_v\Big] + \\ &+(\vec u\cdot\vec v)^2\frac{1}{(1+\gamma_v)}\Big[-2\gamma_u^{-2}+2\gamma_u^{-1}+\gamma_u^{-1}\gamma_v - 3\gamma_u^{-1}\gamma_v^2 + 3 + \gamma_v - 2\gamma_u - 2\gamma_u\gamma_v + 2\gamma_u\gamma_v^2\Big] + \\ &+(\vec u\cdot\vec v)^3\frac{1}{c^2}\frac{1}{(\gamma_v^2-1)}\Big[-\gamma_u^{-1}\gamma_v + 2\gamma_u^{-1}\gamma_v^2 - \gamma_u^{-1}\gamma_v^3 - 2\gamma_v + 2\gamma_v^2 + 2\gamma_u\gamma_v - 2\gamma_u\gamma_v^3\Big] + \\ &+(\vec u\cdot\vec v)^4\frac{1}{c^2v^2}\frac{1}{(1+\gamma_v)}\Big[-1+\gamma_v+2\gamma_u+\gamma_u\gamma_v - 3\gamma_u\gamma_v^2\Big] + \\ &-(\vec u\cdot\vec v)^5\frac{1}{c^2v^4}\gamma_u\gamma_v(1-\gamma_v^{-1})^2 \,. \end{aligned} \tag{161}$$

7.2.5 $\hat{\vec{x}}\big|_{\vec v}\Big]_{\vec x\cdot\vec v}$

Let us denote

$$\bigstar\bigstar \equiv \hat{\vec{x}}\big|_{\vec v}\Big]_{\vec x\cdot\vec v} \,.$$

From (155) we obtain

$$\begin{aligned}\bigstar\bigstar &= \frac{u^2}{u^2v^2-(\vec u\cdot\vec v)^2} - \frac{\Big[1-\gamma_u\gamma_v\Big(1+\frac{\vec u\cdot\vec v}{c^2}\Big)\Big]\Big[\gamma_v-(1-\gamma_v)\frac{\vec u\cdot\vec v}{v^2}\Big]^2}{\gamma_v^2\Big(u^2+v^2+2\vec u\cdot\vec v-\frac{u^2v^2}{c^2}+\frac{(\vec u\cdot\vec v)^2}{c^2}\Big)} \\ &= \Bigg\{\overset{\blacktriangle}{u}{}^4 + \overset{\blacktriangle}{u}{}^2v^2 + 2u^2\,\overset{\blacktriangle\blacktriangle}{\vec u\cdot\vec v} - \frac{\overset{\blacktriangle}{u}{}^4v^2}{c^2} + \frac{u^2}{c^2}\,\overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec u\cdot\vec v)^2} + \\ &\quad - \overset{\blacktriangle}{u}{}^2v^2 - 2(1-\gamma_v^{-1})u^2\,\overset{\blacktriangle\blacktriangle}{\vec u\cdot\vec v} - (1-\gamma_v^{-1})^2\frac{u^2}{v^2}\,\overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec u\cdot\vec v)^2} + \\ &\quad + \gamma_u\gamma_v\overset{\blacktriangle}{u}{}^2v^2 + 2\gamma_u\gamma_v(1-\gamma_v^{-1})u^2\,\overset{\blacktriangle\blacktriangle}{\vec u\cdot\vec v} + \gamma_u\gamma_v(1-\gamma_v^{-1})^2\frac{u^2}{v^2}\,\overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec u\cdot\vec v)^2} \\ &\quad + \gamma_u\gamma_v\frac{u^2v^2}{c^2}\,\overset{\blacktriangle\blacktriangle}{\vec u\cdot\vec v} + 2\gamma_u\gamma_v(1-\gamma_v^{-1})\frac{u^2}{c^2}\,\overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec u\cdot\vec v)^2} + \gamma_u\gamma_v(1-\gamma_v^{-1})^2\frac{u^2}{c^2v^2}\,\overset{\blacksquare}{(\vec u\cdot\vec v)^3} + \\ &\quad + \overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec u\cdot\vec v)^2} + 2(1-\gamma_v^{-1})\frac{1}{v^2}\,\overset{\blacksquare}{(\vec u\cdot\vec v)^3} + (1-\gamma_v^{-1})^2\frac{1}{v^4}\,\overset{\blacksquare\blacksquare}{(\vec u\cdot\vec v)^4} + \\ &\quad - \gamma_u\gamma_v\,\overset{\blacktriangle\blacktriangle\blacktriangle}{(\vec u\cdot\vec v)^2} - 2\gamma_u\gamma_v(1-\gamma_v^{-1})\frac{1}{v^2}\,\overset{\blacksquare}{(\vec u\cdot\vec v)^3} - \gamma_u\gamma_v(1-\gamma_v^{-1})^2\frac{1}{v^4}\,\overset{\blacksquare\blacksquare}{(\vec u\cdot\vec v)^4} + \\ &\quad - \gamma_u\gamma_v\frac{1}{c^2}\,\overset{\blacksquare}{(\vec u\cdot\vec v)^3} - 2\gamma_u\gamma_v(1-\gamma_v^{-1})\frac{1}{c^2v^2}\,\overset{\blacksquare\blacksquare}{(\vec u\cdot\vec v)^4} - \gamma_u\gamma_v(1-\gamma_v^{-1})^2\frac{1}{c^2v^4}\,\overset{\blacksquare\blacksquare\blacksquare}{(\vec u\cdot\vec v)^5}\Bigg\}\Bigg/ \\ &\quad \Bigg/\Bigg\{\Big(u^2v^2-(\vec u\cdot\vec v)^2\Big)\Big(u^2+v^2+2\vec u\cdot\vec v-\frac{u^2v^2}{c^2}+\frac{(\vec u\cdot\vec v)^2}{c^2}\Big)\Bigg\} \,.\end{aligned}$$

We have

$$\begin{aligned}\blacktriangle\big|_{\bigstar\bigstar} &= u^4 + u^2v^2 - \frac{u^4v^2}{c^2} - u^2v^2 + \gamma_u\gamma_v u^2v^2 \\ &= \gamma_u\gamma_v u^2v^2 + u^4 - \frac{u^4v^2}{c^2}\end{aligned}$$

$$\begin{aligned}\blacktriangle\blacktriangle\big|_{\bigstar\bigstar} &= \vec{u}\cdot\vec{v}\,\Big[2u^2 - 2(1-\gamma_v^{-1})u^2 + 2\gamma_u\gamma_v(1-\gamma_v^{-1})u^2 + \gamma_u\gamma_v\frac{u^2v^2}{c^2}\Big] \\ &= \vec{u}\cdot\vec{v}\;u^2\big[2\gamma_v^{-1} - \gamma_u\gamma_v^{-1} - 2\gamma_u + 3\gamma_u\gamma_v\big]\end{aligned}$$

$$\blacktriangle\blacktriangle\blacktriangle\big|_{\bigstar\bigstar} = (\vec{u}\cdot\vec{v})^2\Big[\frac{u^2}{c^2} - (1-\gamma_v^{-1})^2\frac{u^2}{v^2} + \gamma_u\gamma_v(1-\gamma_v^{-1})^2\frac{u^2}{v^2} + 2\gamma_u\gamma_v(1-\gamma_v^{-1})\frac{u^2}{c^2} + 1 - \gamma_u\gamma_v\Big]\,.$$

Using again (160) we end up with

$$\blacktriangle\blacktriangle\blacktriangle\Big|_{\bigstar\bigstar} = (\vec{u}\cdot\vec{v})^2\frac{1}{(1+\gamma_v)}\big[-2\gamma_u^{-2} + 2\gamma_u^{-1} + \gamma_u^{-1}\gamma_v - 3\gamma_u^{-1}\gamma_v^2 + 3 + \gamma_v - 2\gamma_u - 2\gamma_u\gamma_v + 2\gamma_u\gamma_v^2\big]$$

$$\begin{aligned}\blacksquare\big|_{\bigstar\bigstar} &= (\vec{u}\cdot\vec{v})^3\Big[\gamma_u\gamma_v(1-\gamma_v^{-1})^2\frac{u^2}{c^2}\frac{1}{v^2} + 2(1-\gamma_v^{-1})\frac{1}{v^2} - 2\gamma_u\gamma_v(1-\gamma_v^{-1})\frac{1}{v^2} - \gamma_u\gamma_v\frac{1}{c^2}\Big] \\ &= (\vec{u}\cdot\vec{v})^3\frac{1}{v^2}\big[-\gamma_u^{-1}\gamma_v^{-1} + 2\gamma_u^{-1} - \gamma_u^{-1}\gamma_v - 2\gamma_v^{-1} + 2 + 2\gamma_u\gamma_v^{-1} - 2\gamma_u\gamma_v\big]\end{aligned}$$

$$\begin{aligned}\blacksquare\blacksquare\big|_{\bigstar\bigstar} &= (\vec{u}\cdot\vec{v})^4\frac{1}{v^2}\Big[(1-\gamma_v^{-1})^2\frac{1}{v^2} - \gamma_u\gamma_v(1-\gamma_v^{-1})^2\frac{1}{v^2} - 2\gamma_u\gamma_v(1-\gamma_v^{-1})\frac{1}{c^2}\Big] \\ &= (\vec{u}\cdot\vec{v})^4\frac{1}{c^2v^2}\frac{1}{(1+\gamma_v)}\big[-1 + \gamma_v + 2\gamma_u + \gamma_u\gamma_v - 3\gamma_u\gamma_v^2\big]\end{aligned}$$

$$\blacksquare\blacksquare\blacksquare\big|_{\bigstar\bigstar} = -(\vec{u}\cdot\vec{v})^5\frac{1}{c^2v^4}\gamma_u\gamma_v(1-\gamma_v^{-1})^2$$

$\therefore$

$$\begin{aligned}\bigstar\bigstar &= \gamma_u\gamma_v u^2v^2 + u^4 - \frac{u^4v^2}{c^2} + \\ &+\vec{u}\cdot\vec{v}\;u^2\Big[2\gamma_v^{-1} - \gamma_u\gamma_v^{-1} - 2\gamma_u + 3\gamma_u\gamma_v\Big] + \\ &(\vec{u}\cdot\vec{v})^2\frac{1}{(1+\gamma_v)}\Big[-2\gamma_u^{-2} + 2\gamma_u^{-1} + \gamma_u^{-1}\gamma_v - 3\gamma_u^{-1}\gamma_v^2 + 3 + \gamma_v - 2\gamma_u - 2\gamma_u\gamma_v + 2\gamma_u\gamma_v^2\Big] + \\ &+(\vec{u}\cdot\vec{v})^3\frac{1}{c^2}\frac{1}{(\gamma_v^2-1)}\Big[-\gamma_u^{-1}\gamma_v + 2\gamma_u^{-1}\gamma_v^2 - \gamma_u^{-1}\gamma_v^3 - 2\gamma_v + 2\gamma_v^2 + 2\gamma_u\gamma_v - 2\gamma_u\gamma_v^3\Big] + \\ &+(\vec{u}\cdot\vec{v})^4\frac{1}{c^2v^2}\frac{1}{(1+\gamma_v)}\Big[-1 + \gamma_v + 2\gamma_u + \gamma_u\gamma_v - 3\gamma_u\gamma_v^2\Big] + \\ &-(\vec{u}\cdot\vec{v})^5\frac{1}{c^2v^4}\gamma_u\gamma_v(1-\gamma_v^{-1})^2\,.\end{aligned} \tag{162}$$

7.2.6

From (161) and (162) we have

$$R\check{\vec{x}}\big|_{\vec{v}}\Big]_{\vec{x}\cdot\vec{v}} = \bigstar = \bigstar\bigstar = \hat{\vec{x}}\big|_{\vec{v}}\Big]_{\vec{x}\cdot\vec{v}}$$

$\therefore$

$$\boxed{R\check{\vec{x}}\big|_{\vec{v}}\Big]_{\vec{x}\cdot\vec{v}} = \hat{\vec{x}}\big|_{\vec{v}}\Big]_{\vec{x}\cdot\vec{v}}} \tag{163}$$

7.2.7 $\boldsymbol{R\check{\vec{x}}\big|_{\vec{v}}\Big]_t}$

Let us denote

$$\spadesuit \equiv R\check{\vec{x}}\big|_{\vec{v}}\Big]_t .$$

From (154) we have

$$\spadesuit \;=\; -\gamma_v \cos\varphi - \gamma_u\gamma_v(\vec{u}\cdot\vec{v}+u^2)\chi\lambda\frac{\sin\varphi}{|\vec{v}\times\vec{u}|}$$

and using (44) and (45) we obtain

$$\begin{aligned}
\spadesuit \;=\; & \Big\{- \overset{\blacktriangle}{u^2} -\gamma_u^{-1}\gamma_v \overset{\blacktriangle}{v^2} -(\gamma_v+\gamma_u^{-1}\gamma_v+1-\gamma_u^{-1})\, \overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} +(\gamma_u^{-1}-1)\gamma_v \,\overset{\blacktriangle\blacktriangle\blacktriangle}{\frac{(\vec{u}\cdot\vec{v})^2}{u^2}} + \\
& +\gamma_v(\gamma_v^{-1}-1)\,\overset{\blacktriangle\blacktriangle\blacktriangle}{\frac{(\vec{u}\cdot\vec{v})^2}{v^2}} -(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\,\overset{\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^3}{u^2v^2}} -\gamma_u\gamma_v\,\overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} + \\
& -\gamma_u\gamma_v\,\overset{\blacktriangle}{u^2} + \overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} + \overset{\blacktriangle}{u^2} +\gamma_u(\gamma_u^{-1}-1)\gamma_v\,\overset{\blacktriangle\blacktriangle\blacktriangle}{\frac{(\vec{u}\cdot\vec{v})^2}{u^2}} +\gamma_u(\gamma_u^{-1}-1)\gamma_v\,\overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} + \\
& +\gamma_u\gamma_v(\gamma_v^{-1}-1)\,\overset{\blacktriangle\blacktriangle\blacktriangle}{\frac{(\vec{u}\cdot\vec{v})^2}{v^2}} +\gamma_u\gamma_v(\gamma_v^{-1}-1)\frac{u^2}{v^2}\,\overset{\blacktriangle\blacktriangle}{\vec{u}\cdot\vec{v}} -\gamma_u(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\,\overset{\blacksquare}{\frac{(\vec{u}\cdot\vec{v})^3}{u^2v^2}} + \\
& -\gamma_u(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\,\overset{\blacktriangle\blacktriangle\blacktriangle}{\frac{(\vec{u}\cdot\vec{v})^2}{v^2}}\Big\}\Big/\Big\{u^2+v^2+2\vec{u}\cdot\vec{v}-\frac{u^2v^2}{c^2}+\frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big\} .
\end{aligned}$$

We have

$$\begin{aligned}
\blacktriangle\big|_{\spadesuit} &= -u^2-\gamma_u^{-1}\gamma_v v^2-\gamma_u\gamma_v u^2+u^2 \\
&= (\gamma_u^{-1}\gamma_v^{-1}-\gamma_u\gamma_v)c^2
\end{aligned}$$

$$\begin{aligned}
\blacktriangle\blacktriangle\big|_{\spadesuit} &= \vec{u}\cdot\vec{v}\Big[-(\gamma_v+\gamma_u^{-1}\gamma_v+1-\gamma_u^{-1})-\gamma_u\gamma_v+1+\gamma_u(\gamma_u^{-1}-1)\gamma_v+\gamma_u\gamma_v(\gamma_v^{-1}-1)\frac{u^2}{v^2}\Big] \\
&= \vec{u}\cdot\vec{v}\,\frac{1}{(1+\gamma_v)}\big[\gamma_u^{-1}-2\gamma_u\gamma_v-3\gamma_u\gamma_v^2\big]
\end{aligned}$$

$$\begin{aligned}
\blacktriangle\blacktriangle\blacktriangle\big|_{\spadesuit} &= (\vec{u}\cdot\vec{v})^2\Big[(\gamma_u^{-1}-1)\gamma_v\frac{1}{u^2}+\gamma_v(\gamma_v^{-1}-1)\frac{1}{v^2}+\gamma_u(\gamma_u^{-1}-1)\gamma_v\frac{1}{u^2}+ \\
&\qquad +\gamma_u\gamma_v(\gamma_v^{-1}-1)\frac{1}{v^2}-\gamma_u(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\frac{1}{v^2}\Big] \\
&= (\vec{u}\cdot\vec{v})^2\frac{1}{c^2(1+\gamma_v)}\big[-\gamma_u\gamma_v-3\gamma_u\gamma_v^2\big]
\end{aligned}$$

$$\begin{aligned}
\blacksquare\big|_{\spadesuit} &= \frac{(\vec{u}\cdot\vec{v})^3}{u^2v^2}\Big[-(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)-\gamma_u(\gamma_u^{-1}-1)\gamma_v(\gamma_v^{-1}-1)\Big] \\
&= -(\vec{u}\cdot\vec{v})^3\frac{1}{c^4}\frac{\gamma_u\gamma_v^2}{(1+\gamma_v)}
\end{aligned}$$

$\therefore$

$$\spadesuit = \Big\{(\gamma_u^{-1}\gamma_v^{-1} - \gamma_u\gamma_v)c^2 + \frac{(\gamma_u^{-1} - 2\gamma_u\gamma_v - 3\gamma_u\gamma_v^2)}{(1+\gamma_v)}\vec{u}\cdot\vec{v} + \frac{(-\gamma_u\gamma_v - 3\gamma_u\gamma_v^2)}{(1+\gamma_v)}\frac{1}{c^2}(\vec{u}\cdot\vec{v})^2 + -\frac{\gamma_u\gamma_v^2}{(1+\gamma_v)}\frac{1}{c^4}(\vec{u}\cdot\vec{v})^3\Big\}\Big/\Big\{u^2 + v^2 + 2\vec{u}\cdot\vec{v} - \frac{u^2v^2}{c^2} + \frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big\}\,. \tag{164}$$

7.2.8 $\hat{\vec{x}}|_{\vec{v}}\big]_t$

Let us denote

$$\spadesuit\spadesuit \equiv \hat{\vec{x}}|_{\vec{v}}\big]_t\,.$$

From (155) we have

$$\begin{aligned}\spadesuit\spadesuit &= -\gamma_u\gamma_v + \gamma_u(1-\gamma_v)\frac{\vec{u}\cdot\vec{v}}{v^2}\\ &= -\gamma_u\gamma_v - \frac{\gamma_u\gamma_v^2}{(1+\gamma_v)}\frac{\vec{u}\cdot\vec{v}}{c^2}\,.\end{aligned} \tag{165}$$

7.2.9

In order to show that $\spadesuit = \spadesuit\spadesuit$ we first notice that from (164)

$$u^2 + v^2 + 2\vec{u}\cdot\vec{v} - \frac{u^2v^2}{c^2} + \frac{(\vec{u}\cdot\vec{v})^2}{c^2} = c^2\big[1 - \gamma_u^{-2}\gamma_v^{-2}\big] + 2\vec{u}\cdot\vec{v} + \frac{(\vec{u}\cdot\vec{v})^2}{c^2}$$

then we rewrite

$$\spadesuit = \Big\{(\gamma_u^{-1}\gamma_v^{-1} - \gamma_u\gamma_v)c^2 + \frac{(\gamma_u^{-1} - 2\gamma_u\gamma_v - 3\gamma_u\gamma_v^2)}{(1+\gamma_v)}\vec{u}\cdot\vec{v} + \frac{(-\gamma_u\gamma_v - 3\gamma_u\gamma_v^2)}{(1+\gamma_v)}\frac{(\vec{u}\cdot\vec{v})^2}{c^2} + -\frac{\gamma_u\gamma_v^2}{(1+\gamma_v)}\frac{(\vec{u}\cdot\vec{v})^3}{c^4}\Big\}\Big/\Big(c^2\big[1 - \gamma_u^{-2}\gamma_v^{-2}\big] + 2\vec{u}\cdot\vec{v} + \frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big)\,.$$

It is straightforward to show that

$$\Big\{(\gamma_u^{-1}\gamma_v^{-1} - \gamma_u\gamma_v)c^2 + \frac{(\gamma_u^{-1} - 2\gamma_u\gamma_v - 3\gamma_u\gamma_v^2)}{(1+\gamma_v)}\vec{u}\cdot\vec{v} + \frac{(-\gamma_u\gamma_v - 3\gamma_u\gamma_v^2)}{(1+\gamma_v)}\frac{(\vec{u}\cdot\vec{v})^2}{c^2} + -\frac{\gamma_u\gamma_v^2}{(1+\gamma_v)}\frac{(\vec{u}\cdot\vec{v})^3}{c^4}\Big\} = \Big(c^2(1 - \gamma_u^{-2}\gamma_v^{-2}) + 2\vec{u}\cdot\vec{v} + \frac{(\vec{u}\cdot\vec{v})^2}{c^2}\Big)\Big(-\gamma_u\gamma_v - \frac{\gamma_u\gamma_v^2}{(1+\gamma_v)}\frac{\vec{u}\cdot\vec{v}}{c^2}\Big)\,,$$

then we obtain

$$\spadesuit = -\gamma_u\gamma_v - \frac{\gamma_u\gamma_v^2}{(1+\gamma_v)}\frac{\vec{u}\cdot\vec{v}}{c^2}\,,$$

which shows that

$$R\check{\vec{x}}|_{\vec{v}}\big]_t = \spadesuit = \spadesuit\spadesuit = \hat{\vec{x}}|_{\vec{v}}\big]_t$$

$\therefore$

$$\boxed{R\check{\vec{x}}|_{\vec{v}}\big]_t = \hat{\vec{x}}|_{\vec{v}}\big]_t} \tag{166}$$

7.2.10

From (159), (163), (166) and (156) we have

$$\left.\begin{array}{l} \clubsuit = \clubsuit\clubsuit \rightleftharpoons \left. R\check{\vec{x}}\big|_{\vec{v}}\right]_{\vec{x}\cdot\vec{u}} = \left.\hat{\vec{x}}\big|_{\vec{v}}\right]_{\vec{x}\cdot\vec{u}} \\ \bigstar = \bigstar\bigstar \rightleftharpoons \left. R\check{\vec{x}}\big|_{\vec{v}}\right]_{\vec{x}\cdot\vec{v}} = \left.\hat{\vec{x}}\big|_{\vec{v}}\right]_{\vec{x}\cdot\vec{v}} \\ \spadesuit = \spadesuit\spadesuit \rightleftharpoons \left. R\check{\vec{x}}\big|_{\vec{v}}\right]_{t} = \left.\hat{\vec{x}}\big|_{\vec{v}}\right]_{t} \end{array}\right\} \Rightarrow \boxed{R\check{\vec{x}}\big|_{\vec{v}} = \hat{\vec{x}}\big|_{\vec{v}}} \tag{167}$$

7.3 Verifying that $\mathcal{R}\check{\vec{x}}\Big|_{\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}} = \hat{\vec{x}}\Big|_{\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}}$

From (138), (139) we have

$$\mathcal{R}\check{\vec{x}}\Big|_{\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}} = \vec{x}\cdot\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|} = \hat{\vec{x}}\Big|_{\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}}$$

$\therefore$

$$\boxed{\mathcal{R}\check{\vec{x}}\Big|_{\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}} = \hat{\vec{x}}\Big|_{\frac{\vec{v}\times\vec{u}}{|\vec{v}\times\vec{u}|}}} \tag{168}$$

7.4 Closing the proof

From (153), (167), (168) and (140) we have

$$\mathcal{R}\check{\vec{x}} = \hat{\vec{x}}$$

which ends the proof of (37). ■

8 Appendix: A proof of the relativistic law for the addition of velocity

In section 5 we obtained the law for the relativistic addition of velocities (22) as a consequence of the Galilei law for the addition of velocities and the absolute time. Now we wish to reobtain (22) restricting ourselves solely to the elements of the SR, i.e. the Lorentz transformation and the repeated use of the chain rule.

Let us consider frames S, S' and analyse the movement of the origin O'' of frame S''. We have

$$\vec{\tilde{v}}_{SS''} = \frac{d\vec{x}_{O''}}{dt_{O''}}\ . \tag{169}$$

Now, considering the Lorentz transformation relating the frames S, S' we have for the movement of the origin O''

$$\vec{x}_{O''} = \vec{x}'_{O''} - (1-\gamma_{\tilde{v}_{SS'}})\frac{\vec{x}'_{O''}\cdot\vec{\tilde{v}}_{SS'}}{\tilde{v}^2_{SS'}}\vec{\tilde{v}}_{SS'} + \gamma_{\tilde{v}_{SS'}}t'_{O''}\vec{\tilde{v}}_{SS'} \tag{170}$$

$$t_{O''} = \gamma_{\tilde{v}_{SS'}}\left(t'_{O''} + \frac{\vec{x}'_{O''}\cdot\vec{\tilde{v}}_{SS'}}{c^2}\right) \tag{171}$$

and the inverse transformation

$$\vec{x}'_{O''} = \vec{x}_{O''} - (1-\gamma_{\tilde{v}_{SS'}})\frac{\vec{x}_{O''}\cdot\vec{\tilde{v}}_{SS'}}{\tilde{v}^2_{SS'}}\vec{\tilde{v}}_{SS'} - \gamma_{\tilde{v}_{SS}} t_{O''}\vec{\tilde{v}}_{SS'} \tag{172}$$

$$t'_{O''} = \gamma_{\tilde{v}_{SS'}}\Big(t_{O''} - \frac{\vec{x}_{O''}\cdot\vec{\tilde{v}}_{SS'}}{c^2}\Big) . \tag{173}$$

From (170) we have

$$\vec{x}_{O''} = \vec{x}_{O''}(\vec{x}'_{O''}(t'_{O''}), t'_{O''}, \vec{\tilde{v}}_{SS'})$$

and from (173) we have

$$t'_{O''} = t'_{O''}(t_{O''}, \vec{x}_{O''}(t_{O''}), \vec{\tilde{v}}_{SS'})$$

then we have

$$\vec{x}_{O''} = \vec{x}_{O''}\Big(\vec{x}'_{O''}\big(t'_{O''}(t_{O''}, \vec{x}_{O''}(t_{O''}), \vec{\tilde{v}}_{SS'})\big), t'_{O''}\big(t_{O''}, \vec{x}_{O''}(t_{O''}), \vec{\tilde{v}}_{SS'}\big), \vec{\tilde{v}}_{SS'}\Big) ,$$

therefore

$$\begin{aligned}
\vec{\tilde{v}}_{SS''} &= \frac{d\vec{x}_{O''}(t_{O''})}{dt_{O''}} = \frac{d}{dt_{O''}}\vec{x}_{O''}\Big(\vec{x}'_{O''}\big(t'_{O''}(t_{O''}, \vec{x}_{O''}(t_{O''}), \vec{\tilde{v}}_{SS'})\big), t'_{O''}\big(t_{O''}, \vec{x}_{O''}(t_{O''}), \vec{\tilde{v}}_{SS'}\big), \vec{\tilde{v}}_{SS'}\Big) \\
&= \frac{\partial \vec{x}_{O''}}{\partial x'^i_{O''}}\frac{dx'^i_{O''}(t'_{O''})}{dt'_{O''}}\bigg\{\frac{\partial t'_{O''}\big(t_{O''}, \vec{x}_{O''}(t_{O''}), \vec{\tilde{v}}_{SS'}\big)}{\partial t_{O''}}\frac{dt_{O''}}{dt_{O''}} + \frac{\partial t'_{O''}\big(t_{O''}, \vec{x}_{O''}(t_{O''}), \vec{\tilde{v}}_{SS'}\big)}{\partial x^k_{O''}}\frac{dx^k_{O''}(t_{O''})}{dt_{O''}}\bigg\} + \\
&\quad + \frac{\partial \vec{x}_{O''}}{\partial t'_{O''}}\bigg\{\frac{\partial t'_{O''}\big(t_{O''}, \vec{x}_{O''}(t_{O''}), \vec{\tilde{v}}_{SS'})\big)}{\partial t_{O''}}\frac{dt_{O''}}{dt_{O''}} + \frac{\partial t'_{O''}\big(t_{O''}, \vec{x}_{O''}(t_{O''}), \vec{\tilde{v}}_{SS'}\big)}{\partial x^l_{O''}}\frac{dx^l_{O''}(t_{O''})}{dt_{O''}}\bigg\} .
\end{aligned} \tag{174}$$

Using the Lorentz transformation given in (170),(171), (172), (173) we have

$$\frac{\partial t'_{O''}\big(t_{O''}, \vec{x}_{O''}(t_{O''}), \vec{\tilde{v}}_{SS'}\big)}{\partial t_{O''}} = \frac{\partial}{\partial t_{O''}}\bigg(\gamma_{\tilde{v}_{SS'}}\Big(t_{O''} - \frac{\vec{x}_{O''}\cdot\vec{\tilde{v}}_{SS'}}{c^2}\Big)\bigg) = \gamma_{\tilde{v}_{SS'}}$$

$$\frac{\partial t'_{O''}\big(t_{O''}, \vec{x}_{O''}(t_{O''}), \vec{\tilde{v}}_{SS'}\big)}{\partial x^k_{O''}} = \frac{\partial}{\partial x^k_{O''}}\bigg(\gamma_{\tilde{v}_{SS'}}\Big(t_{O''} - \frac{\vec{x}_{O''}\cdot\vec{\tilde{v}}_{SS'}}{c^2}\Big)\bigg) = -\gamma_{\tilde{v}_{SS'}}\frac{1}{c^2}\tilde{v}^k_{SS'}$$

$$\frac{dx^k_{O''}(t_{O''})}{dt_{O''}} = \tilde{v}^k_{SS''}$$

$$\begin{aligned}
\frac{\partial \vec{x}_{O''}}{\partial x'^i_{O''}} &= \hat{e}_m\frac{\partial}{\partial x'^i_{O''}}\bigg(x'^m_{O''} - (1-\gamma_{\tilde{v}_{SS'}})\frac{\vec{x}'_{O''}\cdot\vec{\tilde{v}}_{SS'}}{\tilde{v}^2_{SS'}}\tilde{v}^m_{SS'} + \gamma_{\tilde{v}_{SS'}}t'_{O''}\tilde{v}^m_{SS'}\bigg) \\
&= \hat{e}_m\Big(\delta^m_i - (1-\gamma_{\tilde{v}_{SS'}})\frac{\tilde{v}^i_{SS'}\tilde{v}^m_{SS'}}{\tilde{v}^2_{SS'}}\Big)
\end{aligned}$$

$$\frac{dx'^i_{O''}(t'_{O''})}{dt'_{O''}} = \tilde{v}^i_{S'S''}$$

$$\frac{\partial \vec{x}_{O''}}{\partial t'_{O''}} = \frac{\partial}{\partial t'_{O''}}\bigg(\vec{x}'_{O''} - (1-\gamma_{\tilde{v}_{SS'}})\frac{\vec{x}'_{O''}\cdot\vec{\tilde{v}}_{SS'}}{\tilde{v}^2_{SS'}}\vec{\tilde{v}}_{SS'} + \gamma_{\tilde{v}_{SS'}}t'_{O''}\vec{\tilde{v}}_{SS'}\bigg) = \gamma_{\tilde{v}_{SS'}}\vec{\tilde{v}}_{SS'}$$

then we get

$$\begin{aligned}
&\frac{\partial \vec{x}_{O''}}{\partial x'^i_{O''}}\frac{dx'^i_{O''}(t'_{O''})}{dt'_{O''}}\bigg\{\frac{\partial t'_{O''}\big(t_{O''}, \vec{x}_{O''}(t_{O''}), \vec{\tilde{v}}_{SS'}\big)}{\partial t_{O''}}\frac{dt_{O''}}{dt_{O''}} + \frac{\partial t'_{O''}\big(t_{O''}, \vec{x}_{O''}(t_{O''}), \vec{\tilde{v}}_{SS'}\big)}{\partial x^k_{O''}}\frac{dx^k_{O''}(t_{O''})}{dt_{O''}}\bigg\} = \\
&\quad = \gamma_{\tilde{v}_{SS'}}\bigg(\vec{\tilde{v}}_{S'S''} - (1-\gamma_{\tilde{v}_{SS'}})\frac{\vec{\tilde{v}}_{SS'}\cdot\vec{\tilde{v}}_{S'S''}}{\tilde{v}^2_{SS'}}\vec{\tilde{v}}_{SS'}\bigg)\bigg(1 - \frac{\vec{\tilde{v}}_{SS'}\cdot\vec{\tilde{v}}_{SS''}}{c^2}\bigg)
\end{aligned}$$

and

$$\frac{\partial \vec{x}_{O''}}{\partial t'_{O''}}\left\{\frac{\partial t'_{O''}\big(t_{O''},\vec{x}_{O''}(t_{O''}),\vec{\tilde{v}}_{SS'}\big)\big)}{\partial t_{O''}}\frac{dt_{O''}}{dt_{O''}}+\frac{\partial t'_{O''}\big(t_{O''},\vec{x}_{O''}(t_{O''}),\vec{\tilde{v}}_{SS'}\big)}{\partial x^l_{O''}}\frac{dx^l_{O''}(t_{O''})}{dt_{O''}}\right\}=$$

$$=\gamma^2_{\tilde{v}_{SS'}}\Big(1-\frac{\vec{\tilde{v}}_{SS'}\cdot\vec{\tilde{v}}_{SS''}}{c^2}\Big)\vec{\tilde{v}}_{SS'}$$

which replacing into (174) gives

$$\vec{\tilde{v}}_{SS''} = \gamma_{\tilde{v}_{SS'}}\left(\vec{\tilde{v}}_{S'S''}-(1-\gamma_{\tilde{v}_{SS'}})\frac{\vec{\tilde{v}}_{SS'}\cdot\vec{\tilde{v}}_{S'S''}}{\tilde{v}^2_{SS'}}\vec{\tilde{v}}_{SS'}+\gamma_{\tilde{v}_{SS'}}\vec{\tilde{v}}_{SS'}\right)\left(1-\frac{\vec{\tilde{v}}_{SS'}\cdot\vec{\tilde{v}}_{SS''}}{c^2}\right). \tag{175}$$

Taking the scalar product with $\vec{\tilde{v}}_{SS'}$ we obtain

$$\vec{\tilde{v}}_{SS'}\cdot\vec{\tilde{v}}_{SS''} = \frac{\tilde{v}^2_{SS'}\gamma^2_{\tilde{v}_{SS'}}\left(1+\frac{\vec{\tilde{v}}_{S'S''}\cdot\vec{\tilde{v}}_{SS'}}{\tilde{v}^2_{SS'}}\right)}{1+\frac{\tilde{v}^2_{SS'}}{c^2}\gamma^2_{\tilde{v}_{SS'}}\left(1+\frac{\vec{\tilde{v}}_{S'S''}\cdot\vec{\tilde{v}}_{SS'}}{\tilde{v}^2_{SS'}}\right)}$$

and then

$$1-\frac{\vec{\tilde{v}}_{SS'}\cdot\vec{\tilde{v}}_{SS''}}{c^2} = \frac{1}{\gamma^2_{\tilde{v}_{SS'}}\left(1+\frac{\vec{\tilde{v}}_{SS'}\cdot\vec{\tilde{v}}_{S'S''}}{c^2}\right)}$$

which replacing it back into (175) gives

$$\vec{\tilde{v}}_{SS''}=\frac{\vec{\tilde{v}}_{S'S''}+\gamma_{\tilde{v}_{SS'}}\vec{\tilde{v}}_{SS'}-(1-\gamma_{\tilde{v}_{SS'}})\frac{\vec{\tilde{v}}_{SS'}\cdot\vec{\tilde{v}}_{S'S''}}{\tilde{v}^2_{SS'}}\vec{\tilde{v}}_{SS'}}{\gamma_{\tilde{v}_{SS'}}\left(1+\frac{\vec{\tilde{v}}_{SS'}\cdot\vec{\tilde{v}}_{S'S''}}{c^2}\right)},$$

that is the form for the addition of velocity given in (22). A similar procedure can be used to prove the expression given in (23). ■

9 Conclusion

In the first part of our work the development of the Thomas precession was based on the properties

i. $S \parallel S', S'' \parallel S' \Longrightarrow \vec{\tilde{v}}_{S''S} = -\mathcal{R}^{-1}\vec{\tilde{v}}_{SS''}$

ii. $\mathcal{R}^{-1}\circ L(\vec{\tilde{v}}_{SS''})(t,\vec{x}) = L(\vec{\tilde{v}}_{S'S''})\circ L(\vec{\tilde{v}}_{SS'})(t,\vec{x})$.

The first property indicates there is a rotation between the frames S and S'', which implies parallelism is not a transitive relation; the second property states how this rotation affects the composition of two Lorentz transformations.

Concerning property **i**, we have seen in sections 3.4.1 and 3.4.2 that the form of the rotation $\mathcal{R}_{\widehat{n}}(\varphi)$ given in (44), (45), (46) depends continuously on the relative velocities $\vec{u}$ and $\vec{v}$ and as a consequence the axis of S, S', S'' are mutually parallel whenever $\vec{u} \parallel \vec{v}$ that we have shown corresponds to a configuration resulting in $\mathcal{R}_{\widehat{n}}(\varphi) = 1$. The analysis of how other configurations between $\vec{u}$ and

$\vec{v}$ affect the form of the Thomas rotation may be pursued from the exact form we have obtained for $\mathcal{R}_{\widehat{n}}(\varphi)$. This exact form also allows a further investigation. In fact, in [5] the formula for the relativistic addition of velocities is viewed as a binary operation $\oplus$ in the space of allowable velocities defined by (22) and satisfying the properties

$$\begin{aligned}
&\vec{v}\oplus\vec{u} = \mathcal{R}^{-1}[\vec{v};\vec{u}]\,\vec{u}\oplus\vec{v} \quad \text{[Weak commutative law]}\\
&\vec{w}\oplus(\vec{v}\oplus\vec{u}) = (\vec{w}\oplus\vec{v})\oplus\mathcal{R}^{-1}[\vec{w};\vec{v}]\,\vec{u} \quad \text{[Right weak associative law]}\\
&(\vec{w}\oplus\vec{v})\oplus\vec{u} = \vec{w}\oplus\big(\vec{v}\oplus\mathcal{R}^{-1}[\vec{v};\vec{w}]\,\vec{u}\big) \quad \text{[Left weak associative law]}\\
&0\oplus\vec{u} = \vec{u}\oplus 0 = \vec{u} \quad \text{[Existence of identity]}\\
&(-\vec{u})\oplus\vec{u} = \vec{u}\oplus(-\vec{u}) = 0 \quad \text{[Existence of inverse]}
\end{aligned} \tag{176}$$

where we have used the same terminology of [5] to name the properties. While the "weak commutative law" corresponds to the form we obtained in (39), and that the "existence of identity" and the "existence of inverse" can be immediately verified from (22), we still need to check the validity of both "weak associative laws" from the form we obtained for the Thomas rotation in (44), (45), (46).

Concerning property **ii**, it represents a common algebraic behavior that shows itself in the Lie algebra of the Lorentz group with generators $\{K_i, J_i\}$, e.g.

$$[K_i, K_j] = -i\epsilon_{ijk}J_k$$

that points to the non-closure of the commutator among the Lorentz transformations generators $\{K_i\}$.

In the second part of our work we assumed the local time depends on the state of motion of the frames and showed that the original Thomas precession is characterized by the properties

i′. $\vec{\tilde{v}}_{S''S}(v_{S''S'}) = -\Omega(v_{S''S'}, v_{SS'})\,\vec{\tilde{v}}_{SS''}(v_{SS'})$

ii″. $L(\vec{\tilde{v}}_{SS''}) = K_{\{S'',S\},\{S'',S'\}}L(\vec{\tilde{v}}_{S'S''})K_{\{S',S''\},\{S',S\}}L(\vec{\tilde{v}}_{SS'})K_{\{S,S'\},\{S,S''\}}$

that replaces the previous properties **i** and **ii**.

In what concerns property **i′**, the factor relating the velocities $\vec{\tilde{v}}_{S''S}$ and $\vec{\tilde{v}}_{SS''}$ is now given by a scalar term Ω (97) rather than a rotation, and in this sense the most suitable name describing this situation would be "Thomas dilation". Writting the addition of velocities as $\vec{v}\oplus\vec{u} = \Omega(\vec{v};\vec{u})\vec{u}\oplus\vec{v}$ we still need to check if some of the properties grouped in (176) are satisfied.

Concerning property **ii′**, the presence of the transformation $K_{\{S',S''\},\{S',S\}}$ introduces one more element besides rotations and boosts. We envisage two ways to treat these transformations. One could be to enlarge in a suitable way the Lorentz group in order to acommodate the K-maps together with the rotations and the Lorentz transformation and to find what algebraic structure arise. A second possibility is to consider for every frame S the whole set of allowable local times as one single object $[t_S] := \{t_{SS'} | \forall S'$ such that $\exists L(\vec{\tilde{v}}_{SS'}), S \overset{L(\vec{\tilde{v}}_{SS'})}{\longrightarrow} S'\}$ which involves a redefinition of not only the Lorentz transformation but also the form we register events that instead of $(t, \vec{x})$ becomes now $([t], \vec{x})$. The implications this may bring to the structure of spacetime may deserve a further investigation.

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To the One who created time and space I turn myself in humble obedience and contemplation. This work is done in honour of $\overline{\Pi\mathrm{HP}}$, $\overline{\Upsilon\mathrm{C}}$, $\overline{\Pi\mathrm{NA}}$.

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