Lower bound on a special type of cyclic sums

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"Entia non sunt multiplicanda praeter necessitatem" (Ockam, W.)

"Dios no juega a los dados con el Universo" (Einstein, Albert)

"Te doy gracias, Padre, porque has ocultado estas cosas a los sabios y entendidos y se las has revelado a la gente sencilla" (Mt 11,25)

Abstract

In this brief paper it is proved a theorem regarding the relative value of the cyclic sums of
\[ f(x) = a^{\alpha+1}_{1} + \sum_{k=2}^{n} a^{\alpha+1}_{k} \]
and the sum of its variables.

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1 Introduction

We define a cyclic sum \( \sum_{\text{cyc}} f(a_1, a_2, ..., a_n) \) as equal to
\[ f(a_1, a_2, ..., a_n) + f(a_2, a_3, ..., a_n, a_1) + f(a_3, a_4, ..., a_1, a_2) + ... + f(a_n, a_1, ..., a_{n-1}) \]
Therefore, all the variables are cycled through.

We are interested in studying the sum

\[ \sum_{\text{cyc}} \frac{a^{\alpha+1}_{1}}{k_1 a^{\alpha}_{1} + k_2 a^{\alpha}_{2}} + \frac{a^{\alpha+1}_{2}}{k_1 a^{\alpha}_{2} + k_2 a^{\alpha}_{3}} + ... + \frac{a^{\alpha+1}_{n}}{k_1 a^{\alpha}_{n} + k_2 a^{\alpha}_{1}} \]

And its relative value compared to
\[ \sum_{k=1}^{n} a_k \]

In this regard, in this paper it is proposed and proved the following theorem:
Theorem.
\[ \sum_{cyc} a_1^{\alpha+1} \geq \sum_{k=1}^{n} a_k \]

2 Proof

2.1 Previous Lemmas
We will need firstly the following

Lemma 1.
\[ \sum_{k=1}^{n} a_k^{\alpha+1} \geq \sum_{cyc} a_1 a_2^\alpha \]

Proof.
Applying the Rearrangement inequality on \( a_1, a_2, ..., a_n \) and \( a_1^\alpha, a_2^\alpha, ..., a_n^\alpha \), we have that \( \sum_{cyc} a_1 a_2^\alpha \) is maximized when \( a_1, a_2, ..., a_n \) and \( a_1^\alpha, a_2^\alpha, ..., a_n^\alpha \) are similarly sorted.

Therefore, we can affirm that
\[ \sum_{k=1}^{n} a_k^{\alpha+1} \geq \sum_{cyc} a_1 a_2^\alpha \]

Other hand, we need the following

Lemma 2.
\[ \sum_{cyc} a_1^{\alpha+1} \left( \frac{k_2}{k_1} \right) a_1 a_2^\alpha \leq \sum_{k=1}^{n} a_k \]

Proof.
If we establish that
\[ \frac{a_1^{\alpha+1}}{k_1 a_1^\alpha + k_2 a_2^\alpha} + \frac{n}{m} = \frac{a_1}{k_1} \]
Operating, we get that
\[
\frac{a_1^{a_1 + 1} + n (k_1 a_1^a + k_2 a_2^a)}{m (k_1 a_1^a + k_2 a_2^a)} = \frac{a_1}{k_1}
\]

\[
k_1 (a_1^{a_1 + 1} m + n (k_1 a_1^a + k_2 a_2^a)) = a_1 m (k_1 a_1^a + k_2 a_2^a)
\]

\[
k_1 a_1^{a_1 + 1} m + k_1 n (k_1 a_1^a + k_2 a_2^a) = a_1 m k_1 a_1^a + a_1 m k_2 a_2^a
\]

\[
k_1 n (k_1 a_1^a + k_2 a_2^a) = a_1 m k_2 a_2^a
\]

\[
\frac{n}{m} = \left(\frac{k_2}{k_1}\right) \frac{a_1 a_2^a}{k_1 a_1^a + k_2 a_2^a}
\]

Therefore, we have that

\[
\frac{a_1^{a_1 + 1} + \left(\frac{k_2}{k_1}\right) a_1 a_2^a}{k_1 a_1^a + k_2 a_2^a} = \frac{a_1}{k_1}
\]

And subsequently, repeating the process for each variable, we get that

\[
\sum_{cyc} \frac{a_1^{a_1 + 1} + \left(\frac{k_2}{k_1}\right) a_1 a_2^a}{k_1 a_1^a + k_2 a_2^a} = \frac{\sum_{k=1}^{n} a_k}{k_1}
\]

2.2 Proof

Applying Lemma 1, we derive that

\[
\left(\frac{k_2}{k_1}\right) \sum_{k=1}^{n} a_k^{a_1 + 1} \geq \left(\frac{k_2}{k_1}\right) \sum_{cyc} a_1 a_2^a
\]

Therefore, substituting in the expression of Lemma 2 and operating, we have that

\[
\sum_{cyc} \frac{a_1^{a_1 + 1} + \left(\frac{k_2}{k_1}\right) a_k^{a_1 + 1}}{k_1 a_1^a + k_2 a_2^a} \geq \frac{\sum_{k=1}^{n} a_k}{k_1}
\]

\[
\sum_{cyc} \frac{\left(\frac{k_1}{k_2}\right) a_1^{a_1 + 1} + \left(\frac{k_2}{k_1}\right) a_k^{a_1 + 1}}{k_1 a_1^a + k_2 a_2^a} \geq \frac{\sum_{k=1}^{n} a_k}{k_1}
\]

3
\[
\sum_{cyc} \left( \frac{k_1 + k_2}{k_1} \right) a_k^{\alpha+1} \geq \sum_{k=1}^{n} a_k \left( k_1 + k_2 \right)
\]

As we wanted to prove.