Refutation of foundations of intuitionistic mathematics and logic (FIM)

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Abstract: We evaluate six equations in this historical apology as translated and find the following not tautologous: Church’s thesis; Markov’s principle (MP); virtual order and substitute for comparability, singly and collectively; fan theorem as equivalent to König’s (infinitary) lemma (as from elsewhere, singly and collectively); equality between functors; and Brouwer’s principle. These results foundations of intuitionistic mathematics (FIM) and form a non tautologous fragment of the universal logic $VŁ4$.

We assume the method and apparatus of Meth8/$VŁ4$ with Tautology as the designated proof value, $F$ as contradiction, $N$ as truthity (non-contingency), and $C$ as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

Let:

- $\sim$ Not, $\neg$; + Or, $\lor$, $\cup$; $-$ Not Or; $\&$ And, $\land$, $\cap$; $\otimes$; $\\backslash$ Not And;

- $>$ Imply, greater than, $\rightarrow$, $\Rightarrow$, $\succ$, $\rightarrow$; $<\not\Rightarrow$ Not Imply, less than, $\in$, $\subset$, $\not\in$, $\not\subset$, $\leq$;

- $\equiv$ Equivalent, $\equiv$, $\leftrightarrow$, $\Leftrightarrow$, $\leftarrow$, $\Rightarrow$; $\not\equiv$ Not Equivalent, $\neq$, $\not\equiv$;

- $\%$ possibility, for one or some, $\exists$, $\exists$, $\boxempty$; $\#$ necessity, for every or all, $\forall$, $\square$, $\L$;

- ($z=z$) $T$ as tautology, $T$, ordinal 3; ($z@z$) $F$ as contradiction, $\emptyset$, $Null$, $\bot$;

- $\%z>\#z$ $N$ as non-contingency, $\Delta$, ordinal 1; $\%z<\#z$ $C$ as contingency, $\forall$, ordinal 2;

- $\sim(y < x)$ $(x \leq y)$, $(x \subseteq y)$, $(x \subseteq y)$; $(A=B)$ $(A\sim B)$.

Note for clarity, we usually distribute quantifiers onto each designated variable.


Abstract. The first seeds of mathematical intuitionism germinated in Europe over a century ago in the constructive tendencies of Borel, Baire, Lebesgue, Poincaré, Kronecker and others. The flowering was the work of one man, Luitzen Egbertus Jan Brouwer, who taught mathematics at the University of Amsterdam from 1909 until 1951. By proving powerful theorems on topological invariants and fixed points of continuous mappings, Brouwer quickly built a mathematical reputation strong enough to support his revolutionary ideas about the nature of mathematical activity. These ideas influenced Hilbert and Gödel . and established intuitionistic logic and mathematics as subjects worthy of independent study. Our aim is to describe the development of Brouwer’s intuitionism, from his rejection of the classical law of excluded middle to his controversial theory of the continuum, with fundamental consequences for logic and mathematics. ...

3. Intuitionistic arithmetic $HA$ Heyting . first axiomatized intuitionistic arithmetic, which is called “Heyting arithmetic” in his honor .. Kleene’s version .. of HA has constants and axioms for zero, successor, addition and multiplication, and the unrestricted axiom schema of mathematical induction.

3.3. Church’s thesis. It is possible to express Church’s Thesis in the language of arithmetic. One version (which includes countable choice) is the schema $CT_0$:

$$\forall x \exists y A(x,y) \rightarrow \exists e \forall x \exists w [T(e,x,w) & A(x,U(w))]$$  \hspace{1cm} (3.3.1.1)

where $T(e, x, w)$ (numeralwise) expresses “$w$ is the gödel number of a computation of $\{e\}(x)$,” and $A(x, U(w))$ abbreviates $\forall z U(w, z) \rightarrow A(x, z)$ where $U(w, z)$ (numeralwise) expresses “$z$ is the value, if any, computed by the computation with gödel number $w$; otherwise $z = 0$”. The gödel numbering is primitive recursive, and $T(e,x,w)$ and $U(w,z)$ are quantifier-free. ...
LET q, s, t, u, w, x, y: A, e, T, U, w, x, y.

#(#x&%y)>((t&(%s&(#x&%w)))&(#(u&%w))) ;
TTTT TTTT TTTT TTTT (48)
CCCC CCCC CCCC CCCC (8)
CCCC CCCC CCCC CCCC {3} x2
CCCC CCCC CCCC CCCC

(3.3.1.2)

3.4. Axiomatization and modifications. ... From the intuitionistic point of view Church’s Thesis is restrictive, and probably unacceptable as a general principle. Markov’s Principle MP, the schema

∀x(A(x) V ¬A(x)) → [¬∀x¬A(x) → ∃xA(x)], is also problematic. (3.4.1.1)

((q&#x)+¬(q&#x))>((¬#x&(~w&x))>(%x&(q&x))) ;
TTTT TTTT TTTT TTTT (16)
NNTT NNTT NNTT NNTT (8)
TTTT TTTT TTTT TTTT (8) (3.4.1.2)

4. The intuitionistic theory of the continuum
Brouwer’s main objection to classical mathematics (apart from the unrestricted use of the principle of excluded middle) was “its introduction and description of the continuum.” .. His entire work was motivated by the attempt to describe a construction of the continuum in harmony with his mathematical principles, and to develop a satisfactory mathematics on the basis of that construction. ..

4.2. The intuitionistic continuum.
4.2.4. The continuum cannot be ordered ... [H] defined a refinement of the natural order, the virtual order

r<s ⇔ r>s & r≠s, which also fails to be total. (4.2.4.1.1)

(r<s)=((~r>s)&(r@s)) ;
TTTT FFFF TTTT TTTT (4.2.4.1.2)

... the most useful substitute for comparability is the property

r<s → r<t V t<s. (4.2.4.2.1)

The lack of order is not a “technical” problem, but rather a consequence of the fact that most real numbers are incompletely determined objects.

(r<s)>((r<t)&(t<s)) ;
TTTT FFFF TTTT TTTT (4.2.4.2.2)

Remark 4.2.4: Eqs. 4.2.4.1.2 and 4.2.4.2.2 as rendered is not tautologous. The authors state the former fails to be total for virtual order and the latter is a property of comparability. However, both produce the same truth table values. Hence the claim that lack of order is not a problem also is also refuted, with real numbers being incompletely determined objects as a specious reason.

4.3. The basic theorems and implicit principles.
4.3.3. The Fan theorem. [We refute this, and König’s (infinitary) lemma, elsewhere, aside from the fact that induction is invoked.]
4.4. FIM: a formal system for intuitionistic analysis. ... In addition to the terms, which are the formal expressions of FIM [foundations of intuitionistic mathematics] for the natural numbers, in this system there are formal expressions, called functors, for (total) functions from ω to ω. The two notions are defined by simultaneous induction: ... The atomic or prime formulas are expressions of the form \( s = t \) where \( s \) and \( t \) are terms. Equality between functors is not prime, but is defined by

\[
u = v \equiv \forall x(u(x) = v(x)), \text{ where } x \text{ does not appear free in } u, v. \tag{4.4.1.1}\]

\[
\text{LET } u, v, x: p, q, r.
\]

\[
(p=q)=((p\&#r)=(q\&r)); \quad \text{TFT} \quad \text{TNTN} \quad \text{TFF} \quad \text{TNTN} \tag{4.4.1.2}
\]

**Remark 4.4.1.2:** Eq. 4.4.1.2 is *not* tautologous, refuting the seminal definition of functors and hence system FIM.

[Brouwer's] principle is false for classical mathematics. For example, in .. it is used to prove

\[
\vdash \neg \forall x (x = 0 \lor \neg x (x = 0)) \tag{4.4.4.1}
\]

(from which it follows that equality between choice sequences must be undecidable), and the negation of the universal closure of the least number principle.

\[
\neg((\#((\#r=(s@s))+(\neg\#r=(s@s)))=(s=s)) = (s=s) \\tag{4.4.4.2}
\]

**Remark 4.4.4.2:** Eq. 4.4.4.2 is *not* contradictory (all F's) as rendered, hence refuting the claim, and by extension Brouwer's principle and intuitionistic logic system.